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ON FUNCTIONS PRESERVING ALMOST RADIALITY AND THEIR RELATIONS TO RADIAL AND PSEUDO-RADIAL SPACES.

G. D. Dimov, R. Isler and G. Tironi (*)

Abstract: Some questions asked by A.V. Arhangel'skii, concerning pseudo-radial and almost radial spaces, are examined in this paper. It is shown that the Alexandroff compactification of the space constructed by Ostaszewski under Jensen's combinatorial principle is a compact pseudo-radial, not almost radial space. Three classes of mappings preserving almost radiality are found and relations between them and one other class of mappings closely connected with the class of almost radial spaces and lying between the classes of pseudo-open maps and quotient maps are fully examined. Also the relations between these four classes of mappings and the classes of pseudo-radial, almost radial and radial spaces are studied. In particular, it is shown that almost radiality is not preserved by pseudo-open or even open mappings. Some new generalizations of sequential and Fréchet spaces are introduced. Finally relations between closed mappings and some of the found mappings, preserving almost radiality, are considered.

Key words and phrases: Pseudo-radial spaces, almost radial spaces, t- and t_i-maps (i=1,2,3), pseudo-open, closed mappings, tightness.

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1. **Introduction**. The definitions of all notions mentioned in this section, if not explicitly given, can be found in section 2 of this paper or in [E].

In [AIT1] A. V. Arhangel'skii, R. Isler and G. Tironi introduced a new class of spaces, the class of almost radial spaces, which lies between the classes of radial and pseudo-radial spaces. They proved that a topological T_1 space is sequential if and only if it is almost radial and has countable tightness. This result gives a complete answer to the question of A. V. Arhangel'skii from [A1], question 3, if every pseudo-radial space with countable tightness is sequential. Let us recall that I. Jané, P. R. Meyer, P. Simon and R. G. Wilson in [JMSW] constructed three examples of non-sequential pseudo-radial spaces with countable tightness: one of them is T_2 but CH is required in its construction (it is based on an example given in [JKR]); the other two are T_c spaces (i.e. every convergent chain-net has a unique limit), so they are T_1 but not T_2. It was a question of Arhangel'skii if there exists a regular (normal, compact) pseudo-radial space with countable tightness which is not almost radial. Recently examples were produced of Hausdorff spaces with countable tightness, which are pseudo-radial but non-sequential (and hence not almost radial). One such example is given by P. Simon and G. Tironi [ST].

Independently a similar example was given by I. Juhász and W. Weiss [JW]. These examples can be easily shown to be even normal. As far as we know till now no such compact space is available. Some other questions were raised in discussions with A. V. Arhangel'skii about almost radial spaces: first, if every pseudo-radial compact space is almost radial and second, which classes of mappings preserve almost radiality; in particular if almost radiality is preserved under quotient, pseudo-open or open mappings.

In this paper we answer these questions. First of all we remark that the Alexandroff one-point compactification of the space Ω_0 constructed by A. S. Ostaszewski under Jensen's Combinatorial Principle $\Diamond$ in [O] gives an example of a compact Hausdorff pseudo-radial space with countable tightness which is not almost radial (and consequently is not sequential), answering in such a way the first two questions. Then we construct an example of an onto open mapping $f : X \to Y$, such that X is almost radial and Hausdorff, Y is T_2 but not almost radial. We describe also the "largest" subclass of the class of quotient mappings which preserves almost radiality (the elements of this class are called qt_1-maps) and define also two proper subclasses of it, the classes of qt_2-maps and of qt_3-maps. The class of almost radial spaces is precisely the image of the class of all orderable spaces under qt_i-maps, for every fixed $i = 1, 2, 3$. These classes of mappings are closed with respect to the composition of functions.

Let us recall that in [A2] A. V. Arhangel'skii characterized Fréchet spaces among Hausdorff spaces as pseudo-open images of metric spaces and in [F] S. P. Franklin proved that sequential spaces, and only these, are quotients of metric spaces. Analogous characterization of the class of radial and pseudo-radial spaces were given by H. Herrlich in [H]. He proved that pseudo-radial spaces and only these are quotients of orderable spaces and that radial spaces and only these are pseudo-open images of orderable spaces (this last result is not explicitly formulated in [H], but, as was pointed out in [A3], the proof is contained in Herrlich's paper). Since the class of almost-radial spaces contains the class of radial spaces and is contained in the pseudo-radial ones, one might expect, in the light of the results cited above, that there exists some class $\mathcal{F}$ of mappings which lies between the classes of quotient and pseudo-open mappings and is such that almost radial spaces, and only these, are images of the orderable spaces under mappings from $\mathcal{F}$. From the mentioned example of an open mapping not preserving almost radiality, we obtain immediately that such a class $\mathcal{F}$ cannot be closed with respect to the composition of maps. Such class of maps exists, it is here described, and its elements are called qt-maps. The relations between qt-maps and qt_i-maps ($i = 1,2,3$) are studied. In particular, it is shown that if $f : X \to Y$ is an qt-map and X is radial, then f is a qt_2-map, but even when the domain X is radial, these two classes do not coincide (see example 3.27).

It is true, however, that if $f : X \to Y$ is a qt_3-map and X is radial, then f is a qt-map, but the converse is not true even when f is a perfect map (see example 3.56).

Some other properties of qt-maps related to the preservation of the primitive tightness are studied (see 3.28 - 3.32).

In [A2] Arhangel'skii showed that if $f : X \to Y$ is a quotient map and Y is a T_2 Fréchet space, then f is a pseudo-open map (see also [F] and [E]). In particular, every quotient map between two Fréchet spaces is pseudo-open. It is natural to ask if some analogous result can be proved when X and Y are radial spaces. Let us note that every quotient map between two almost radial spaces must be a qt_1-map. But in the case of radial spaces the situation is more complicated. We give an example of a qt-map between two orderable spaces (and consequently radial), which is not pseudo-open (see 3.45). Then we prove that if $f : X \to Y$ is a quotient map between two radial spaces such that $t(X) \leq \aleph_1$ and Y is Hausdorff, then f is a qt-map. It is impossible to drop here the requirement that $t(X) \leq \aleph_1$. In fact an example is given of a quotient map $f : X \to Y$ such that X is orderable, $t(X) = \aleph_2$, Y is radial Hausdorff, but f is not qt. However it is easy to see that when the codomain of a quotient map is a radial space, then the map must be qt_2.

A class of spaces, called gF-spaces, is introduced, such that every quotient map from some topological space onto a gF-space is pseudo-open. Hausdorff Fréchet spaces are a proper subclass of gF-spaces, but not all orderable (even compact) spaces are gF-spaces.

Finally some simple observations concerning the relations between closed and qt_3-maps are made. Some of the results included here were announced (without proofs) in [DIT].

2. Old and new definitions. Preliminary results and observations.

In this paper all spaces are supposed to be T_1. Our standard notations and notions are as in [E], but tightness is denoted by $t(X)$.

Pseudo-radial or chain-net spaces as well as radial or Fréchet chain-net spaces were introduced by Herrlich in [H]. The same class of topological spaces was then considered by P. Meyer, S. Mrówka, M. Rajagopalan, K. Malliha Devi, T. Soundararajan in [DMR] and in [MRS], and systematically examined by A. V. Arhangel'skii in [A1] and [A3]. Some questions presented there stimulated the publication of other papers [JMSW], [T1], [FIT], [AIT1], [IT], [T2]. In [AIT2] some new cardinal invariants are introduced and their properties are studied in the class of chain-net spaces and in [AIT1] the notion of almost radial space is introduced and studied.

Let us first recall the basic equivalent definitions of pseudo-radial, radial and almost radial spaces.

2.1. Definition. For any cardinal number λ a <u>λ-sequence</u> $S = (x_\alpha)_{\alpha<\lambda}$ in a topological space X is a function from λ into X. The set of all limit points of S will be denoted by lim S.

2.2. Definition. [H] A topological space X is called <u>pseudo-radial</u> or <u>chain-net</u> if for every non-closed subset A of X there are a point $x \in \overline{A} \setminus A$ and a λ-sequence $(x_\alpha)_{\alpha<\lambda}$ in A converging to x.

2.3. Proposition. [A1]. A topological space X is pseudo-radial if and only if for every non closed subset A of X there exist a point $x \in \overline{A} \setminus A$ and a family $\mathcal{P}_x$ of subsets of X such that $\mathcal{P}_x$ is linearly ordered by inclusion and:

i) $P \cap A \neq \emptyset$ for every P in $\mathcal{P}_x$;
ii) for every neighbourhood U of x there is P in $\mathcal{P}_x$ such that $P \subset U$;
iii) $\cap \mathcal{P}_x = \{x\}$.

2.4. Definition. [A1], [AIT1]. A subset B of a topological space X is said to be topologically directed (in X) if there exists a point x in X such that for every neighbourhood U of x, $|B \setminus U| < |B|$. Such a point x will be called an end of B. If X is a Hausdorff space then there is only one end of B. The set of all end points of a subset B of X will be denoted by end B. So B is topologically directed if and only if end $B \neq \emptyset$.

2.5. Proposition. [A1] X is a pseudo-radial space if and only if for every non-closed subset A of X there exist a point $x \in \overline{A} \setminus A$ and a subset B_x of A which is topologically directed in X, has regular cardinality and is such that $x \in$ end B_x.

2.6. Definition. [H] A topological space X is called a radial or Fréchet chain-net space if for every A, subset of X, and every $x \in \overline{A}$ there exist a λ-sequence in A converging to x.

2.7. Remark. [A1] If in Proposition 2.3 and in Proposition 2.5 we substitute " there exist a point $x \in \overline{A} \setminus A$ and " with " and for every point $x \in \overline{A} \setminus A$ there exists " we obtain two different characterizations of radial spaces.

2.8. Definition. [AIT1] Let X be a topological space and let be given a subset A and a point x of X. The point x is said to be a target point for A if for any subset B of A the following assertions are equivalent:
(i) $x \in \overline{B}$;
(ii) $|B| = |A|$.
This property will be denoted by x (tar) A.

2.9.Definition. [AIT1] A topological space X is called almost radial if for every non closed subset A of X there exist a subset B of A and a point $x \in \overline{A} \setminus A$ such that $|B|$ is a regular cardinal number and x (tar) B.

2.10. Definition. Let $S = (x_\alpha)_{\alpha<\lambda}$ be a λ-sequence of points of a topological space X and x a point of the space. The pair (S,x) is called a $t\lambda$-sequence if λ is an initial and regular ordinal number, $x \in \lim S$, $x_\alpha \neq x_\beta$

for $\alpha \neq \beta$, $\alpha, \beta < \lambda$, and $x \notin \overline{\{x_\alpha \in S : \alpha < \beta\}}$ for every $\beta < \lambda$. If A is a subset of X and $S \subset A$, we shall say that (S,x) is a tλ-sequence in A. It will be convenient to consider every one-point sequence as a t1-sequence.

2.11. Proposition. [AIT1] X is almost radial if and only if for every non closed subset A of X there exist a point $x \in \overline{A} \setminus A$ and a λ-sequence S of points of A, such that (S,x) is a tλ-sequence.

2.12. Facts. [AITi] Let X be a topological space and $A \subset X$, $x \in X$.

a) If $|A| > 1$, then x (tar) A implies $x \in \overline{A} \setminus A$.

b) If A is infinite and x (tar) A, then $x \in$ end A.

c) If X is a Hausdorff space and x, y are target points for A, then $x = y$.

2.13. Theorem. [AIT1] The following hold:

a) Every radial space is almost radial.

b) Every almost radial space is pseudo-radial.

c) Every sequential space is almost radial.

2.14. Theorem. [AIT1] A space X is sequential if and only if it is almost radial and its tightness is countable.

2.15. Notations. Let X be a topological space and $A \subset X$. We shall use the following notations:

Lim A = { $x \in X$: there exists a λ-sequence S of points in A, such that $x \in \lim S$};

t-Lim A = { $x \in X$: there exists a λ-sequence S of points in A, such that (S,x) is a tλ-sequence}.

2.16. Using these notations we can reformulate the definitions 2.2, 2.6 and 2.9 as follows (recall that the original Herrlich's definitions of pseudo-radial and radial spaces [H] were in fact as in the proposition below):

Proposition. [H] The following hold:

a) A topological space X is pseudo-radial if and only if every subset A of X, for which Lim $A \subset A$ holds, is closed in X.

b) A topological space X is radial if and only if for every subset A of X, Lim $A = \overline{A}$ holds.

2.17. It is obvious that the following proposition is true:

Proposition. A topological space X is almost radial if and only if every subset A of X, for which t-Lim $A \subset A$ holds, is closed in X.

2.18. Let us mention that in [AIT1] the following proposition is in fact proved:

Proposition. A topological space X is radial if and only if for every subset A of X t-Lim $A = \overline{A}$ holds.

2.19. Definition. [J], [A4]. Let X be a topological space, A a subset of X and $x \in \overline{A} \setminus A$. The <u>primitive tightness of x with respect to A</u> is the least infinite cardinality of a set $B \subset A$ such that $x \in \overline{B}$; it is denoted by pt(x,A). (Writing "pt(x,A)" we shall always assume that $x \in \overline{A} \setminus A$ holds.)

2.20. Definition. Let X be a topological space, A, B disjoint subsets of X such that $A \cap \overline{B} \neq \emptyset$. The <u>primitive tightness of A with respect to B</u> is the following cardinal number: $pt(A,B) = \min\{ pt(a,B) : a \in A \cap \overline{B} \}$.
(Writing pt(A,B) we shall always assume that $A \cap B = \emptyset$ and $A \cap \overline{B} \neq \emptyset$ hold.)

2.21. Remark. Let us first recall the following definition given by R. Engelking in [E]:

The <u>tightness of a set A in a topological space X</u> is the smallest infinite cardinal number τ with the property that if $A \cap \overline{C} \neq \emptyset$, then there exist $C_0 \subset C$ such that $|C_0| \leq \tau$ and $A \cap \overline{C_0} \neq \emptyset$. This cardinal number is denoted by $\tau(A,X)$.

Let now $A \cap \overline{X \setminus A} \neq \emptyset$. Then it is easy to see that the following equalities are true:

$$\tau(A,X) = \sup\{ pt(A,C) : C \subset X, \overline{C} \cap A \neq \emptyset, C \cap A = \emptyset \} =$$
$$= \sup\{ \inf\{ |B| : B \subset C, \overline{B} \cap A \neq \emptyset \} : C \subset X \setminus A, \overline{C} \cap A \neq \emptyset \}.$$

From the Definition 2.20 we obtain immediately that if $A \cap \overline{X \setminus A} \neq \emptyset$, then $pt(A,X \setminus A) = \inf\{ \sup\{ |B| : B \subset C, \overline{B} \cap A \neq \emptyset \} : C \subset X \setminus A, \overline{C} \cap A \neq \emptyset \}$.

The comparison of these two formulas shows that while $pt(A,X \setminus A)$ can be defined as inf sup of some family of sets, $\tau(A,X)$ in turn can be defined as sup inf of the same family of sets.

2.22. In order to prove one simple result on the cardinal invariant pt(A,B) in radial spaces, we shall recall some definitions and some related results.

Definition. [AIT1] Let X be a topological space and x a point of it. The cardinal number

$q\chi(x,X) = \min\{ \tau :$ for any $A \subset X$ such that $x \in \overline{A} \setminus A$ there is a family γ of subsets of A such that $|\gamma| \leq \tau$, $x \notin \overline{P}$ for any P in γ, but $x \in \overline{\cup \gamma}$ $\}$

is called the quasi-character of X at x. The quasi-character of X is the cardinal number $q\chi(X) = \sup\{ q\chi(x,X) : x \in X\}$.

2.23. Theorem. [AIT1] For every almost radial space X the equality $q\chi(X) = t(X)$ holds.

2.24. Definition. [AIT2] Let X be a topological space and $x \in \overline{A} \setminus A$. The cardinal number

$pq\chi(x,A) = \min\{ \tau :$ there exists a family γ of subsets of A such that $|\gamma| \leq \tau$, $x \notin \overline{P}$ for any P in γ, but $x \in \overline{\cup \gamma}$ $\}$

is called primitive quasi-character of the point x with respect to the subset A.

2.25. Theorem. [AIT2] The following hold:

a) $q\chi(x,X) = \sup\{ pq\chi(x,A) : A \subset X$ and $x \in \overline{A} \setminus A \}$;

b) If X is radial then $pq\chi(x,A) = pt(x,A)$ for every subset A of X and every $x \in \overline{A} \setminus A$; consequently $q\chi(x,X) = t(x,X)$ for every x in X.

2.26. Definition. Let X be a topological space, A and B disjoint subsets of X such that $A \cap \overline{B} \neq \emptyset$. The cardinal number

$pq\chi(A,B) = \min\{ |\gamma| : \gamma \subset \exp B, \overline{P} \cap A = \emptyset$ for every P in γ, but $A \cap \overline{\cup\gamma} \neq \emptyset \}$

(where exp B is the set of all subsets of B) is called the primitive quasi-character of the subset A with respect to the subset B.

2.27. Proposition. Let X be a radial space, A a closed subset of it and B a subset of X such that $A \cap \overline{B} \neq \emptyset$ and $A \cap B = \emptyset$. Then $pq\chi(A,B) = pt(A,B)$.

Proof. Let $Y = X/A$, i.e. Y is the quotient space of X corresponding to the decomposition of X into the set A and the one-point sets $\{x\}$, when $x \notin A$. Then the natural mapping $q : X \to Y$ is a closed map and consequently Y is a radial space. Then, from 2.25 and the corresponding definitions we obtain that $pq\chi(A,B) = pq\chi(q(A),q(B)) = pt(q(A),q(B)) = pt(A,B)$.

2.28. Definition. A continuous map $f : X \to Y$ is called a $\underline{t_1\text{-map}}$ if for every $A \subset Y$, t-Lim $A \subset A$ implies t-Lim $f^{-1}A \subset f^{-1}A$. If, in addition, f is a quotient map, then it is called a $\underline{qt_1\text{-map}}$ (let us recall that by definition (see [E]) every quotient map is an onto map).

2.29 Definition. A continuous map $f : X \to Y$ is called a $\underline{t_2\text{-map}}$ if for every $x \in X$ and every $A \subset X$, such that x (tar) A with $|A|$ regular, there exists a subset $B \subset f(A)$, such that $|B|$ is regular and f(x) (tar) B. If, in addition, f is a quotient map, then f is called $\underline{qt_2\text{-map}}$.

2.30. It is easy to prove the following proposition:

Proposition. A continuous map $f : X \to Y$ is a t_2-map if and only if for every $t\lambda$-sequence (S,x) in X, there is a $t\lambda'$-sequence (S',f(x)) in the subset f(S) of Y.

2.31. Definition. A continuous map $f : X \to Y$ is called a $\underline{t_3\text{-map}}$ if for every $t\lambda$-sequence (S,x) in X the λ-sequence f(S) contains a cofinal λ'-sequence S', such that (S',f(x)) is a t λ'-sequence in Y. If, in addition, f is a quotient map, then f is called a $\underline{qt_3\text{-map}}$.

2.32. Definition. A continuous map $f : X \to Y$ is called a <u>t-map</u> if for every subset A of Y and every $x \in \overline{f^{-1}A} \setminus f^{-1}A$ there exists a subset B of A such that $f(x) \in \overline{B}$, $|B| = pt(f(x),B)$ and $f^{-1}f(x) \cap \overline{f^{-1}B} \neq \emptyset$. If, in addition, f is a quotient map, then f is called a <u>qt-map</u>.

2.33. Definitions.

a) A topological space X is called a <u>gF-space</u> if for every subset A of X and for every $x \in \overline{A} \setminus A$, there exists a subset B of A such that $\{x\} = \overline{B} \setminus B$.

b) A topological space X is called a <u>gs-space</u> if for every non closed subset A of X there exist a point $x \in \overline{A} \setminus A$ and a subset B of A such that $\{x\} = \overline{B} \setminus B$.

2.34. Example. [ST] There exists a first countable, Hausdorff, locally countable, locally compact space Z with cardinality $2^{\aleph_0}$ such that if $Y = Z \cup \{p\}$ with the neighbourhood base at p in Y consisting of all sets of the form $\{p\} \cup (Z \setminus A)$, where $A \subset Z$, A is closed in Z and $|A| \leq \aleph_0$, then Y is a pseudo-radial, Hausdorff, normal, non-sequential space with countable tightness. Let us remark also that in the space Z there exists a countable set A with $|\overline{A}| = 2^{\aleph_0}$.

2.35. Remark. The results listed in this section will be used continuously and often without explicit reference in the next section.

3. Theorems and examples.

3.1. Proposition. Let X be a locally compact, Hausdorff, locally countable, hereditarily separable, countably compact, non-compact, first-countable space. Then the Alexandroff one-point compactification αX of X is a Hausdorff compact pseudo-radial, which is not almost radial space and has countable tightness.

Proof. It is clear that $\alpha X = X \cup \{p\}$ is a Hausdorff compact space, which is hereditarily separable and hence $t(\alpha X) = \aleph_0$. Let us now show that αX is pseudo-radial. Since X is first-countable it is enough to consider sets A in αX, such that $\{p\} = \overline{A}^{\alpha X} \setminus A$. Hence A is a closed subset of X. But then the cardinality of A is greater than $\aleph_0$, since, if not, A would be countably compact and Lindelöf, i. e. compact, and hence $p \notin \overline{A}^{\alpha X}$.

Consequently A is an uncountable subset of X. Since the complements in αX of compact countable subsets of X form an open basis in αX at the point p, it is clear that every minimal well-ordering of the set A will be a λ-sequence convergent to p in αX. Hence αX is a pseudo-radial space.

We shall prove that αX is not almost radial. Let A be a closed subset of X such that $\overline{A}^{\alpha X} \setminus A = \{p\}$. Since X is countably compact no ω-sequence in A can converge to p. So, if $S = (x_\alpha)_{\alpha<\lambda}$ is a λ-sequence in A which converges to p, and λ is an initial regular ordinal number, then $\lambda > \omega_0$. Since X is hereditarily separable, then there exists a countable subset C of S, such that $\overline{S}^{\alpha X} = \overline{C}^{\alpha X}$. Consequently, there exists some ordinal number $\beta < \lambda$ such that $p \in \overline{\{x_\alpha \in S : \alpha < \beta\}}$, which shows that (S,p) is not a t λ-sequence. Hence t-Lim $A \subset A$, but A is not closed in αX. So, αX is not almost radial.

3.2. Example. ($\Diamond$) There exists a Hausdorff compact, pseudo-radial space with countable tightness, which is not almost radial (and hence is not sequential).

Proof. Let Ω_0 be the Ostaszewski's space from [O, p. 506], which is constructed under the Jensen's Combinatorial Principle $\Diamond$. Then Ω_0 satisfies all the hypothesis of the Proposition 3.1. Hence the Alexandroff compactification Y of Ω_0 is the desired example.

3.3. Remark. Our Proposition 3.1 was suggested by Example 1 from [JMSW]. Example 3.2 answers the first two questions of A. V. Arhangel'skii, which we already mentioned in the Introduction. In the rest of this paper we shall answer the third question of A. V. Arhangel'skii and prove some related results (see the Introduction).

3.4. Theorem. Let $f : X \to Y$ be a quotient mapping of an almost radial space X onto a topological space Y. Then Y is almost radial if and only if f is a t_1-map.

Proof. Let A be a subset of Y. We have that (A is a closed subset of Y) if and only if ($f^{-1}A$ is a closed subset of X) if and only if (t-Lim $f^{-1}A \subset f^{-1}A$).

If f is a t_1-map, then t-Lim $A \subset A$ implies t-Lim $f^{-1}A \subset f^{-1}A$ and hence A is a closed subset of Y. So Y is almost radial.

If Y is an almost radial space, then t-Lim $A \subset A$ implies that A is a closed subset of Y and hence t-Lim $f^{-1}A \subset f^{-1}A$.

Consequently, f is a t_1-map.

3.5. Remark. The previous theorem is analogous to Theorem 4 of A. V. Arhangel'skii from [A2] relative to the class of Hausdorff Fréchet spaces.

3.6. Proposition. The following are true:

a) The composition of two t_1-maps is a t_1-map; hence, the composition of two qt_1-maps is a qt_1-map.

b) Every t_2-map is a t_1-map; hence every qt_2-map is a qt_1-map.

c) There exists a quotient map which is not a qt_1-map.

Proof.

a) It is obvious.

b) Let $f : X \to Y$ be a t_2-map and let A be a subset of Y, such that t-Lim $A \subset A$. If t-Lim $(f^{-1}A)$ is not contained in $f^{-1}A$, then there exist a $t\lambda$-sequence (S,x) such that S is a λ-sequence in $f^{-1}A$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$. From Proposition 2.30 we obtain that there exists a t λ'-sequence $(S',f(x))$ in Y, such that S' is a λ'-sequence in A. Hence $f(x) \in$ t-Lim $A \subset A$, which is a contradiction since $x \notin f^{-1}A$. So, t-Lim $f^{-1}A \subset f^{-1}A$, and f is a t_1-map.

c) Let Y be the space, which was described in 2.34; i.e. Y is a T_2 pseudoradial space, which is not almost radial. Then, by Herrlich's theorem ([H], Theorem 1) there exist an ordered space X and a quotient onto map $f : X \to Y$. Since X is almost radial, from 3.4 we have that f cannot be a t_1-map.

3.7. Theorem. A topological space X is almost radial if and only if there exist an orderable space $\underline{X}$ and a qt_1-map $f : \underline{X} \to X$.

Proof. Since every orderable space is radial and hence almost radial, from 3.4 we obtain that, if $f : \underline{X} \to X$ is a qt_1-map, where $\underline{X}$ is orderable, then X is almost radial. Let X be an almost radial space. Let us consider the set tar-Lim X $= \{ (S^\alpha, x^\alpha) : \alpha \in \mathfrak{A} \}$, where (S^α, x^α) are $t\lambda_\alpha$-sequences for all $\alpha \in \mathfrak{A}$ and $(S^\alpha, x^\alpha) \neq (S^\beta, x^\beta)$ if $\alpha \neq \beta$ (but it is possible that $x^\alpha = x^\beta$ for $\alpha \neq \beta$ and $S^\gamma = S^\delta$ for $\gamma \neq \delta$). Let $S^\alpha = \{ x^\alpha_\beta : \beta < \lambda_\alpha \}$ and $\underline{X}^\alpha = \{ (x^\alpha_\beta, \alpha) : \beta < \lambda_\alpha \} \cup$ $\{ (x^\alpha, \alpha) \}$. Let us topologize $\underline{X}^\alpha$, for every $\alpha \in \mathfrak{A}$, letting all points of type (x^α_β, α) to be discrete and the open neighborhood basis at (x^α, α) to be given by sets $O^\alpha_\delta ((x^\alpha, \alpha)) = \{ (x^\alpha, \alpha) \} \cup \{ (x^\alpha_\beta, \alpha) \in \underline{X}^\alpha : \delta < \beta < \lambda_\alpha \}$, where $\delta < \lambda_\alpha$. Let $\underline{X}$ be the topological sum of all spaces $\{ \underline{X}^\alpha : \alpha \in \mathfrak{A} \}$ and let $f : \underline{X} \to X$ be defined by $f((x^\alpha_\beta, \alpha)) = x^\alpha_\beta$ and $f((x^\alpha, \alpha)) = x^\alpha$, for $\beta < \lambda_\alpha$, $\alpha \in \mathfrak{A}$.

From Herrlich's Lemma (see [H]) it follows that $\underline{X}$ is an orderable space. Since the one-point sequences are in tar-Lim X, then f is an onto mapping. Obviously f is continuous.

Let us prove that f is a quotient mapping. Let $A \subset X$ and $f^{-1}A$ be closed in $\underline{X}$. We shall show that A is closed in X, verifying that t-Lim $A \subset A$. Let (S,x) be a $t\lambda$-sequence in A (i.e. $S \subset A$). We must show that x is in A. But (S,x) is in tar-Lim X and hence there exists $\alpha \in \mathfrak{A}$ such that $S = S^\alpha$ and $x = x^\alpha$. Then $\{ (x^\alpha_\beta, \alpha) : \beta < \lambda_\alpha \} \subset f^{-1}A$. Since $f^{-1}A$ is closed in $\underline{X}$, then $f^{-1}A \cap \underline{X}^\alpha$ is closed in $\underline{X}^\alpha$. Hence $(x^\alpha, \alpha) \in f^{-1}A \cap \underline{X}^\alpha$, so $x = f((x^\alpha, \alpha)) \in A$ and A is closed in X, i. e. f is a quotient map.

Since $\underline{X}$ is radial and hence almost radial, and X is almost radial, we obtain, from 3.4, that f is a t_1-map.

3.8. Remark. Theorem 3.7 as well as its proof are analogous to the Theorem 1 in [H].

3.9. Proposition. Let $f : X \to Y$ be a t_1-map, (S,x) be a $t\lambda$-sequence in X and $f(x) \notin f(S)$. Then t-Lim (f(S)) $\subset$ f(S) (where f(S) is regarded now as a set).

Proof. Let $A = f(S)$. Since $x \notin f^{-1}A$, but $x \in$ t-Lim $f^{-1}A$, we obtain, using the definition of t_1-map, that t-Lim $A \subset A$.

3.10. Remark. Let us mention, on the occasion of 3.9, that if $f : X \to Y$ is a t_2-map and (S,x) is a $t\lambda$-sequence in X, then $f(x) \in$ t-Lim $f(S)$, where $f(S)$ is regarded as a set. Hence, if $f(x) \notin f(S)$ we have that $f(x) \in$ (t-Lim $f(S)) \setminus f(S)$ when f is a t_2-map, while when f is a t_1-map we can affirm only that (t-Lim $f(S)) \setminus f(S) \neq \emptyset$. This remark helps us to construct the following example.

3.11. Example. There exists a qt_1-map $f : X \to Y$, where X is an orderable space and Y is almost radial, which is not a qt_2-map.

Proof. Let R^* denote a copy of the real line R, disjoint from it and let $Y = R \cup R^* \cup \{p\}$, p being not a point of $R \cup R^*$. Let the topology on Y be the following: the topology on R is the natural topology; if $\pi : R^* \to R$ is the "identity" map then the basic open neighborhoods in Y of a point $x \in R^*$ are of the form $\{x\} \cup (O(\pi x) \setminus \{\pi x\})$, where $O(\pi x)$ is some open neighborhood of the point $\pi(x)$ in R. Let us fix some minimal well-ordering on R^*, i.e. $R^* = \{x_\alpha\}_{\alpha<}$; then the basic open neighborhoods of the point p in Y are as follows: let M be some countable closed (in R) subset of R and let α_M be an ordinal number such that $x_{\alpha_M} > x_\alpha$ for every $x_\alpha \in \pi^{-1}(M)$. Since $\aleph_0 < cf(2^{\aleph_0})$ (see for example [AP]), such α_M exists. Then put

$$U^M(p) = (R \setminus M) \cup \{p\} \cup \{ x_\alpha \in R^* : \alpha > \alpha_1\}, \text{ where } \alpha_1 > \alpha_M.$$

Let us prove now that Y is a T_1 almost radial space. Obviously Y is a T_1 topological space. Since the subspace $Y \setminus \{p\}$ is first-countable, then, in order to prove that Y is almost radial, it is enough to consider subsets A of $Y \setminus \{p\}$, such that $\{p\} = \overline{A}^Y \setminus A$. If $A \cap R = \emptyset$, then A must be a cofinal subset of the well-ordered set R^* and it is clear that $p \in$ t-Lim A.

Let now $A \cap R \neq \emptyset$. If $|A \cap R| = \aleph_0$, then $M = A \cap R$ is a countable closed subset of R and hence $p \notin \overline{M}^Y$. So, in this case, $p \in \overline{A \cap R^*}$ and we can argue as in the previous case. Finally let $|A \cap R| > \aleph_0$. Since $A \cap R$ is a closed subset of the real line with its natural topology, then $|A \cap R| = 2^{\aleph_0}$ and there are no more than $\aleph_0$ isolated points in the subspace $A \cap R$ (see for example [E]). So there are $2^{\aleph_0}$ accumulation points in $A \cap R$. If x is an accumulation point of $A \cap R$, then $\pi^{-1}x \in \overline{A \cap R}^Y \cap R^*$. Hence $|A \cap R^*| = 2^{\aleph_0}$ and so $A \cap R^*$ is cofinal in R^*. Now we can argue as in the first case.

Hence Y is an almost radial and, consequently, a pseudo-radial space. Let X be the Herrlich's space, which corresponds to Y; i. e. X is the topological sum of all convergent λ-sequences, topologized as in the proof of 3.7 (see [H]). Let $f: X \to Y$ be the natural map (see [H] or the proof of 3.7). Then, as it is proved in [H], f is a quotient map and, since X is an orderable space, and so almost radial, we obtain from 3.4 that f is a qt_1-map.

Let us prove that f is not a qt_2-map. Let us take some subset S of R with cardinality $\aleph_1$ and fix some minimal well-ordering on it. Then obviously the ω_1-sequence $S = \{x_\alpha\}_{\alpha<\omega_1}$ is convergent to p. So $S \cup \{p\}$ is an element of the topological sum X of all convergent λ-sequences in Y. Let us denote the set S and its points $\{x_\alpha : \alpha < \omega_1\}$ when they are regarded as subsets of X by S' and $\{x'_\alpha : \alpha < \omega_1\}$ and the point p of Y regarded as a point of X by p'. Since the topology on $S' \cup \{p'\}$ is as follows: points of S' are isolated, while the basic neighborhoods of p' are of the form $\{p'\} \cup \{x'_\alpha : \alpha > \beta\}$, for $\beta < \omega_1$, we can immediately see that (S',p') is a t ω_1-sequence in X. We have that $f(S') = S$ and $f(p') = p$. Since every ω_1-sequence in the set S converging to p cannot be a $t\omega_1$-sequence and since no sequence in the set S converges to p, we have that f is not a t_2-map.

3.12. Theorem. A topological space X is almost radial if and only if there exist an ordered space $\underline{X}$ and a qt_2-map $f: \underline{X} \to X$.

Proof. If such a map $f: \underline{X} \to X$ exists, then from 3.6 b) and 3.4 we obtain that X is almost radial. Let now X be almost radial. Then the space $\underline{X}$ and the onto map $f: \underline{X} \to X$ are constructed as in the proof of Theorem 3.7. It only remains to prove that f is a qt_2-map. We know from 3.7 that f is a quotient map and so we need to show that f is t_2. Let (S,x) be a $t\lambda$-sequence in $\underline{X}$. If $\underline{x}$ is an isolated point in $\underline{X}$, then S is the one-point sequence and all is clear. Suppose now that $\underline{x}$ is not an isolated point in $\underline{X}$. Then $\underline{x} = (x^\alpha, \alpha)$ for some $\alpha \in \mathfrak{A}$. Let $S' = S \cap \underline{X}^\alpha$. Then, obviously, $(S', \underline{x})$ is again a $t\lambda$-sequence in $\underline{X}$. Since $f(\underline{X}^\alpha) = (S^\alpha, x^\alpha)$, then $f((S',\underline{x}))$ is a λ-subsequence of (S^α, x^α) and hence, $(f(S'), x^\alpha)$ is a $t\lambda$-sequence in X. Since $f(S') \subset f(S)$, it follows from 2.30, that f is a t_2-map.

3.13. Proposition. The following hold:

a) The composition of two t_2-maps is a t_2-map; the same holds for qt_2-maps.

b) If $f : X \to Y$ is a qt_2-map and X is almost radial, then Y is almost radial.

c) Every t_3-map is a t_2-map; hence every qt_2-map is a qt_2-map.

Proof. a) follows easily from 2.30, for example. b) follows from 3.6 b) and 3.4. c) is obvious from 2.30.

3.14. Lemma. Let $f : X \to Y$ be a map, $S = (x_\alpha)_{\alpha<\lambda}$ be a λ-sequence in X and λ be a regular cardinal number. Then there exists a cofinal λ-subsequence S' of S, such that the map $f|S' : S' \to f(S')$ is either one-to-one or constant.

Proof. Let us suppose that there is no cofinal λ-subsequence of S on which f is constant. Then for every $\alpha<\lambda$ we shall have $|\{ x_\beta \in S : f(x_\alpha) = f(x_\beta)\}| < \lambda$. Using this fact and the regularity of λ, we can define by transfinite induction a cofinal λ-subsequence of S on which f is a one-to-one mapping, as follows. Let $z_1 = x_1$ and let us put $\alpha_1 = 1$. Hence, $z_1 = x_{\alpha_1}$. Let β be less than λ and let us suppose that the points $\{ x_{\alpha_\delta} : \delta < \beta\}$ be already defined. Then we put $\alpha_\beta = \min \{ \alpha < \lambda :$ for every $\alpha' \geq \alpha, f(x_{\alpha'}) \neq f(x_{\alpha_\delta})$, for every $\delta < \beta\}$ and $z_\beta = x_{\alpha_\beta}$. In such a way we define a λ-subsequence $S' = (z_\beta)_{\beta<\lambda}$ of S and, obviously, $f|S' : S' \to f(S')$ is one-to-one.

3.15. Proposition. The following are true:

a) If Y is a radial space and $f : X \to Y$ is a continuous map, then f is a t_2-map.

b) If X is a Fréchet space and $f : X \to Y$ is a continuous map, then f is a t_2-map.

Proof.

a) Let $S = (x_\alpha)_{\alpha<\lambda}$ and (S,x) be a $t\lambda$-sequence in X. If there exists a cofinal λ-subsequence of S on which f is constant, then all is clear. If not, then, by Lemma 3.14, there exists a cofinal λ-subsequence S' of S on which f is one-to-one. If $(f(S'),fx)$ is not a $t\lambda$-sequence in Y, then there exists a $\beta < \lambda$, such that $f(x) \in \overline{\{ f(x_\alpha) : \alpha<\beta \}}$. Since Y is radial, we have that $f(x) \overset{\bullet}{\in}$ t-Lim $(\{f(x_\alpha) : \alpha < \beta\})$. Hence f is a t_2-map.

b) Let $A \subset X$ and $x \in X$ be such that x(tar)A and $|A|$ is regular. Then $x \in \overline{A}$ and there exists a sequence $(x_n)_{n<\omega}$ in A converging to x. Obviously $(f(x_n))_{n<\omega}$ converges to $f(x)$ and $f(x)$ (tar) $(f(x_n))_{n<\omega}$. Hence f is a t_2-map.

3.16. Example. There exists a qt_2-map $f : X \to Y$, where X is orderable and Y is Hausdorff radial, which is not a qt_3-map.

Proof. Let Y be a set of cardinality $\aleph_1$. The topology on Y will be the following: all points of Y except one, say p, are isolated; the basic open neighborhoods in Y at p are the complements in Y of finite sets (which, of course, do not contain the point p). Obviously, Y is a Fréchet and, hence, a radial space. Let X be the space constructed in the proof of Theorem 1 of [H], starting with the space Y (see also the proof of 3.11). Then the "natural map" $f : X \to Y$, also constructed in [H], is a pseudo-open map and hence a quotient map. Now from 3.15 a) we obtain that f is a qt_2-map. Let us prove now that f is not a qt_3-map. If we fix some minimal well-ordering on $Y \setminus \{p\}$, i.e. $Y \setminus \{p\} = (y_\alpha)_{\alpha < \omega_1} = S$, then the ω_1-sequence S is convergent to p and hence it is contained in the topological sum X of all convergent λ-sequences in Y. But $S \cup \{p\}$ has a different topology as a subspace of X, namely, the points of S are isolated and the basic open neighborhoods at p in X are of the form $\{p\} \cup \{ y_\alpha \}_{\beta < \alpha < \omega_1}$, for every $\beta < \omega_1$. Hence (S,p) is a $t\,\omega_1$-sequence in X. Since $f(S) = S$ and since every cofinal λ-subsequence in S is of cardinality $\aleph_1$, and since p belongs to the closure in Y of every countable subset of Y, we obtain that f is not a qt_3-map.

3.17. Theorem. A topological space X is almost radial if and only if there exist an orderable space $\underline{X}$ and a qt_3-map $f : \underline{X} \to X$.

Proof. If such a map exists, then from 3.13 c) and 3.12 we obtain that X is almost radial. Let now X be almost radial. Then we construct the orderable space $\underline{X}$ and the onto map $f : \underline{X} \to X$ as in the proof of 3.12. In order to see that f is a qt_3-map it is enough to add to the proof of 3.12 the remark, that in the case when (S,x) is a $t\lambda$-sequence in $\underline{X}$ and x is not an isolated point in $\underline{X}$, the set S' (in the notations of the proof of 3.12) is cofinal in $\underline{X}^\alpha$ and hence f(S') is cofinal in f(S). Consequently, f is a qt_3-map.

3.18. Proposition. The following hold:

a) The composition of two t_3-maps is a t_3-map; the same is true for qt_3-maps.

b) If $f : X \to Y$ is a qt_3-map and X is an almost radial space, then Y is almost radial too.

Proof. a) It is obvious.

b) It follows, for example, from 3.13 c) and 3.13 b).

3.19. Proposition. If X is radial and $f : X \to Y$ is a t_3-map, then f is a t-map.

Proof. Let $A \subset Y$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$. Then, since X is radial, there exists a $t\lambda$-sequence (S,x) in $f^{-1}A$. Hence, there exists a $t\lambda$-sequence $(S', f(x))$ in A, since f is a t_3-map and since we can use the Lemma 3.14. If we put $B = S'$, then we obtain that $f(x) \in \overline{B}$, $B \subset A$, $f^{-1} f(x) \cap \overline{f^{-1}B} \neq \emptyset$ and $|B| = pt(f(x),B)$ (since $(S',f(x))$ is a $t\lambda$-sequence). Consequently, f is a t-map.

3.20. Proposition. If X is a radial space and $f : X \to Y$ is a t-map,then f is a t_2-map.

Proof. Let $A \subset X$ and $x \in X$ be such that x (tar) A and $|A|$ is regular. Let $A' = f(A)$. Then $x \in \overline{A} \subset \overline{f^{-1}f(A)} = \overline{f^{-1}A'}$ If $x \in f^{-1}A'$, then $f(x) \in f(A)$ and all is clear. Let $x \notin f^{-1}A'$. Since f is a t-map, then there exists a set $B \subset A'$, such that $f(x) \in \overline{B}$, $f^{-1} f(x) \cap \overline{f^{-1}B} \neq \emptyset$ and $|B| = pt(f(x),B)$.

Let $x_1 \in f^{-1} f(x) \cap \overline{f^{-1}B}$. If $x_1 \in f^{-1}B$ then $f(x_1) = f(x) \in B$, which is impossible (see 2.19). Hence $x_1 \in \overline{f^{-1}B} \setminus f^{-1}B$. Since X is radial, then there exists a $t\lambda$-sequence (S, x_1), where $S \subset f^{-1}B$. From 3.14 we obtain that there exists a cofinal λ-sequence S' of S on which f is one-to-one. Then $f(S')$ is a λ-sequence in Y which converges to $f(x)$ and $f(S') \subset B$. Let us put $B_1 = f(S')$. Then we have that:

a) $|B_1|$ is regular (since $|B_1| = |S'| = |S| = \lambda$ and λ is regular);
b) $B_1 \subset f(A)$ (since $A' = f(A)$) and
c) $f(x)$ (tar) B_1 (indeed, if $C \subset B_1$ and $|C| = |B_1| = \lambda$, then, since λ is regular, it follows that C is cofinal in $f(S')$ and hence $f(x) \in \overline{C}$; if $D \subset B_1$ and $f(x) \in \overline{D}$, then, since $|B| = pt(f(x),B)$, we have that $|B_1| = |B|$ and $|D| = |B_1|$). Hence f is a t_2-map.

3.21. Theorem. A topological space X is almost-radial if and only if there exist an orderable space $\underline{X}$ and a qt-map $f : \underline{X} \to X$.

Proof. If there exist an orderable space $\underline{X}$ and a qt-map $f : \underline{X} \to X$, then, by 3.20, f is a qt_2-map and from 3.12 X is almost radial.

Let now X be almost radial. Then from 3.17 and 3.19 it follows that there exist an orderable space $\underline{X}$ and a qt-map $f : \underline{X} \to X$.

3.22. Fact. Every pseudo-open map is a qt-map.

Proof. It is well known (see, for example, [AP]), that $f : X \to Y$ is pseudo-open if and only if for every subset A of Y and every $y \in \overline{A}$, $f^{-1}(y) \cap \overline{f^{-1}A} \neq \emptyset$ holds. Let now $A \subset Y$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$. Then $f(x) \in \overline{A}$ and there exists $B \subset A$, such that $f(x) \in \overline{B}$ and $|B| = pt(f(x),B)$. Since f is pseudo-open and $f(x) \in \overline{B}$, we obtain that $f^{-1}f(x) \cap \overline{f^{-1}B} \neq \emptyset$. So f is a t-map. Since every pseudo-open map is a quotient map, then f is a qt-map.

3.23. We are going now to construct an example of an open mapping which does not preserve almost radiality. For doing this we shall use the example of P. Simon and G. Tironi from [ST], which was described in 2.34 here.

We shall present our construction in the following abstract form (let us remark that all the hypothesis of the proposition stated below are fulfilled in many examples of pseudo-radial non-sequential spaces with countable tightness (see examples 1 and 2 from [JMSW] and the example from [JW])):

Proposition. Let τ be an infinite cardinal, $Y = Z \cup \{p\}$, where $|Z| \geq \tau^+$, Z is a Hausdorff, almost radial space and the neighbourhood base in Y at p consists of all sets of the form $\{p\} \cup (Z \setminus A)$, where A is a closed subset of Z and $|A| \leq \tau$. Then there exist a Hausdorff almost radial space X and an open onto mapping $f : X \to Y$. Moreover, if there exists a subset B of Z such that $|B| \leq \tau$ and $|\overline{B}^Z| \geq \tau^+$ then X is not radial. (Obviously, X is not radial also in the case that Y is not radial.)

Proof. Let Z_α be a copy of Z and let $\pi_\alpha : Z_\alpha \to Z$ be the "identity" map for every $\alpha < \tau^+$. Let Z' be the topological sum of all spaces Z_α for $\alpha < \tau^+$ and let us put $X = Z' \cup \{q\}$, where $q \notin Z'$. Let the open neighbourhood base in X at the point q consists of all sets of the form $\{q\} \cup \bigcup \{ Z_\alpha \setminus M_\alpha : \beta < \alpha < \tau^+\}$, where $\beta < \tau^+$, M_α is a closed subset of Z_α for every $\beta < \alpha < \tau^+$ and $|\bigcup \{\pi_\alpha(M_\alpha): \beta < \alpha < \tau^+\}| \leq \tau$. Let, finally, every open subset of Z' be open also in X.

In such a way we defined a topology on the set X, which is obviously Hausdorff. We shall show that X is almost radial. Indeed, let $A \subset X$. Since every Z_α is almost radial, it is sufficient to consider the case when $\{q\} = \overline{A}^X \setminus A$. Hence A is closed subset of Z'. Then $A_\alpha = A \cap Z_\alpha$ is a closed subset of Z_α for every $\alpha < \tau^+$. Since $q \in \overline{A}^X$, we have that $|\bigcup \{ \pi_\alpha(M_\alpha) : \beta < \alpha < \tau^+\}| \geq \tau^+$ for every $\beta < \tau^+$. Using this fact, one can easily construct by transfinite induction a τ^+-sequence S in A such that $|S \cap A_\alpha| \leq 1$ for every $\alpha < \tau^+$, $\pi_\alpha(S \cap A_\alpha) \cap \pi_\beta (S \cap A_\beta) = \emptyset$ for $\alpha \neq \beta$, $\alpha, \beta < \tau^+$ and if $x_{\beta_1} \in A_{\alpha_1}$, $x_{\beta_2} \in A_{\alpha_2}$ and $\alpha_1 < \alpha_2$, then $\beta_1 < \beta_2$ (for $\alpha_i, \beta_i < \tau^+$, $i = 1, 2$). Then, since τ^+ is a regular cardinal, (S, q) is a $t\tau^+$-sequence. Hence $q \in$ t-Lim A. Consequently, X is almost radial space.

Let us define now the map $f : X \to Y$ by the rule $f(q) = p$ and $f| Z_\alpha = \pi_\alpha$ for every $\alpha < \tau^+$. Then, obviously, f is a continuous onto map. Since the image of every basic open neighbourhood of every point of X is an open set in Y, the map f is open.

Let now B be a subset of Z with $|B| \leq \tau$ and $|\overline{B}^Z| \geq \tau^+$. We shall show that in this case X is not radial. Indeed, let $A = f^{-1}B$. Then, obviously, $q \in \overline{A}^X \setminus A$. Let us suppose that there exists a t λ-sequence (S,q) in A (see 2.18). Since τ^+ is a regular cardinal, we have that $\lambda \geq \tau^+$. On the other hand, since $|S| \leq |A|$ (see 2.10) and $|A| \leq \tau . \tau^+ = \tau^+$, it follows that $\lambda = |S| \leq \tau^+$. Hence, $\lambda = \tau^+$.
Since $|B| \leq \tau$, there exists $b \in B$ such that $|S \cap f^{-1}(b)| = \tau^+$. Then $S' = S \cap f^{-1}(b)$ is a cofinal subsequence of S. Hence S' converges to q. Since there exists a neighbourhood of q disjoint from S', we obtain a contradiction. Hence X is not radial.

3.24. Example. There exist a Hausdorff almost radial space X, a Hausdorff normal space Y and an open onto map $f : X \to Y$, such that Y is not almost radial; hence f is a qt-map but not a t_1-map (and, consequently, neither t_2 nor t_3).
Proof. Let Y be the space described in 2.34. Hence Y is Hausdorff and normal. Obviously, the space Y satisfies all the hypothesis of Proposition 3.23 for $\tau = \aleph_0$. Hence, there exist an almost radial Hausdorff space X and an open onto mapping $f : X \to Y$. Since Y is not sequential, but $t(Y) = \aleph_0$, we obtain from 2.14 that Y is not almost radial.

3.25.Example. The composition of two qt-maps is not, in general, a qt-map.
Proof. Let $f : X \to Y$ be as in 3.24. Since X is almost radial, then there exist an orderable space $\underline{X}$ and a qt-map $\underline{f} : \underline{X} \to X$ (see 3.21). Since every open map is a qt-map (see 3.22), $\underline{f}$, f and their composition $f \circ \underline{f}$ give the required example. If $f \circ \underline{f}$ was a qt-map, then the image Y had to be almost-radial (see 3.21).

3.26. Example. There exists a perfect onto qt_2-map $f : X \to Y$, such that Y is orderable and X is a compact Hausdorff but not pseudo-radial space. Hence the preimage of an orderable space under a qt_2-map or a qt-map is not, in general, a pseudo-radial space.
Proof. The continuous onto map $f : \beta N \setminus N \to W(\omega_1 + 1)$ constructed in [FR] has all the properties which we need (it follows from 3.15 a) that f is a t_2-map).

3.27. Example. There exist an orderable space X with $t(X) = \aleph_2$, a Hausdorff radial space Y and a qt_2-map $g : X \to Y$, such that g is not a t-map.

Proof. Let $Y_o = [0, \aleph_2]$, i. e. Y_o is the ordered set of all ordinal numbers not greater than $\aleph_2$. Let Y'' be the set of all non-isolated points of Y_o with countable character in the usual order-topology $\mathcal{T}$ on Y_o. We shall denote the closure of a subset M of Y_o in the space $(Y_o, \mathcal{T})$ by cl(M). Let Y' be a copy of Y'' disjoint from it and let $\pi : Y' \to Y_o$ be the "identity" embedding of Y' in Y_o. Hence, $\pi(Y') = Y''$.

Let now Y be the disjoint sum of the sets Y_o and Y'. We shall define a topology on Y as follows:

a) All points of the set $Y_o \setminus \{ \omega_2 \}$ will be isolated in Y;

b) If $y \in Y'$ then the open neighbourhood basis in Y at y will consist of all sets of the form $U_\delta(y) = \{y\} \cup \{ \alpha \in Y_o : \delta < \alpha < \pi(y) \}$, for $\delta < \pi(y)$, $(\delta \in Y_o)$;

c) The open neighbourhood basis in Y at ω_2 will consist of all sets of the form $U_B(\omega_2) = (Y_o \setminus B) \cup \pi^{-1}(Y_o \setminus cl(B))$, where B is a countable subset of Y_o.

Let us remark that cl(B) is also a countable set.

Obviously in such a way we defined a topology on the set Y. It is easy to see that the space Y is Hausdorff and radial. Let now $\mathcal{A}$ be the family of all ordered subsets of order-type ω_1 of $Y_o \setminus \{ \omega_2 \}$. For every $A \in \mathcal{A}$, let $Y(A) = \pi^{-1}(cl(A))$.

We are going now to construct the space X.

Let X_o be a copy of Y_o. Let $A \in \mathcal{A}$, $y \in Y(A)$ and $(y_n)_{n < \omega}$ be a sequence in A converging to $\pi(y)$ such that $\pi(y) = \sup\{ y_n : n < \omega \}$. Then we put

$$X_{A,(y,(y_n))} = \{y\} \cup \{y_n : n < \omega \}.$$

Let us put, for every $A \in \mathcal{A}$, $X_A = Y(A) \cup \{ \omega_2 \}$. From now on we shall denote the point ω_2 of X_A with ω_2^A in order to be explicit that it is regarded as a point of X_A.

Let X be the disjoint sum of X_o and all sets of the form $X_{A,(y,(y_n))}$ and X_A, which we just described above. We shall define a topology on the set X as follows:

1) All points of $X_o \setminus \{ \omega_2 \}$ are isolated and the open base in X at ω_2 consists of the sets of the form $U_\beta = \{ \alpha \in X_o : \alpha > \beta \}$, for $\beta \in X_o \setminus \{ \omega_2 \}$;

2) All points of $X_{A,(y,(y_n))} \setminus \{y\}$ are isolated and the open base in X at y consists of the sets of the form $U_n = \{y_n : n > n_o\}$, for $n_o \in \omega$;

3) All points of $X_A \setminus \{\omega_2^A\}$ are isolated and the open base of $\omega_2^A \in X_A$ in X consists of the sets of the form $U_\xi = \{y \in Y(A) : \pi(y) > \xi\} \cup \{\omega_2^A\}$, where $\xi \in Y_o$

and ξ < sup A. It follows from the Herrlich's lemma that X is an orderable space and hence a Hausdorff radial space.

For convenience, points of X will be denoted with a bar above, from now on.

Let $f : X \to Y$, be the natural "identity" map, i. e. $f(X_0) = Y_0$ and $f(\bar{\alpha}) = \alpha$, for every $\bar{\alpha}$ in X_0. Furthermore, for every $X_{A,(y,(y_n))}$, $f(\bar{y}) = y \in Y(A)$ and $f(\bar{y}_n) = y_n \in A$; for every X_A, $f(\bar{\omega}_2^A) = \omega_2 \in Y_0$ and $f(\bar{y}) = y \in Y(A)$.

We shall show that f is a quotient map. Obviously, f is a continuous onto map. Let $C \subset Y$ and let $f^{-1}C$ be closed. Let $C_0 = C \cap Y_0$ and $C_1 = C \cap Y'$. If $|C_0| \leq \aleph_0$, then C_0 is closed in Y_0. Hence if $y \in \overline{C_0}^Y \setminus C_0$, then there exists a sequence $S = \{y_n \in C_0 : n < \omega\}$, such that $\pi(y) = \sup S$. Obviously, we can define a subset A_y of Y_0, such that $S \subset A_y$ and $A_y \in \mathcal{A}$. Then we shall have that $X_{A,(y,(y_n))} \setminus \{y\} \subset f^{-1}C$ but, since $f^{-1}C$ is closed in X, it follows that $\bar{y} \in f^{-1}C$. Hence $y \in C$. So, in this case, $\overline{C_0} \subset C$. If $|C_0| = \aleph_1$, then, as in the previous case, we can show that if $y \in \overline{C_0} \cap Y'$ then y is in C. Hence we need only to prove that $\omega_2 \in C$, since, obviously, $\omega_2 \in \overline{C_0}$.

Since $|C_0| = \aleph_1$, we can find some $A \in \mathcal{A}$, such that $A \subset C_0$. From the previous remark, it follows that $Y(A) \subset C$. But then $X_A \setminus \{\omega_2^A\} \subset f^{-1}C$ and, since $f^{-1}C$ is closed in X, it follows that $\bar{\omega}_2^A \in f^{-1}C$. Hence $\omega_2 \in C$ and, again, $\overline{C_0} \subset C$.

Let now $|C_0| = \aleph_2$. Then C_0 is cofinal in Y_0 and hence $f^{-1}C \cap X_0$ is cofinal in X_0. Since $f^{-1}C$ is closed in X, it follows that $\bar{\omega}_2 \in f^{-1}C$. Hence, $\omega_2 \in C$. If $y \in \overline{C_0} \cap Y'$, then, arguing as above, we obtain that $y \in C$. So $C_0 \subset C$.

If $\omega_2 \in C_1$, then there exists $A \in \mathcal{A}$, such that $\omega_2 \in \overline{Y(A) \cap C}$. Then $X_A \cap f^{-1}C$ will be cofinal in X_A and from the closedness of $f^{-1}C$ we obtain that $\bar{\omega}_2^A \in f^{-1}C$. Hence $\omega_2 \in C$.

Since, obviously, $\overline{C_1} \subset C_1 \cup \{\omega_2\}$, we proved that $\overline{C_1} \subset C$. But $\overline{C} = \overline{C_0} \cup \overline{C_1}$. Hence $\overline{C} \subset C$; i. e. C is closed.

Now from 3.15 a) we obtain, since Y is radial, that f is a qt_2-map. We shall prove now that f is not a t-map.

Let $C = Y_0 \setminus \{\omega_2\}$. Then $X_0 \setminus \{\bar{\omega}_2\} \subset f^{-1}C$ and so $\bar{\omega}_2 \in \overline{f^{-1}C}^X$. We shall show that for any subset D of C, one of the following three conditions is not fulfilled:

(1) $\omega_2 \in \overline{D}^Y$,

(2) $pt(\omega_2, D) = |D|$,

(3) $f^{-1}(\omega_2) \cap \overline{f^{-1}D} \neq \emptyset$.

Indeed, let $D \subset C$. If $|D| = \aleph_0$, then $\omega_2 \notin \overline{D}$, and hence condition (1) is not fulfilled. If $|D| = \aleph_2$, then $\omega_2 \in \overline{D}$, but, since every subset of D having cardinality $\aleph_1$ has ω_2 in its closure, we obtain that $pt(\omega_2, D) = \aleph_1 \neq |D|$ and so the condition (2) is not fulfilled. If $|D| = \aleph_1$, then $\omega_2 \in \overline{D}$ and $pt(\omega_2, D) = |D|$, but we shall show that $f^{-1}(\omega_2) \cap \overline{f^{-1}D} = \emptyset$ and hence the condition (3) will not be satisfied.

Since $|D| = \aleph_1$, D is not a cofinal subset of Y_0 and hence $f^{-1}D \cap X_0$ is not cofinal in X_0, which means that $f^{-1}(\omega_2) \cap \overline{X_0 \cap f^{-1}D} = \emptyset$. Since $D \subset Y_0 \setminus \{\omega_2\}$, we have that $f^{-1}D \cap X_A = \emptyset$, for every $A \in \mathcal{A}$. Hence $f^{-1}(\omega_2) \cap \overline{f^{-1}D \cap X_A} = \emptyset$.

Obviously, $f^{-1}(\omega_2) \cap \overline{f^{-1}D \cap X_{A,(y,(y_n))}} = \emptyset$, for every component of the space X of the form $X_{A,(y,(y_n))}$.

Since $f^{-1}D = \overline{X_0 \cap f^{-1}D} \cup \bigcup\{\overline{f^{-1}D \cap X_A} : A \in \mathcal{A}\} \cup$

$\bigcup\{\overline{f^{-1}D \cap X_{A,(y,(y_n))}} : A \in \mathcal{A}, y \in Y(A), (y_n)_{n<\omega} \subset A, \pi(y) = \sup\{y_n : n \in \omega\}\}$, we obtain that $f^{-1}(\omega_2) \cap \overline{f^{-1}D} = \emptyset$.

Consequently, f is not a t-map, since $f^{-1}(\omega_2) = f^{-1} f(\overline{\omega_2})$.

3.28. Theorem. Let X be a radial space and $f : X \to Y$ be a continuous function. Then the following conditions are equivalent:
a) f is a t-map;
b) for every $A \subset Y$ and for every $x \in \overline{f^{-1}A} \setminus f^{-1}A$, there exists $B \subset A$, such that $pt(f(x),B) = pt(f^{-1}f(x), f^{-1}A)$ $(= |B|)$.

Proof. a) implies b): Let $A \subset Y$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$. Let us put $\tau_x = pt(f^{-1}f(x), f^{-1}A)$. Then there exists a point $\overline{x} \in f^{-1}f(x) \cap \overline{f^{-1}A}$, such that $pt(\overline{x}, f^{-1}A) = \tau_x$. So there is a set $C \subset f^{-1}A$, such that $\overline{x} \in \overline{C}$ and $|C| = \tau_x$. If $A_1 = f(C)$, then $|A_1| \leq \tau_x$, $A_1 \subset A$ and $\overline{x} \in \overline{f^{-1}A_1}$.

Since f is a t-map, then there exists a set $B \subset A_1$ such that $f(x) \in \overrightarrow{B}$, $|B| = pt(f(x),B)$ and $f^{-1}f(x) \cap \overline{f^{-1}B} \neq \emptyset$. Obviously $|B| \leq \tau_x$. We shall prove that $|B| = \tau_x$. Let $x' \in f^{-1}f(x) \cap \overline{f^{-1}B}$. Then, using 3.14 and the fact that X is radial, we can obtain a $t\lambda$-sequence (F,x') in $f^{-1}B$ such that $f|F$ is a one-to-one mapping. Since $x' \in f^{-1}f(x) \cap \overline{f^{-1}B}$, we have that $|F| \geq \tau_x = pt(f^{-1}f(x), f^{-1}A)$. Then, from $f(F) \subset B$ we have $|B| \geq \tau_x$ and, finally, $|B| = \tau_x$. So $|B| = pt(f(x),B) = pt(f^{-1}f(x), f^{-1}A)$.

b) implies a): Let us first remark that the following simple result holds

Claim. Let Z be a topological space; let H, D and E be subsets of Z such that $D \subset E$ and $H \cap \overline{D} = \emptyset$. Then $pt(H,E) \leq pt(H,D)$.

Let now $A \subset Y$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$. Since X is radial, using 3.14, we can find a set $C \subset f^{-1}A$ such that $|C| = pt(f^{-1}f(x), f^{-1}A)$, $f|C$ is one-to-one and x (tar) C for some $\overline{x} \in f^{-1}f(x) \cap \overline{f^{-1}A}$. Let $A_1 = f(C)$. Then $\overline{x} \in \overline{f^{-1}A_1}$ and $f(\overline{x}) = f(x)$. If $\overline{x} \in f^{-1}A_1$, then $f(x) \in f(C) \subset A$, i. e. $x \in f^{-1}A$, which is not true. Hence $\overline{x} \in \overline{f^{-1}A_1} \setminus f^{-1}(A_1)$ and we can use b). Consequently, there exists $B_1 \subset A_1$, such that $f(x) \in \overline{B_1}$ and $pt(f(x), B_1) = pt(f^{-1}f(x), f^{-1}A_1)$.

Let us put $\tau_x = pt(f^{-1}f(x), f^{-1}A_1)$. Then there exists $B \subset B_1$ such that $f(x) \in \overline{B}$ and $|B| = pt(f(x), B_1)$. Then $\tau_x = |B| = pt(f(x), B)$. We shall prove now that $f^{-1}f(x) \cap \overline{f^{-1}B} \neq \emptyset$. Since $B \subset B_1 \subset A_1 = f(C)$ and $f|C$ is one-to-one, then there is $C_1 \subset C$ such that $f(C_1) = B$ and $|C_1| = \tau_x$. Since $A_1 \subset A$ and $\overline{x} \in f^{-1}f(x) \cap \overline{f^{-1}A_1}$, it follows from the "claim" above that $\tau_x = |C_1| \leq |C| = pt(f^{-1}f(x), f^{-1}A) \leq pt(f^{-1}f(x), f^{-1}A_1) = \tau_x$.
Hence $|C_1| = |C|$ and, since $\overline{x}$ (tar) C, it follows that $\overline{x} \in \overline{C_1} \subset \overline{f^{-1}B}$.
So $f^{-1}f(x) \cap \overline{f^{-1}B} \neq \emptyset$, which implies that f is a t-map.

3.29. Example. There exist an almost radial Hausdorff space X, a pseudo-radial Hausdorff, normal space Y and an open onto mapping $f : X \to Y$ such that there exist $A \subset Y$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$ for which $pt(f(x),B) \neq pt(f^{-1}f(x),f^{-1}A)$, for every subset B of A, which contains f(x) in its closure.

(This example shows that the hypothesis of the radiality of the space X in Theorem 3.28 cannot be omitted.)

Proof. Let X, Y and $f : X \to Y$ be as in 3.24. Let us put $A = Z \subset Y$ and $x = q \in X$. Then $x \in \overline{f^{-1}A} \setminus f^{-1}A$, $f^{-1}f(x) = \{x\}$, $f(x) = p$ and $pt(f^{-1}f(x), f^{-1}A) = pt(x, f^{-1}A) = \aleph_1$. But for every $B \subset A$, such that $f(x) \in \overline{B}$, we have $pt(f(x),B) = pt(q,B) = \aleph_0 \neq pt(f^{-1}f(x), f^{-1}A)$.

3.30. Theorem. If X is a radial space and $f : X \to Y$ is a continuous map, then the following conditions are equivalent:

a) f is a t-map;

b) For every $A \subset Y$ and every $x \in \overline{f^{-1}A} \setminus f^{-1}A$, there exists $B \subset A$, such that $pt(f(x),B) = pt(f^{-1}f(x), f^{-1}B) = pt(f^{-1}f(x), f^{-1}A)$ $(= |B|)$.

Proof. a) implies b).

Let $A \subset Y$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$. Since f is a t-map, 3.28 b) is fulfilled. Then, as in the proof of the part "b) implies a)" of 3.28, we can find a set $B \subset A$, such that $|B| = pt(f(x),B) = pt(f^{-1}f(x), f^{-1}A)$ and $f^{-1}f(x) \cap \overline{f^{-1}B} \neq \emptyset$. From the claim in 3.28 we obtain that

$$pt(f^{-1}f(x), f^{-1}A) \leq pt(f^{-1}f(x), f^{-1}B).$$

Let $x' \in f^{-1}f(x) \cap \overline{f^{-1}B}$. Then $B \subset Y$, $x \in \overline{f^{-1}B} \setminus f^{-1}B$ and since f is a t-map, from 3.28 it follows that there exists a set $B_2 \subset B$, such that $|B_2| = pt(f(x), B_2) = pt(f^{-1}f(x), f^{-1}B)$. Since $B_2 \subset B$, we have:

$$pt(f^{-1}f(x), f^{-1}B) = |B_2| \leq |B| = pt(f^{-1}f(x), f^{-1}A).$$

Hence, $|B| = pt(f(x),B) = pt(f^{-1}f(x), f^{-1}B) = pt(f^{-1}f(x), f^{-1}A)$.

b) implies a) follows from 3.28.

3.31. Remark. Example 3.29 shows that the hypothesis of radiality of the domain space X in 3.30 cannot be omitted.

3.32. Corollaries. Let X be a radial space and $f : X \to Y$ a qt-map. Then:
a) If $x \in X$ is such that $f(x)$ is not isolated in Y, then there exists a set $B \subset Y \setminus \{f(x)\}$, such that $pt(f(x),B) = pt(f^{-1}f(x), f^{-1}B) = |B|$;
b) If $x \in X$ is such that $f^{-1}f(x) = \{x\}$ and $f(x)$ is not isolated in Y, then there exists $B \subset Y \setminus \{f(x)\}$, such that $pt(f(x),B) = pt(x, f^{-1}B) = |B|$;
c) If $x \in X$ is such that $f^{-1}f(x) = \{x\}$ and x is not isolated in X, then there exists $B \subset Y \setminus \{f(x)\}$, such that $pt(f(x),B) = pt(x, f^{-1}B) = |B|$;
d) If $x \in X$ is such that $f^{-1}f(x) = \{x\}$ and $x \in \overline{C} \setminus f^{-1}f(C)$ for some $C \subset X$, then there exists $B \subset f(C)$, such that $|B| = pt(f(x),B) = pt(x,f^{-1}B) = pt(x,f^{-1}f(C))$;
e) If $x \in X$ is such that $f^{-1}f(x) = \{x\}$ and x is not isolated in X, then there exists $B \subset Y \setminus \{f(x)\}$, such that $pt(f(x),B) = pt(x,f^{-1}B) = |B| = pt(x,X\setminus\{x\})$.

3.33. Proposition. Let $f : X \to Y$ be a pseudo-open onto map, $g : Y \to Z$ be a map and $h = g \circ f$. Then h is a t-map (a qt-map) if and only if g is a t-map (a qt-map).

3.34. Proposition. Let X be a radial space, $f : X \to Y$ be a pseudo-open onto map, $g : Y \to Z$ be a map and $h = g \circ f$. Then h is a t_2-map (a qt_2-map) if and only if g is a t_2-map (a qt_2-map).
Proof. Let g be a t_2-map. Then it follows from 3.22, 3.20 and 3.13 that h is a t_2-map. If h is a t_2-map, then it is not difficult to see that g is also a t_2-map.

3.35. Proposition. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps and $h = g \circ f$ be a t_3-map. Then f is a t_3-map.

3.36. Given a pseudo-radial space X let us recall that on the disjoint sum $\underline{X}$ of all convergent λ-sequences in X, the Herrlich's topology is defined as follows: if $S = (x_\alpha)_{\alpha < \lambda}$ is a convergent λ-sequence in X and $x \in \lim S$, then we take the set $S \cup \{x\}$ and introduce a topology on it taking all points of S as isolated, while basic neighborhoods of x are of the form $\{x_\alpha : \alpha > \beta\}$

$\cup \{x\}$, for every $\beta < \lambda$; if $x_1 \neq x$ and $x_1 \in \lim S$ the set $S \cup \{x_1\}$ is also considered. $\underline{X}$ is the topological sum of all these spaces.

Proposition. Let X be a pseudo-radial space, $\underline{X}$ be as above and $p : \underline{X} \to X$ be the natural "identity" map. Let $f : X \to Y$ be a continuous map and let $\underline{Y}$ and $q : \underline{Y} \to Y$ be the analogous to $\underline{X}$ and p. Since the image of a convergent λ-sequence in X is a convergent λ-sequence in Y, we can define in a natural way a map $\underline{f} : \underline{X} \to \underline{Y}$. Let us denote by $\underline{q}$ the restriction of q to $\underline{f}(\underline{X})$ Then we have:

a) $\underline{f}$ is an open map;

b) The following diagram is commutative;

$$\begin{tikzcd}
\underline{X} \arrow[r, "\underline{f}"] \arrow[d, "p"'] & \underline{f}(\underline{X}) \arrow[r, hook] \arrow[d, "\underline{q}"'] & \underline{Y} \arrow[dl, "q"] \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

c) f is pseudo-open (quotient) if and only if $\underline{q}$ is pseudo-open (quotient).

If X is supposed to be radial, then from [H] it follows that p is pseudo-open. Moreover we have that:

d) f is a t-map (a qt-map) if and only if $\underline{q}$ is a t-map (a qt-map).

e) f is a t_2-map (a qt_2-map) if and only if $\underline{q}$ is a t_2-map (a qt_2-map).

Proof. The proof is obvious from 3.33 and 3.34.

3.37. Definition. Let $f : X \to Y$ be a continuous map and let $A \subset Y$. We use the following notations:

a) f-Lim A $= \{y \in Y :$ there exists a λ-sequence F in A converging to y, which is the image of a converging λ'-sequence in X$\}$;

b) ft-Lim A $= \{y \in Y :$ there exists a t λ-sequence (F,y) in A which is the image of a t λ'-sequence in X$\}$.

3.38. Proposition. Let X be a pseudo-radial space and $f : X \to Y$ be a continuous onto map. Then the following conditions are equivalent:

a) f is a quotient map;

b) f-Lim $A \subset A$ implies Lim $A \subset A$, for every subset A of Y.

Proof. The proof is obvious from 3.36.

3.39. Theorem. Let X be a radial space and $f : X \to Y$ be a continuous map. The following conditions are equivalent:

a) f is a t-map (a qt-map);

b) $f(\overline{f^{-1}A}) \subset$ ft-Lim A, for every $A \subset Y$ (f is onto and A $f(\overline{f^{-1}A}) \subset$ ft-Lim A, for every non closed subset A of Y).

Proof. The proof is obvious from 3.36.

3.40. Definition. Let $f : X \to Y$ be a map and P be a topological property of f (we will write that f is a P-map). f is said to be globally hereditarily P-map (hereditarily P-map) if $f|A : A \to f(A)$ is a P-map for every $A \subset X$ (such that $A = f^{-1}f(A)$).

3.41. Remark. It is easy to show for a map $f : X \to Y$ that if it is a t_2-map, then it is globally hereditarily t_2-map, and if it is a t-map, then it is hereditarily such.

3.42. Proposition. Let $f : X \to Y$ be an onto map. Then:

a) f is hereditarily qt-map if and only if it is a pseudo-open map;

b) f is hereditarily qt_2-map if and only if it is a pseudo-open t_2-map;

c) if X or Y are radial, then f is hereditarily qt_2-map if and only if f is pseudo-open.

Proof. The proof follows from 3.41.

3.43. Theorem. Let X and Y be radial spaces, Y be Hausdorff, $t(X) \leq \aleph_1$ and $f : X \to Y$ be a quotient map. Then f is a t-map.

Proof. Let $A \subset Y$ and $x \in \overline{f^{-1}A} \setminus f^{-1}A$. Since $t(X) \leq \aleph_1$ and since X is radial, there exists a $t\lambda$-sequence (F,x) in $f^{-1}A$ with $\omega_0 \leq \lambda \leq \omega_1$. Since $f(x) \notin A$, using 3.14, we can think that F is chosen in such a way that $f|F$ is one-to-one. Then, if $B = f(F)$, we obtain that $|B| = \lambda \leq \omega_1$. If $pt(f(x),B) = \lambda$, then all is proved. Since $pt(f(x),B) \leq |B| = \lambda$, we must consider now only the case $\lambda = \omega_1$ and $pt(f(x),B) = \aleph_0$. Since Y is radial, there exists a sequence $(y_n : n < \omega_0)$ in B converging to $f(x)$. Then $C = \{y_n : n < \omega_0\} \cup \{f(x)\}$ is a compact subset of Y and hence, since Y is Hausdorff, C is closed in Y. Since f is quotient, the set $D = f^{-1}(\{y_n : n < \omega_0\})$ is not closed in X. Let $x' \in \overline{D} \setminus D$. Then there exists a $t\lambda'$-sequence (S,x') in D on which f is one-to-one (by 3.14). Since $f(S) \subset \{y_n : n < \omega_0\}$, we obtain that $\lambda' = \omega_0$. If we put now $B' = f(S)$ we have:

1) $B' \subset B \subset A$,

2) $|B'| = \aleph_0$,

3) $f(x) \in \overline{B'}$ and hence $|B'| = pt(f(x),B')$.
We shall prove now that $f^{-1}f(x) \cap \overline{f^{-1}B'} \neq \emptyset$. Indeed, since S converges to x' and f is continuous, $f(S) = B'$ converges to $f(x') \in \overline{B'}$. But $\overline{B'} = B' \cup \{f(x)\}$ and hence $f(x') = f(x)$, i. e. $x' \in f^{-1}f(x)$. Consequently, $x' \in f^{-1}f(x) \cap \overline{f^{-1}B'}$. This shows that f is a t-map.

3.44. Example. There exists an orderable space X with $t(X) = \aleph_2$, a Hausdorff radial space Y with $t(Y) = \aleph_1$ and a quotient map $f : X \to Y$, which is not a t-map. Hence, the hypothesis "$t(X) \leq \aleph_1$" in Theorem 3.43 cannot be omitted.
Proof. The spaces X and Y and the map $f : X \to Y$ from 3.27 are exactly what we need here.

3.45. Example. There exist an orderable space X with $t(X) \leq \aleph_1$, an orderable compact space Y and a qt-map $f : X \to Y$, such that f is not pseudo-open. Hence, the conclusion in 3.43 cannot be strengthened to "f is pseudo-open".
Proof. Let Y be the space of all ordinal numbers not greater than ω_1 with the natural topology, induced by the order. Let us denote by X_α the subset of Y consisting of all ordinal numbers less then or equal to α, where $\alpha \in Y$ and X' be the subset of Y consisting of all limit ordinal numbers in Y.
Let $X = \bigoplus \{ X_\alpha : \alpha \in X' \setminus \{\omega_1\} \} \oplus X'$ with the following topology: all points of $X_\alpha \setminus \{\alpha\}$ and $X' \setminus \{\omega_1\}$ are isolated in X; the open basic neighborhoods of $\alpha \in X_\alpha$ in X are of the form $U_\beta = \{\gamma \in X_\alpha : \gamma > \beta\}$, where $\beta < \alpha$ and $\alpha \in X' \setminus \{\omega_1\}$; the open basic neighborhoods of ω_1 in X are of the form $V_\beta = \{\gamma \in X' : \gamma > \beta\}$, for $\beta < \omega_1$. It will be convenient to denote the points of X with a bar below. Now we can define a natural onto map $f : X \to Y$ by $f(\underline{\beta}) = \beta$. Obviously Y is a compact orderable space, X is an orderable space (see [H]), $t(X) = \aleph_1$ and f is a continuous map. We shall show now that f is a quotient map (and hence, by 3.43, f will be a qt-map) and that f is not a pseudo-open map. Let $A \subset Y$ and $f^{-1}A$ be closed in X. In order to show that A is closed in Y it is enough to prove that $\mathrm{Lim}\, A \subset A$. Let $(y_\alpha : \alpha < \lambda)$ be a λ-sequence in A convergent to some point y. We shall show that $y \in A$. Indeed, if y is an isolated point of Y, then $y_\alpha = y$ for all $\alpha > \alpha_0$, for some $\alpha_0 < \lambda$. Hence $y \in A$. Let y be a limit ordinal. If $y \neq \omega_1$, then $y \in X_y$. Since the λ-sequence $(y_\alpha : \alpha < \lambda)$ can be considered as lying in X_y and, since $\{ \underline{y}_\alpha : \alpha < \lambda \} \subset f^{-1}A$, from the closedness of $f^{-1}A$ we obtain that $\underline{y} \in f^{-1}A$ and hence $y \in A$. Let now $y = \omega_1$. If $(y_\alpha : \alpha < \lambda)$ or some cofinal subsequence of it is in X', then arguing as before, we shall obtain that $y \in A$. In the opposite case

we can regard $(y_\alpha : \alpha < \lambda)$ as contained in $Y \setminus X'$. Since $Y \setminus \{\omega_1\}$ is a countably compact space we can obtain a new sequence $(y'_\alpha : \alpha < \lambda = \omega_1)$ converging to ω_1 and consisting of points y'_α of Y which are limit points of sequences in $\{y_\alpha : \alpha < \lambda\}$. Since, arguing as in previous cases,we can obtain that all these points y'_α belong to A, we come to the case already considered ($\{ y'_\alpha : \alpha < \lambda\} \subset X'$) and hence we conclude again that $y \in A$. Consequently, f is a quotient mapping.

In order to show that f is not pseudo-open map, we shall show that f is not hereditary quotient map. Indeed, let A be the subset of Y consisting of the set B of all isolated points of Y and of the point $\{\omega_1\}$, i. e. $A = B \cup \{\omega_1\}$. Since $f^{-1}A = \{\omega_1\} \oplus \bigoplus \{ X_\alpha^d : \alpha \in X' \setminus \{\omega_1\}\}$, where $X_\alpha^d = \{ \beta \in X_\alpha : \beta$ is isolated in Y$\}$, and since $f^{-1}B = \bigoplus \{ X_\alpha^d : \alpha \in X' \setminus \{\omega_1\}\}$, we have that $f^{-1}B$ is closed in $f^{-1}A$. But, obviously, B is not closed in A. Hence $f|f^{-1}A : f^{-1}A \to A$ is not a quotient map. Consequently, f is not a pseudo-open map.

3.46. Proposition. The following hold:

a) Every Hausdorff Fréchet space is a gF-space.

b) Every Hausdorff sequential space is a gs-space.

c) If $X = Y \cup \{p\}$ and all points of Y are isolated in X, then X is a gF-space.

d) The topological sum of gF-spaces is a gF-space.

e) Every subspace of a gF-space is a gF-space.

3.47. Proposition. The image of a gF-space under a closed mapping is a gF-space.

3.48. Theorem. If Y is a gF-space and $f : X \to Y$ is a quotient map, then f is a pseudo-open map.

Proof. Let $A \subset Y$ and $y \in \overline{A} \setminus A$. Since Y is a gF-space, there exists a set $C \subset A$ such that $\{y\} = \overline{C} \setminus C$. Since f is quotient, $f^{-1}C$ is not closed in X. Let $z \in \overline{f^{-1}C} \setminus f^{-1}C$. Then there exists a net $(z_\alpha : \alpha \in D)$ in $f^{-1}C$ which converges to z. Then $(f(z_\alpha))_{\alpha \in D}$ is a net in C convergent to f(z). Since $f(z) \notin C$, it follows that $f(z) = y$. Hence $z \in f^{-1}(y)$. Consequently, $f^{-1}(y) \cap \overline{f^{-1}A} \neq \emptyset$, i. e. f is a pseudo-open map.

3.49. Example. There exists a gF-space X which is not pseudo-radial.

Proof. $X = N \cup \{p\}$, where $p \in \beta N \setminus N$, with the subspace topology from βN is the required example.

3.50. Example. There exists a compact Hausdorff orderable space X, which is not a gF-space.

Proof. The space X of all ordinal numbers not greater then ω_1 with the order topology is such an example. Indeed, let Z be the subset of X consisting of all isolated points in X. Then $\omega_1 \in \overline{Z}$, but it is impossible (from the countably compactness of $[0, \omega_1)$) to find a subset B of Z such that $\overline{B} \setminus B = \{ \omega_1 \}$. Hence X is not a gF-space.

3.51. Remark. Let $\{X_\alpha : \alpha \in A\}$ be gF-spaces and let $X = \bigoplus \{X_\alpha : \alpha \in A\}$. Let p be a point not belonging to X and let $Y = X \cup \{p\}$. We shall introduce now a topology on Y as follows: let us fix some well-ordering on A; the basic open neighborhoods of the point p in Y will be of the form $U_\beta = \{p\} \cup \bigcup \{X_\alpha : \alpha > \beta\}$; the points of X will have the same basic neighborhoods in Y as in X. It easy to see that Y is a gF-space.

Using this construction we can obtain some non-trivial examples of gF-spaces.

3.52. Proposition. Let $f : X \to Y$ be a closed map, $S = (x_\alpha : \alpha < \lambda)$ and (S,x) be a $t\lambda$-sequence in X. Then the following conditions are equivalent:

a) $(f(S),f(x))$ is a $t\lambda$-sequence in Y;

b) $\overline{\{x_\alpha : \alpha < \alpha_0\}} \cap f^{-1}(f(x)) = \emptyset$, for every $\alpha_0 < \lambda$.

3.53. Corollary. Let $f : X \to Y$ be a closed map, (S,x) be a $t\lambda$-sequence in X and $f^{-1}(f(x)) = \{x\}$. Then $(f(S),f(x))$ is a $t\lambda$-sequence in Y.

3.54. Definition. Let $f : X \to Y$ be a map. Let $S = (x_\alpha : \alpha < \lambda)$ be a λ-sequence in X converging to a point x of X. The pair (S,x) will be called <u>$t_f\lambda$-sequence</u>, if λ is an initial regular ordinal and $\overline{\{x_\alpha : \alpha < \alpha_0\}} \cap f^{-1}(f(x)) = \emptyset$, for every $\alpha_0 < \lambda$.

3.55. Proposition. Let $f : X \to Y$ be a closed map. Then f is a t_3-map if and only if every $t\lambda$-sequence in X contains a cofinal $t_f\lambda$-sequence.

3.56. Example. There exist compact Hausdorff radial spaces X and Y with $t(X) = \aleph_1$, and a continuous (hence closed, perfect) map $f : X \to Y$, which is not a t_3-map.

Proof. Let X be the space $[0, \omega_1]$ of all ordinals not greater then ω_1 with the order topology. Let Z be the subset of X consisting of all limit ordinals and $Y = X/Z$, i. e. Y is the quotient space of X corresponding to the decomposition of

X into the set Z and one-point sets $\{x\}$, when $x \notin Z$. Let $f : X \to Y$ be the natural quotient map. Let $S = (\alpha : \alpha < \omega_1, \alpha \notin Z)$. Then (S, ω_1) is a $t\omega_1$-sequence in X. Obviously, every cofinal ω_1-subsequence of this ω_1-sequence is not a $t_f\,\omega_1$-sequence.

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G.D. Dimov: Mathematical Institute of the Bulgarian Academy of Sciences, P.O. Box 373, 1090 Sofia, Bulgaria.

R. Isler: Department of Applied Mathematics, University of Trieste, Piazzale Europa 1, 34100 Trieste, Italy.

G. Tironi: Department of Mathematical Sciences, University of Trieste, Piazzale Europa 1, 34100 Trieste, Italy.

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