Revista Colombiana de Matemáticas
Volumen II, 1968, págs. 45-49

ON THE METHOD OF THE STEEPEST DESCENT

by

Guillermo RESTREPO

Let H be a Hilbert space over the reals and let $f: H \to R^1$ be a function of class C^1. We have shown in |1| that the differential equation

$$\frac{du}{dt} = - T(u(t)), \quad u(0) = u_o \tag{1}$$

(T = grad f) has global solutions if

i) $\langle T(x)-T(y), x-y\rangle \geq c\|x-y\|^2, \quad c > 0$;

ii) f is bounded from below,

iii) T is locally Lipschitzian.

To be precise, in [1;Th.3] we heve assumed f to be of class C^2 and f'' to be locally bounded. However, the hypothesis f'' is locally bounded implies that $f' = T$ is locally Lipschitzian, and this is what matters to show existence and uniqueness.

The condition i) implies that f is strictly convex and i) and ii) together imply that f has a unique critical point. If $u(t) = u(t;u_o)$, $t \geq 0$, is the solution of (1) through u_o then $u(t)$ converges in the norm topology to the critical point of f when t tends to infinity.

In practice it is not possible to solve the differential equation (1), so the theorem stated above does not give a method for finding the solutions of $Tu = 0$. This is why we need an iterative procedure. In this paper we study the convergence of the so called <<steepest descent>> aproximations of $Tu = 0$. From now on we make the following assumptions on f and $T = f'$:

i) $\langle T(x)-T(y), x-y\rangle \geq c\|x-y\|^2, \; c > 0$ (2)

i.e., T is strongly monotone.

ii) $\|T(x) - T(y)\| \leq k\cdot\|x-y\| \;, \quad k > 0,$ (3)

i.e., T satisfies a global Lipschitz condition.

iii) f is bounded from below.

Let $x_o \in H$ and let $u_o = T(x_o)$. Inductively, we define

$$x_{n+1} = x_n - t_n u_n \tag{4}$$

where $u_n = T(x_n)$ and t_n is defined by the condition

$$\inf\left\{f(x_n - tu_n) \;;\; t \geq 0\right\} = f(x_n - t_n u_n) \tag{5}$$

We will give condition under which (x_n) converges to the critical point of f.

LEMMA 1 <u>The sequence (u_n), $u_n \in H$, defined above satisfies the orthogonality condition</u>

$$\langle u_n, u_{n+1}\rangle = 0 \tag{6}$$

Proof. Define $g(t) = f(x_n - tu_n)$. Then

$$g'(t_n) = \langle T(x_n - t_n u_n), -u_n\rangle = 0 = \langle u_{n+1}, -u_n\rangle$$

(by (5), t_n is a critical point of g).

LEMMA 2 i) <u>The sequence (u_n) satisfies the condition</u>

$$0 \leq \|u_{n+1}\|^2 \leq \|u_n\|^2(k^2 t_n^2 - 1) \tag{7}$$

ii) The sequence (t_n) satisfies

$$\frac{1}{c} \leq t_n \leq \frac{1}{c} \qquad (8)$$

Proof. We have

$$<u_{n+1} - u_n, u_{n+1} - u_n> = \|u_{n+1}\|^2 + \|u_n\|^2 ,$$

from (6),

$$\leq k^2 \| x_{n+1} - x_n \|^2 \quad , \quad \text{from } (3)$$

$$= k^2 t_n^2 \|u_n\|^2 \; , \quad \text{from } (4).$$

Thus, $0 \leq \|u_{n+1}\|^2 \leq \|u_n\|^2 (k^2 t_n^2 - 1)$ and $t_n \geq 1/k$.

Now, from (2) and (6) we get:

$$<u_{n+1} - u_n, x_{n+1} - x_n> = <u_{n+1} - u_n, -t_n u_n> \quad .$$

$$= t_n \|u_n\|^2 \geq c \|x_{n+1} - x_n\|^2 = c t_n^2 \|u_n\|^2 \; .$$

Thus,

$$t_n - c t_n^2 \geq 0. \qquad (8')$$

LEMMA 4 The sequence (u_n) converges to zero and

$$\|u_n\|^2 \leq \frac{2k^2}{c} (f(x_n) - f(x_{n+1})) \qquad (9)$$

Proof. We have

$$f(x_n) - f(x_{n+1}) = f(x_n) - f(x_n - t_n x_n)$$

$$= - \int_0^{t_n} \frac{d}{dt} f(x_n - t u_n) dt$$

$$= \int_0^{t_n} <T(x_n - t u_n), u_n> dt$$

$$= \int_0^{t_n} <T(x_n - t u_n) - T(x_{n+1}), (t_n - t) u_n> (t_n - t)^{-1} dt$$

(this follows from (6))

$$\geq \int_c^{t_n} c\|u_n\|^2(t_n - t)dt, \quad \text{from } (2)$$

$$= c\|u_n\|^2 t_n^2 (2^{-1})$$

$$\geq c\cdot 2^{-1}\cdot k^{-2}\|u_n\|^2 \quad , \quad \text{from } (8).$$

This concludes the proof of (9).

THEOREM <u>Let H be a Hilbert space and let</u> $f: H \to R^1$ <u>be a function of class</u> C^1, <u>bounded from below and satisfying conditions</u> (2) <u>and</u> (3). <u>Let</u> (x_n) <u>be the sequence defined in</u> (4). <u>Then</u> (x_n) <u>converges to the critical point</u> v <u>of</u> f <u>and</u>

$$\|v - x_n\| \leq \frac{\|u_n\|}{c} \leq \frac{k\sqrt{(2)}}{c\sqrt{(c)}}(f(x_n) - f(x_{n+1})) \qquad (10)$$

Proof. Since $T(v) = 0$, we obtain from (2):

$$c\|v - x_n\|^2 \leq \langle T(v)-T(x_n), v-x_n\rangle \leq \|T(x_n)\|\cdot\|v - x_n\|.$$

Thus, using (9)

$$\|v - x_n\| \leq \frac{1}{c}\|u_n\| \leq \frac{k\sqrt{(2)}}{c\sqrt{(c)}}(f(x_n) - f(x_{n+1})).$$

I would like to thank Professors V. ALBIS and J. LESMES for some improvements in the presentation of this paper.

REFERENCES

1. G. RESTREPO, La ecuación $d\varphi/dt = -\text{grad } f(\varphi(t))$ en un espacio de Hilbert, *Revista Colombiana de Matemáticas*, II (1967), 21-25

2. Djairo Guedes de FIGUEIREDO, *Topics in Nonlinear Functional Analysis*, Lecture Ser. no.48, University of Maryland, (1967)

3. A. LAGENBACH, Uber Gleichugen mit Potential-Opera-

toren und Minimalfolgen nichtquadratisher Functionale, *Math.Nach.* (1966)

University of Puerto Rico
Mayaguez, Puerto Rico

(Recibido en marzo de 1968)

ERRATA: The proof of lema 4 must be completed by adding the following statement: Since $f(x_n)$ is a bounded decreasing sequence we have that $\|u_n\| \to 0$.