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SOME COMBINATORIAL RESULTS ABOUT THE OPERATORS WITH JUMPING NONLINEARITIES

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Abstract: In this article various examples of the operators with jumping nonlinearities are constructed by means of a combinatorial method, developed in [1]. Among others, the following is proved: There exist operators with jumping nonlinearities $S_{\lambda,\mu}:R^n \to R^n$ such that the corresponding equation

$$S_{\lambda,\mu}(u)=f$$

has at least $\binom{n-1}{[(n-1)/2]}$ distinct solutions for almost every $f\in R^n$ (in the sense of the n-dimensional Lebesgue measure).

Key words: Jumping nonlinearity, Brouwer degree, multiplicity of solutions, n-dimensional cube.

Classification: 47H15, 55M25, 52A25, 05A15, 90C33

Introduction. This article can be regarded as a second part of [1], hence we shall not give any bibliographical comments here. They can be found in [1]. We shall also use the notation which was introduced in [1]. Nevertheless for the convenience of the reader, both the notation and the main results of [1] will be briefly repeated here.

The brackets [...] are used in a double sense: [a,b] is a closed interval of real numbers, [c] is the integer part of the real number c.

$\bar{n}=\{1,2,3,\dots,n\}$.

card ω is the number of the elements of the set ω.

For every vector $u=(u_i)_{i\in\bar{n}}\in R^n$ we can define two vectors $u^+=(u_i^+)_{i\in\bar{n}}\in R^n$ and $u^-=(u_i^-)_{i\in\bar{n}}\in R^n$ as follows:

$$u_i^+=\max\{u_i,0\},\quad u_i^-=\max\{-u_i,0\}$$

for every $i\in\bar{n}$. (Then $u=u^+-u^-$.)

Definition 1. Let $S:R^n\to R^n$ be a linear operator, let λ and μ be two real numbers. Then the equation

$$S_{\lambda,\mu}(u)=u+\lambda Su^{+}-\mu Su^{-}$$

defines the operator

$$S_{\lambda,\mu}:R^{n}\longrightarrow R^{n},$$

Any operator of this type is said to be an operator with jumping nonlinearity.

We are interested in the solvability of the equation

$$(1)\qquad S_{\lambda,\mu}(u)=f$$

for various $f\in R^{n}$.

Definition 2. Let $k(S_{\lambda,\mu},f)$ be the number of the distinct solutions to (1). Let

$$k(S_{\lambda,\mu})=\operatorname*{inf\,ess}_{f\in R^{n}}k(S_{\lambda,\mu},f)=\sup_{\mathcal{M}\in O_{n}}\inf_{f\in R^{n}-\mathcal{M}}k(S_{\lambda,\mu},f),$$

where O_{n} is the system of all the subsets of R^{n} which have zero n-dimensional Lebesgue measure.

Using the positive homogeneity of $S_{\lambda,\mu}$, we obtain easily from the general Brouwer degree theory

Theorem 1. Let $B\subset R^{n}$ be an open ball containing the origin 0. Let the Brouwer degree $\deg(S_{\lambda,\mu},0,B)$ of $S_{\lambda,\mu}$ w.r.t. the point 0 and the ball B be defined. Then

$$k(S_{\lambda,\mu})\geq|\deg(S_{\lambda,\mu},0,B)|$$

and

$$k(S_{\lambda,\mu})-\deg(S_{\lambda,\mu},0,B)\text{ is even.}$$

If $\deg(S_{\lambda,\mu},0,B)\neq 0$, then (1) has at least one solution for every $f\in R^{n}$.

Proof of this theorem can be found in [2].

For any $\omega\subset\bar{n}$ let us define the point $C_{\omega}=(c_{i}^{\omega})_{i\in\bar{n}}\in R^{n}$ by means of the formulae

$$c_{i}^{\omega}=-1\text{ if }i\in\omega,$$
$$c_{i}^{\omega}=1\text{ if }i\in\bar{n}-\omega.$$

The points C_{ω}, $\omega\subset\bar{n}$ are just all the vertices of the n-dimensional cube C^{n}. For every $\omega\subset\bar{n}$ we define the index of C_{ω}

$$i(C_{\omega})=(-1)^{\operatorname{card}\omega}.$$

(Then the indices of the vertices of C^{n} define a colouring of C^{n} in the sense of the graph theory.)

For every $i\in\bar{n}$ there are in C^{n} 2^{n-1} one-dimensional edges parallel to the x_i-coordinate axis in R^n. These edges are called i-edges in the sequel.

Convention. The word hyperplane will be used in a restricted sense. Namely, the word hyperplane without any additional specification will always denote an $(n-1)$-dimensional hyperplane in R^n which does not contain any vertex C_ω of C^n. If $\rho \subset R^n$ is such a hyperplane, then ρ^+ is the open half-space of R^n w.r.t. ρ which contains the points $(a,a,a,\dots,a)$ for all sufficiently large positive values of a. ρ^- is the opposite open half-space.

Definition 3. For any hyperplane $\rho \subset R^n$ (in the sense of Convention) let

$$d(\rho)=\Big|\sum_{C_\omega\in\rho^+} i(C_\omega)\Big|=\Big|\sum_{C_\omega\in\rho^-} i(C_\omega)\Big|.$$

For $i\in\bar{n}$ let $k_i(\rho)$ be the number of the i-edges of C^n which are intersected by ρ. Let

$$k(\rho)=\min\{k_i(\rho)\,|\,i\in\bar{n}\}.$$

Definition 4. An (n,d,k)-hyperplane is a hyperplane $\rho\subset R^n$ such that $d(\rho)=d$ and $k(\rho)=k$.

The main result of [1] is

Theorem 2. If there exists an (n,d,k)-hyperplane then there exists a linear operator $S:R^n\to R^n$ and two real numbers λ and μ such that

$$|\deg(S_{\lambda,\mu},0,B)|=d$$

and

$$k(S_{\lambda,\mu})=k.$$

For $n\leq 3$ and S symmetric, the converse implication is also true.

Remark 1. Let us recall that the proof of Theorem 1 is constructive.

Section 1. Three simple results

Example 1. Let n=1. $C^1=[-1,1]\subset R$, a hyperplane is a point. The point ρ is either an interior point of C^1 in which case $d(\rho)=k(\rho)=1$ or is a point outside $[-1,1]$ in which case $d(\rho)=k(\rho)=0$. Thus for n=1 there exist only (1,0, 0)- and (1,1,1)-hyperplanes.

Example 2. Let n=2. Then ρ is a straight line and there are only three substantially different possibilities for the position of ρ w.r.t. C^2. (See Fig. 1.) In the cases A and C in Fig. 1 $d(\rho)=k(\rho)=0$. In the case B

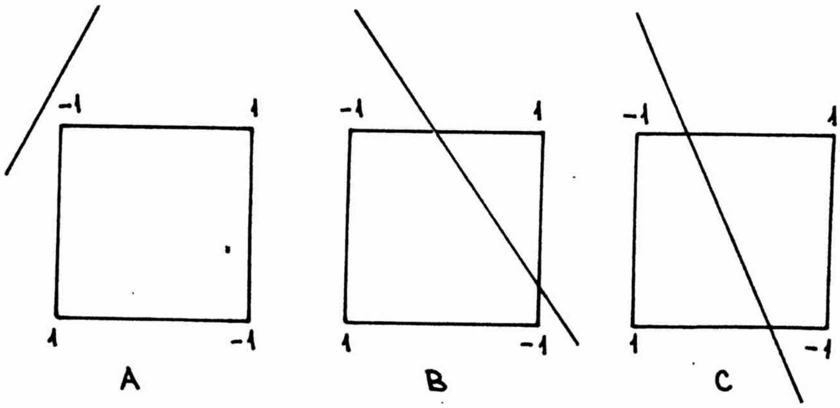

FIG. 1

obviously $d(\varrho)=k(\varrho)=1$. Thus for $n=2$ there exist only $(2,0,0)$- and $(2,1,1)$-hyperplanes.

Lemma 1. If there exists an (n,d,k)-hyperplane, then (m,d,k)-hyperplanes exist for every $m>n$.

Proof. It is sufficient to show that the existence of an (n,d,k)-hyperplane implies the existence of an $(n+1,d,k)$-hyperplane. The proof of this assertion is illustrated in Fig. 2.

Let

$$C_+^n = C^{n+1} \cap \{x \in R^{n+1} \mid x_{n+1}=1\},$$
$$C_-^n = C^{n+1} \cap \{x \in R^{n+1} \mid x_{n+1}=-1\}. \tag{2}$$

Both C_+^n and C_-^n are n-dimensional cubes.

Let $E: R^n \longrightarrow R^{n+1}$ be the mapping

$$E((x_1,x_2,\dots,x_n))=(x_1,x_2,\dots,x_n,1).$$

Then

$$E(R^n)=\{x \in R^{n+1} \mid x_{n+1}=1\},$$
$$E(C^n)=C_+^n \tag{3}$$

and the index of any vertex $C_\omega \in C^n$, $\omega \subset \bar{n}$ is equal to the index of the corresponding vertex $E(C_\omega) \in C_+^n \subset C^{n+1}$.

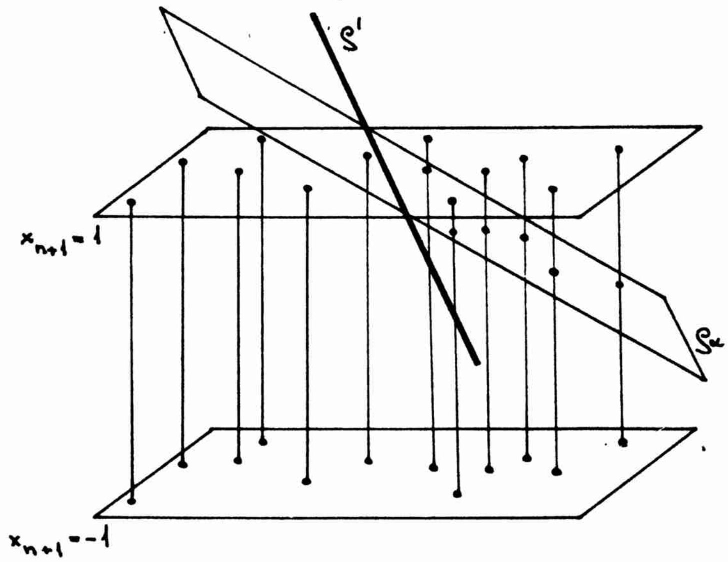

FIG. 2.

Let $\wp \subset R^n$ be an (n,d,k)-hyperplane, let

(4) $$\sum_{i\in\bar{n}} a_i x_i = b$$

be its equation. Then $\wp' = E(\wp)$ is an (n,d,k)-hyperplane w.r.t. $C_+^n = E(C^n)$ and the hyperplane (3), dim $\wp'$ =n-1. The equations of $\wp'$ are (4) and

$$x_{n+1}=1.$$

We can define for every $\alpha \in R$ a hyperplane $\wp_\alpha$ by the equation

(5) $$\alpha\Big(\sum_{i\in\bar{n}} a_i x_i - b\Big) = x_{n+1}-1.$$

Then

(6) $$\wp_\alpha \cap E(R^n) = \wp',$$

whenever $\alpha \neq 0$. But we shall investigate only those $\wp_\alpha$ for which

(7) $$0 < |\alpha| < 2/\Big(\sum_{i\in\bar{n}} |a_i| + b\Big).$$

If $x \in C_-^n$, then $|x_i| < 1$ for all $i \in \bar{n}$ and $x_{n+1} = -1$, hence

$$\Big|\alpha\Big(\sum_{i\in\bar{n}} a_i x_i - b\Big)\Big| < 2,$$

$$|x_{n+1}-1| = 2$$

and (5) cannot be fulfilled. Thus

(8) $$\rho_\alpha \cap C^n_- = \emptyset.$$

The last relation together with (6) implies that for $i \in \bar{n}$ (and α as in (7)) ρ_α intersects only those i-edges of C^{n+1} which are intersected by ρ' in C^n_+. But $\rho' = E(\rho)$ and $C^n_+ = E(C^n)$, hence

(9) $$k_i(\rho_\alpha) = k_i(\rho) \text{ for all } i \in \bar{n}.$$

The value of the term

(10) $$\sum_{i \in n} a_i x_i - b$$

is constant on every (n+1)-edge in C^{n+1} and it is on every such edge nonzero, because it is simultaneously the value of the same term in a vertex of C^n, ρ is given by (4) and must not contain any vertex of C^n. Now, we can see that the (n+1)-edges, for which (10) is positive, are intersected by ρ_α for $\alpha < 0$, the (n+1)-edges, for which (10) is negative, are intersected by ρ_α for $\alpha > 0$, because in both cases according to (7) and $|x_i| < 1$ for $x \in C^{n+1}$

$$-2 < \alpha(\sum_{i \in n} a_i x_i - b) < 0.$$

Now (5) implies $x_{n+1} \in]-1,1[$. Hence we can choose α so that ρ_α intersects at least one half of all the (n+1)-edges, that means

(11) $$k_{n+1}(\rho_\alpha) \geq 2^n/2 = 2^{n-1}.$$

On the other hand, there are only 2^{n-1} i-edges in C^n for every $i \in \bar{n}$, thus

(12) $$k_i(\rho) \leq 2^{n-1} \text{ for all } i \in \bar{n}.$$

The relations (9), (11) and (12) imply that

(13) $$k(\rho_\alpha) = k(\rho).$$

for a suitable α.

In one of the half-spaces of R^{n+1} w.r.t. ρ_α there are just the vertices of C^{n+1} which are in one of the n-dimensional half-spaces of (3) w.r.t. ρ'. To see it, one only needs to recall (8). Taking into account that the indices in C^n_+ are as in C^n, we obtain immediately

(14) $$d(\rho_\alpha) = d(\rho).$$

According to (13) and (14) ρ_α is an (n+1,d,k)-hyperplane.

Remark 2. A similar result for the operators with jumping nonlinearities is trivial. Given an operator $S_{\lambda,\mu}: R^n \to R^n$, one only needs to join to the matrix $S=(s_{ij})_{i,j\in\bar{n}}$ the entries $s_{i,n+1}=s_{n+1,j}=s_{n+1,n+1}=0$ for $i,j\in\bar{n}$ in order to obtain a matrix $\tilde{S}$. Then

$$\tilde{S}_{\lambda,\mu}:R^{n+1}\longrightarrow R^{n+1},\ d(\tilde{S}_{\lambda,\mu})=d(S_{\lambda,\mu}),\ k(\tilde{S}_{\lambda,\mu})=k(S_{\lambda,\mu}).$$

Lemma 2. If there exists an $(n+1,d,k)$-hyperplane which does not intersect at least one of the n-dimensional faces of C^{n+1}, then there exists an (n,d,k)-hyperplane, too.

Proof. We can assume that the $(n+1\ ,d,k)$-hyperplane $\tilde{\rho}$ does not intersect C^n_-,

(15) $$C^n_-\cap\tilde{\rho}=\emptyset,$$

because of the symmetry of C^{n+1}. We can also assume that $\tilde{\rho}$ is not parallel to (3). Otherwise it would be an $(n+1,0,0)$-hyperplane and $(n,0,0)$-hyperplane exists according to Example 1 and Lemma 1.

We can define the $(n-1)$-dimensional hyperplane

$$\rho'=\tilde{\rho}\cap E(R^n)\subset R^{n+1}$$

and the hyperplane

$$\rho=E^{-1}(\rho')\subset R^n.$$

We shall prove that ρ is an (n,d,k)-hyperplane.

In the proof of Lemma 1 we have deduced (14) from (8). The same argument applied to (15) gives

$$d(\tilde{\rho})=d(\rho).$$

Also, we obtain the equations

$$k_i(\tilde{\rho})=k_i(\rho)\ \text{for}\ i\in\bar{n}$$

which correspond to (9). Hence we only need to show that

(16) $$k_{n+1}(\tilde{\rho})\geq k_i(\tilde{\rho})\ \text{for every}\ i\in\bar{n}.$$

Let us choose an $i\in\bar{n}$ and an i-edge of C^{n+1} which is intersected by $\tilde{\rho}$. W.r.t. (15), it must be in C^n_+. Let A and B be its end-points. Each of them is also an end-point of an $(n+1)$-edge, let these edges be AA' and BB'. $A'B'$ is also an i-edge of C^{n+1},

(17) $$A'B'\in C^n_-.$$

The codimension of $\tilde{\rho}$ is 1, $\tilde{\rho}$ intersects AB, thus it must intersect another edge of the square $ABB'A'$. (15) and (17) imply that $\tilde{\rho}$ intersects either AA' or BB'. So we can define a mapping from the set of all the intersected i-edges into the set of all the intersected $(n+1)$-edges. If two intersected i-edges AB and CD are different, then the corresponding (n+1)-edges are also different, because every vertex of C^{n+1} is an end-point of just one i-edge.

Thus we have (16).

Remark 3. From this proof, a modification of the proof of Lemma 1 obviously follows. In fact, it is not important, whether we choose ρ_α with $\alpha > 0$ or with $\alpha < 0$, only (7) is important.

Remark 4. One can prove an analogous "reduction lemma" for the operators with jumping nonlinearities. But the proof of the "reduction lemma" in the case of general operators with jumping nonlinearities is rather complicated. It will be published elsewhere. Let us only mention that for the special operators which are investigated in [1] (see also (25), (26), (27)), the assumption (15) corresponds, roughly speaking, to the assumption that the values $a_{n+1}+\varepsilon$ and $a_{n+1}-\varepsilon$ are positive, but $|\varepsilon|$ is big enough w.r.t. a_{n+1}. (Cf. also Remark 2.)

Definition 5. Two vertices $C_{\omega_1} \in C^n$ and $C_{\omega_2} \in C^n$, $\omega_1, \omega_2 \subset \bar{n}$ are said to be neighbours, if there exists an edge in C^n which joins them.

Lemma 3. Let $\rho \subset R^n$ be a hyperplane. The following conditions are equivalent:

(i) There exist two opposite vertices C_ω and $C_{\bar{n}-\omega}$ in C^n such that C_ω and all its neighbours are in ρ^+ and $C_{\bar{n}-\omega}$ and all its neighbours are in ρ^-.

(ii) ρ intersects all (n-1)-dimensional faces of C^n.

Proof. Let (4) be the equation of ρ. Because of the symmetry of C^n we can assume that

(18) $$a_i \geq 0 \text{ for all } i \in \bar{n}.$$

Let

$$\varphi(x) = \sum_{i \in \bar{n}} a_i x_i - b,$$

then the equation of ρ can be rewritten in the form

(19) $$\varphi(x) = 0.$$

$$\max\{\varphi(C_\omega) \mid \omega \subset \bar{n}\} = \varphi(C_\emptyset) = \varphi((1,1,1,\dots,1)),$$

$$\min\{\varphi(C_\omega) \mid \omega \subset \bar{n}\} = \varphi(C_{\bar{n}}) = \varphi((-1,-1,-1,\dots,-1))$$

and

$$\varphi(C_\emptyset) > 0, \quad \varphi(C_{\bar{n}}) < 0,$$

if ρ intersects C^n.

Let, e.g., $\varphi(C_{\{n\}})=\varphi((1,1,1,\ldots,1,-1))<0$. ($C_{\{n\}}$ is a neighbour of $C_\emptyset$.) Then

(20) $\varphi(C_\omega)<0$ whenever $n\in\omega$

according to (18). But the convex hull of these points is just the (n-1)-dimensional face C_-^{n-1} (see (2)) and ρ does not intersect it according to (19) and (20).

Hence (the special case (18) of) (ii) implies (i) (with $\omega=\emptyset$). Now we shall assume (i) (but not necessarily (18)). Let, e.g.,

(21) $\varphi(C_\emptyset)>0$,

(22) $\varphi(C_\omega)>0$, whenever card $\omega=1$,

(23) $\varphi(C_{\bar{n}})<0$,

(24) $\varphi(C_\omega)<0$, whenever card $\omega=n-1$.

All (n-1)-dimensional faces of C^n are contained in the hyperplanes

$$\rho_i^+=\{x\in R^n \mid x_i=1\},$$

$$\rho_i^-=\{x\in R^n \mid x_i=-1\},\ i\in\bar{n}.$$

The face contained in ρ_i^+ contains $C_\emptyset$ and $C_{\bar{n}-\{i\}}$ and according to (19), (21), (24) ρ intersects it. The face contained in ρ_i^- contains $C_{\bar{n}}$ and $C_{\{i\}}$ and is intersected by ρ, according to (19), (22), (23).

Hence, (i) implies (ii).

Example 3. Let n=2. Any two opposite vertices of C^2 have common neighbours, hence according to Lemma 3 and Lemma 2, a (2,d,k)-hyperplane exists only if (1,d,k)-hyperplane exists. On the other hand, according to Lemma 1, if a (1,d,k)-hyperplane exists, a (2,d,k)-hyperplane exists, too. (Cf. Example 1 and 2.)

Example 4. Let n=3. Let ρ intersect all faces of C^3. Then there are two opposite vertices A and B in C^3 which satisfy (i) of Lemma 3. Thus A together with all its neighbours A_1, A_2, A_3 is in ρ^+, B together with its neighbours B_1, B_2, B_3 is in ρ^- and ρ must be as in Fig. 3. (In Fig. 3 we have a parallel projection of C^3 into R^2. The direction of the projection is parallel to ρ.) Then $d(\rho)=k(\rho)=2$ and this is the only case which can take place in R^3, but not in R^2. (C f. Lemma 2.) Hence, for n=3 there exist just 3 types of hyperplanes, namely (3,0,0)-, (3,1,1)- and (3,2,2)-hyperplanes. Of course, the last type is the most interesting one.

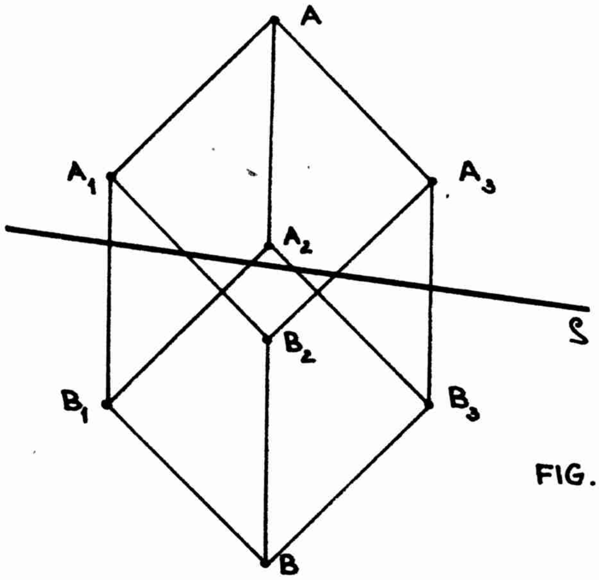

FIG. 3

Remark 5. Theorem 1 implies the following result: If $S: R^3 \longrightarrow R^3$ is a linear symmetric operator, then $d(S_{\lambda,\mu}) = k(S_{\lambda,\mu})$ and the common value of $d(S_{\lambda,\mu})$ and $k(S_{\lambda,\mu})$ is either 0 or 1 or 2. Nevertheless, the last assertion is true not only for $S_{\lambda,\mu}$ with a symmetric S, but also for general operators with jumping nonlinearities in R^3.

Section 2. The hyperplanes in R^4.

Definition 6. Let j be an integer, $1 \leq j \leq n$. The j-th level of C^n consists of all the vertices C_ω of C^n, for which card $\omega = j$. (Cf. Fig. 4, where a two-dimensional parallel projection of C^4 has been constructed.)

According to Example 4 and Lemma 1, (4,0,0)-, (4,1,1)- and (4,2,2)-hyperplanes exist. Any other hyperplane must satisfy (i) of Lemma 3 according to Example 4 and Lemma 2. W.r.t. the symmetries of C^4 we can assume that the two opposite vertices of Lemma 3(i) are $C_\emptyset$ and $C_{\bar{4}}$. Thus the 0-th and the first level of C^4 are in $\wp^+$, the third and the fourth level of C^4 are in $\wp^-$ and

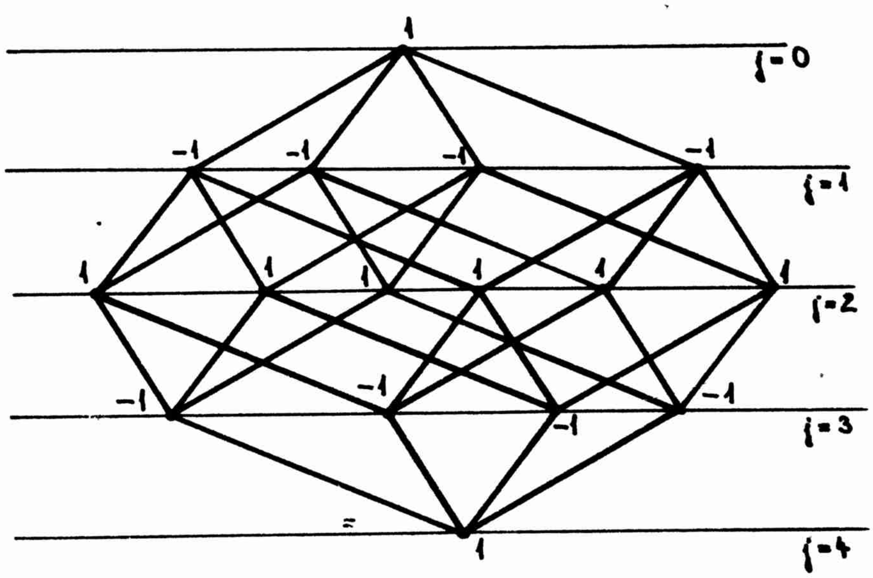

FIG. 4

$\wp$ can split only the second level. Because of the central symmetry of C^4 w.r.t. the origin O, we can further assume that $\wp^+$ contains at most as many vertices of C^4 as $\wp^-$. Hence from the six points of the second level at most three are in $\wp^+$.

If $\wp^+$ does not contain any point of the second level, then $\wp$ splits C^4 between the first and the second level intersecting just all the edges joining these two levels. We can easily calculate the numbers $d(\wp)$ and $k(\wp)$ and we obtain the existence of (4,3,3)-hyperplanes.

If $\wp^+$ contains three points of the second level, then obviously $d(\wp)=$ $=0$. Because of the symmetry of C^4 there are only three possibilities, how to divide six vertices of the second level into two triples. There are namely only three possibilities, how three vertices of the second level can be connected with the first level. These three cases are drawn in Fig. 5 and one can easily see that the other three points of the second level of C^4 which belong to $\wp^-$, are always connected with the third level in a way which is completely symmetric to the connection between the first three points and the first level.

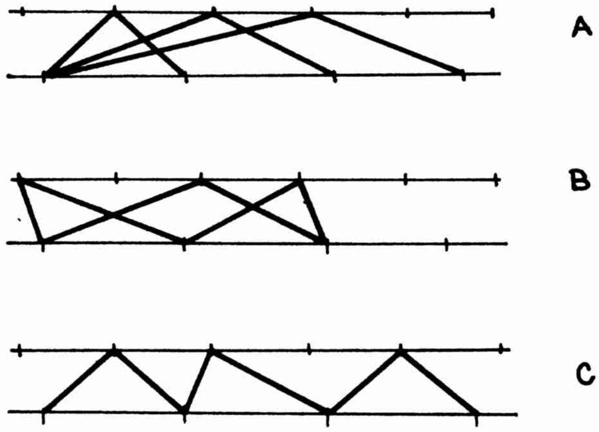

FIG. 5

An easy examination of the case C shows that this case is impossible. Namely, a partition of C^4 which corresponds to C in Fig. 5, can be carried through by means of some hypersurface, but not by a hyperplane.

In Fig. 6A, resp. 6B we can see one half of the edges which are not intersected by $\wp$. These figures correspond to Fig. 5A, resp. 5B.

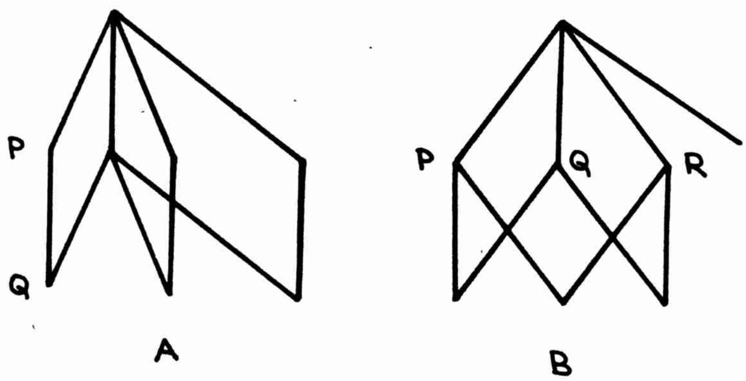

FIG. 6

In Fig. 6A there are four edges of the direction PQ. Hence ρ does not intersect 8 edges of this direction. But there are only 8 edges of this direction in the whole C^4, thus $k(\rho)=0$ in this case and ρ is a (4,0,0)-hyperplane.

Let us count the edges of different directions in Fig. 6B. We get the numbers 3,3,3,1. Multiplying by two gives 6,6,6,2, hence the number of the edges of different directions which are intersected by ρ, are 2,2,2,6 and $k(\rho)=2$. So we have found a way, how to construct (4,0,2)-hyperplanes.

The case, in which ρ^+ contains either two or one vertex of the second level, can be investigated similarly, but we shall not obtain any other type of hyperplanes. Hence for n=4 there exist just 5 types of hyperplanes, namely (4,0,0)-, (4,1,1)-, (4,2,2)-, (4,3,3)- and (4,0,2)-hyperplanes.

According to Theorem 1 we can construct an operator $S_{\lambda,\mu}: R^4 \to R^4$ such that $d(S_{\lambda,\mu})=0$ and $k(S_{\lambda,\mu})=2$. The construction given in Section 5 of [1] is inductive and leads to an operator of the type

$$\varepsilon u+Su^{+}+Su^{-}, \tag{25}$$

where

$$S = \begin{pmatrix} -1+a_1, & -1, & -1, & -1 \\ -1, & -1+a_2, & -1, & -1 \\ -1, & -1, & -1+a_3, & -1 \\ -1, & -1, & -1, & -1+a_4 \end{pmatrix} \tag{26}$$

and

$$a_i+\varepsilon>0,\ a_i-\varepsilon<0 \text{ for every } i\in\overline{4}. \tag{27}$$

The points P, Q, R in Fig. 6B are completely equivalent, hence we can seek for a matrix (26) with $a_1=a_2=a_3=a$, $a_4=b$.

Example 5. If ρ is as in Fig. 7, then certain inequalities for all the terms ϑ_ω, $\omega\subset\overline{4}$ must take place (for the definition of ϑ_ω see (38) in [1]). Each ϑ_ω corresponds to C_ω and must be either positive, if $C_\omega\in\rho^+$, or negative, if $C_\omega\in\rho^-$.

If we decide to seek for

$$\varepsilon>0, \tag{28}$$

then according to (27) we obtain

$$a>\varepsilon,\ b>\varepsilon. \tag{29}$$

Hence, all the inequalities for ϑ_ω, $\omega\subset\overline{4}$, which must be fulfilled, can be reduced to the following four of them:

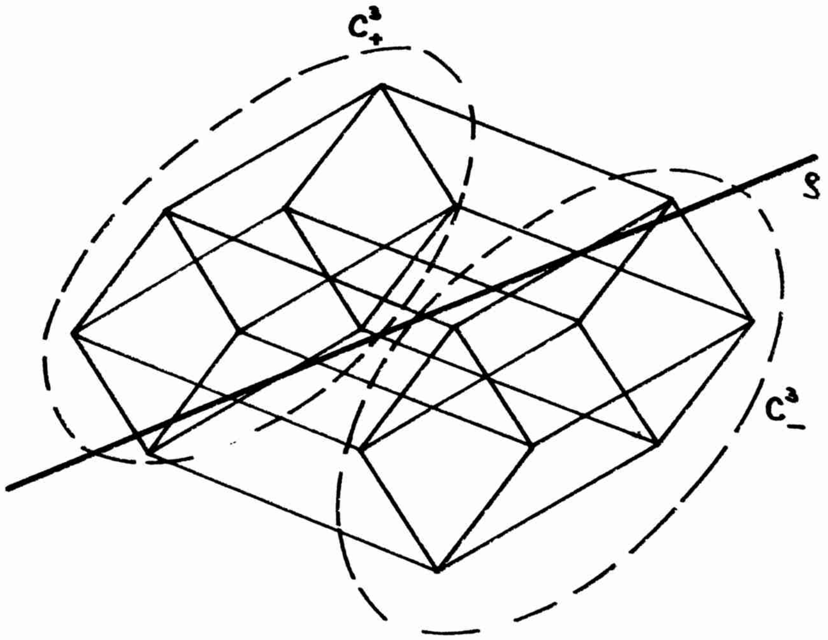

FIG. 7

$$
(30)\qquad
\begin{aligned}
&1-\frac{2}{a-\varepsilon}-\frac{1}{a+\varepsilon}-\frac{1}{b+\varepsilon}>0,\\
&1-\frac{3}{a+\varepsilon}-\frac{1}{b-\varepsilon}>0,\\
&1-\frac{3}{a-\varepsilon}-\frac{1}{b+\varepsilon}<0,\\
&1-\frac{2}{a+\varepsilon}-\frac{1}{a-\varepsilon}-\frac{1}{b-\varepsilon}<0.
\end{aligned}
$$

All the other inequalities for ϑ_ω are consequences of (28), (29) and (30). The inequalities (30) are fulfilled, if, e.g., $a=\frac{9}{2}$, $b=\frac{7}{2}$, $\varepsilon=1$. This choice of a, b and ε gives the example of the section 6 of [2].

On the other hand, by the method, developed in [1], one can construct to these values of a, b and ε the corresponding (4,0,2)-hyperplane ρ in R^4. The equation of ρ can be calculated to be

$$180(x_1+x_2+x_3)+308x_4= -43$$

and one can show that $\wp$ really intersects C^4 as in Fig. 7.

Example 6. We can also construct an analogous example following exactly Section 5 of [1]. We can choose $\wp$ as in Fig. 7. Let us notice the partition of C^4 into C^3_+ and C^3_-. For $\wp$ we can choose, e.g., the hyperplane

$$x_1+x_2+x_3+2x_4=0.$$

This hyperplane intersects C^3_+ so that it divides the vertex (-1,-1,-1,1) from all the other vertices of C^3_+. Further, this hyperplane passes through the centres of all the edges which join the vertex (-1,-1,-1,1) to the other vertices of C^3_+. With respect to the symmetry we see that $a_1=a_2=a_3$, hence we shall begin the inductive construction, described in Section 5 of [1], in the dimension 3.

$$C^3_0=C^4\cap\{x\in R^4\,|\,x_4=0\},$$

$$\wp_0=\wp\cap\{x\in R^4\,|\,x_4=0\},$$

hence the equations of $\wp_0$ are

(31) $$x_1+x_2+x_3=0,\ x_4=0.$$

After a transformation of the form

(32) $$3d\,\xi_i=x_i+d,\ i\in\overline{3},$$
$$3d\,\xi_4=2x_4,$$

we will get the new coordinates of the vertices of C^3_0

$$\xi_i=\frac{1}{\tilde{a}\pm\tilde{\varepsilon}},\ i\in\overline{3};\quad \xi_4=0,$$

where

$$\tilde{a}=\frac{3d^2}{d^2-1},$$

$$\tilde{\varepsilon}=\frac{3d}{d^2-1}.$$

According to (31) and (32), the new equations of $\wp$ will be

$$\xi_1+\xi_2+\xi_3=1,$$
$$\xi_4=0.$$

The relations corresponding to (27) should be satisfied, thus

$$d>1$$

f we want to get $\tilde{\varepsilon}>0$.

Let us choose, e.g. d=2. Now, we can make the "induction step" as in

Section 5 of [1]. We obtain the values

$a=4+2\sqrt{10}/3$,

$b=5/3+\sqrt{10}$,

$\varepsilon=2+\sqrt{10}/3$.

Hence, the matrix

$$\begin{pmatrix} 3+2\sqrt{10}/3 & -1 & -1 & -1 \\ -1 & 3+2\sqrt{10}/3 & -1 & -1 \\ -1 & -1 & 3+2\sqrt{10}/3 & -1 \\ -1 & -1 & -1 & 2/3+\sqrt{10} \end{pmatrix}.$$

and $\varepsilon=2+\sqrt{10}/3$ give another example of an operator with jumping nonlinearity of the form (25), for which $d(S_{\lambda,\mu})=0$ and $k(S_{\lambda,\mu})=2$.

For $d=7$ we obtain the rational values

$$a=\frac{217}{48}, \quad b=\frac{155}{48}, \quad \varepsilon=\frac{31}{48}$$

which also give an example of $S_{\lambda,\mu}$ with $d(S_{\lambda,\mu})=0$, $k(S_{\lambda,\mu})=2$. Other rational values a, b, ε can be obtained for $d=41$ and $d=239$. For $d=9/2$ we obtain values which are very near to the values of Example 5.

Section 3. The hyperplanes in R^n

Lemma 4. There exist $(n, \binom{n-1}{p}, \binom{n-1}{p})$-hyperplanes for every $n\in N$ and every integer $p\geq 0$. The equation of such a hyperplane $\wp_{n,p}$ is

(33) $$\sum_{i\in n} x_i=n-2p-1.$$

Proof. $\wp_{n,p}$ intersects C^n between the p-th and the $(p+1)$-th level, because

(34) $$\sum_{i\in n} x_i=n-2p$$

for the vertices of the p-th level and

$$\sum_{i\in n} x_i=n-2p-2$$

for the vertices of the $(p+1)$-th level. Hence, the levels from 0 to p are contained in $\wp^+_{n,p}$. In the j-th level of C^n there are $\binom{n}{j}$ vertices with the index $(-1)^j$, so we have

$$d(\wp_{n,p})=\left|\sum_{j=0}^{p}(-1)^j\binom{n}{j}\right|.$$

But

$$\sum_{j=0}^{p}(-1)^j\binom{n}{j}=(-1)^p\binom{n-1}{p},$$

thus

(35) $$d(\rho_{n,p})=\binom{n-1}{p}.$$

Every C_ω, $\omega\subset\bar{n}$, is connected by edges with all its neighbours and each neighbour of C_ω differs from C_ω in just one coordinate. So if C_ω is in the p-th level, then its neighbours are in the (p-1)-th and the (p+1)-th level. If $C_{\bar\omega}$ from the (p-1)-th level is a neighbour of C_ω from the p-th level, then card ω=p, card $\bar\omega$=p-1 and we see that $C_{\bar\omega}$ can be obtained from C_ω by changing the sign of one of the negative coordinates of C_ω. Hence C_ω is connected with the (p-1)-th level by one i-edge for every $i\in\omega$. Similarly one can show that C_ω is connected with the (p+1)-th level by one i-edge for every $i\in\bar{n}-\omega$.

Let $i\in\bar{n}$ be fixed. By i-edges, those C_ω in the p-th level are connected with the (p+1)-th level, for which $i\in\bar{n}-\omega$. There are $\binom{n-1}{p}$ vertices C_ω with $\omega\subset\bar{n}-\{i\}$, card ω=p, hence there are just $\binom{n-1}{p}$ i-edges connecting the p-th and the (p+1)-th level of C^n. But just these i-edges are intersected by $\rho_{n,p}$, thus

$$k_i(\rho_{n,p})=\binom{n-1}{p}.$$

According to the definition of $k(\rho_{n,p})$ this implies

(36) $$k(\rho_{n,p})=\binom{n-1}{p},$$

and the equations (35) and (36) prove the lemma.

$\max\left\{\binom{n-1}{p}\,\middle|\,p\geq 0\right\}=\binom{n-1}{[(n-1)/2]}$, hence a special case of Lemma 4 and (33) is

Lemma 5. There exist $\left(n,\binom{n-1}{[n-1/2]}\binom{n-1}{[n-1/2]}\right)$-hyperplanes for every $n\in\mathbf{N}$. The equation of such a hyperplane ρ_n is

(37) $$\sum_{i\in\bar{n}}x_i=n-2[(n-1)/2]-1.$$

Now we are able to prove

Theorem 3. Let $n\in\mathbf{N}$ be fixed. There exist (n,d,d)-hyperplanes for every integer d such that

$$0\leq d\leq\binom{n-1}{[(n-1)/2]}.$$

Proof. $\rho_n=\rho_{n,[(n-1)/2]}$ intersects C^n between the $[(n-1)/2]$-th and the $([(n-1)/2]+1)$-th level. Similarly (see (33)) the hyperplane $\sigma_n=\rho_{n,[(n-1)/2]-1}$ with the equation

(38) $$\sum_{i\in\bar{n}}x_i=n-2[(n-1)/2]+1$$

intersects C^n between the $[(n-1)/2]$-th and the $([(n-1)/2]-1)$-th level. If the coefficients on the left-hand sides of (37) and (38) are subjected to an arbitrary sufficiently small change, the resulting hyperplanes $\tilde{\wp}_n$, resp. $\tilde{\sigma}_n$ obviously have all the properties of the hyperplanes $\wp_n$, resp. σ_n. This means not only that the relative numbers k and d remain unchanged, but $\tilde{\wp}_n$, resp. $\tilde{\sigma}_n$ intersect just those edges which are intersected by $\wp_n$, resp. σ_n. Let

(39)
$$\sum_{i\in\pi} a_{n,i} x_i = n-2[(n-1)/2]-1$$

and

(40)
$$\sum_{i\in\pi} a_{n,i} x_i = n-2[(n-1)/2]+1,$$

where

(41)
$$|a_{n,i}-1| < \varepsilon$$

and $\varepsilon > 0$ is sufficiently small, be equations of $\tilde{\wp}_n$, and $\tilde{\sigma}_n$, resp.

All pairs of vertices $C_{\omega_1}, C_{\omega_2} \in C^n$ define finitely many directions and we can choose $a_{n,i}$, satisfying (41) so that neither of the hyperplanes $\wp_n(t)$

(42)
$$\sum_{i\in\pi} a_{n,i} x_i = n-2[(n-1)/2]+t, \quad t\in[-1,1]$$

is parallel to any of these directions. (Cf. (39), 40).) Hence, any $\wp_n(t)$ can contain at most 1 of the vertices of C^n. $\wp_n(-1)=\tilde{\wp}_n$ and $\tilde{\wp}_n^+$ contains the levels from 0 to $[(n-1)/2]$. If t increases from -1 to +1, the vertices of the $[(n-1)/2]$-th level pass one after another through $\wp_n(t)$ from the plus into the minus half-space of R^n w.r.t. $\wp_n(t)$, because $\wp_n(1)=\tilde{\sigma}_n$ and $\tilde{\sigma}_n^+$ contains only the levels from 0 to $[(n-1)/2]-1$. Let

(43)
$$t_1 < t_2 < t_3 < \dots < t_\nu ,$$

where

$$\nu = \binom{n}{[(n-1)/2]},$$

be all the values of $t \in [-1,1]$, for which $\wp_n(t)$ contains a vertex of the $[(n-1)/2]$-th level. The sum of the indices of the vertices in $\wp_n(t)^+$ is

(44)
$$(-1)^{[(n-1)/2]}\binom{n-1}{[(n-1)/2]}, \text{ if } t=-1,$$

(45)
$$(-1)^{[(n-1)/2]-1}\binom{n-1}{[(n-1)/2]-1}, \text{ if } t=1$$

and it changes by 1 or -1, whenever t growing from -1 to +1 passes across one of the values (43). Hence the sum of the indices of the vertices in $\wp_n(t)^+$

attains all the integer values between (44) and (45), when t varies over $[-1,1]$. The values (44) and (45) have opposite signs, thus $d(\varphi_n(t))$ attains all integer values between 0 and $\binom{n-1}{[(n-1)/2]}$ (some of them even twice!).

It remains to show that $d(\varphi_n(t))=k(\varphi_n(t))$ for all $t\in[-1,1]$ except (43). But this is not necessarily true unless we make an additional assumption about $\varphi_n(t)$. So let

$$a_{n,n}-1<-\sum_{i\in\overline{n-1}}|a_{n,i}-1|. \tag{46}$$

Let C_ω be in the $[(n-1)/2]$-th level of C^r According to (34) its coordinates fulfil the equation

$$\sum_{i\in\overline{n}}x_i=n-2[(n-1)/2],$$

hence

$$\sum_{i\in\overline{n}}a_{n,i}x_i=\sum_{i\in\overline{n}}(a_{n,i}-1)x_i+\sum_{i\in\overline{n}}x_i=\sum_{i\in\overline{n}}(a_{n,i}-1)x_i+n-2[(n-1)/2]$$

and

$$C_\omega\in\varphi_n\Big(\sum_{i\in\overline{n}}(a_{n,i}-1)x_i\Big)$$

according to (42). So the values t_r, $r\in\overline{\nu}$ in (43) are the values of

$$\sum_{i\in\overline{n}}(a_{n,i}-1)x_i$$

in the vertices C_ω of $[(n-1)/2]$-th level. If $n\in\omega$, then in C_ω

$$\sum_{i\in\overline{n}}(a_{n,i}-1)x_i=\sum_{i\in\overline{n-1}}(a_{n,i}-1)x_i-(a_{n,n}-1)\geq-\sum_{i\in\overline{n-1}}|a_{n,i}-1|-(a_{n,n}-1)>0,$$

because $|x_i|=1$ for $i\in\overline{n-1}$, $x_n=-1$ and we assume (46). If $n\notin\omega$, then we obtain similarly

$$\sum_{i\in\overline{n}}(a_{n,i}-1)x_i<0.$$

Hence we have (see (43))

$$-1<t_1<t_2<\ldots<t_{\nu_1}<0<t_{\nu_1+1}<\ldots<t_\nu<-1, \tag{47}$$

where

$$\nu_1=\binom{n-1}{[(n-1)/2]}$$

and we have just shown that the values $t_r<0$ in (47) correspond to the points C_ω with $n\notin\omega$ and the values $t_r>0$ correspond to the points C_ω with $n\in\omega$.

Let $t_0=-1$ and let us choose some $r\in\overline{\nu}_1$. In the interval (t_{r-1},t_r), $k_i(\varphi(t))$ is constant for every $i\in\overline{n}$. Let us look, what happens, when t passes through the value t_r.

If $C_{\omega(r)}$ is the vertex contained in $\varphi_n(t_r)$, then for $t<t_r$ $C_{\omega(r)}\in\varphi_n(t)^+$

and $\wp_n(t)$ intersects the edges connecting $C_{\omega(r)}$ with the $([(n-1)/2]+1)$-level, for $t > t_r$ $C_{\omega(r)} \in \wp_n(t)^-$ and $\wp_n(t)$ intersects the edges connecting $C_{\omega(r)}$ with the $([(n-1)/2]-1)$-th level. Hence for any two values $\tau_1, \tau_2 \in (t_{r-1}, t_{r+1})$ such that $\tau_1 < t_r < \tau_2$, all the edges which do not contain $C_{\omega(r)}$ are intersected by $\wp_n(\tau_1)$ if and only if they are intersected by $\wp_n(\tau_2)$. The edges which contain $C_{\omega(r)}$ are intersected by $\wp_n(\tau_1)$ if and only if they are not intersected by $\wp_n(\tau_2)$. Thus passing from τ_1 to τ_2, some of the values $k_i(\wp(t))$, $i \in \overline{n}$ increase, the other decrease by 1. But $n \notin \omega(r)$, hence the n-edge goes from $C_{\omega(r)}$ to the $([(n+1)/2]+1)$-th level and $k_n(\wp_n(t))$ decreases for each $r \in \overline{\nu}_1$. By induction w.r.t. r we can show that for each

$$t \in (t_{r-1}, t_r),\ r \in \overline{\nu}_1,$$

$$k_n(\wp_n(t)) = \min\{k_i(\wp_n(t)) \mid i \in \overline{n}\} = k(\wp_n(t)).$$

Thus, $k(\wp_n(t))$ drops by 1, whenever t passes through any of the values t_r, $r \in \overline{\nu}_1$. The same happens with $d(\wp_n(t))$, as we have seen above. For $t = -1$

$$d(\wp_n(t)) = k(\wp_n(t)), \tag{48}$$

hence, the equation (48) is true for any $t \in [-1,0]$ different from the values (43).

Now, it remains to show that our assumptions, concerning the coefficients $a_{n,i}$, $i \in \overline{n}$, are consistent, but it is easy and is left to the reader.

In order to get a better insight into the relation between $d(\wp)$ and $k(\wp)$, we shall investigate another type of hyperplanes in $\mathbf{R}^n$.

Lemma 6. There exist $(n,0,2\binom{n-2}{n/2})$-hyperplanes for every even positive integer n. The equation of such a hyperplane $\varkappa_n$ is

$$\sum_{i \in \overline{n-1}} x_i + 2x_n = 0. \tag{49}$$

Proof. Let n be even. The case n=2 is trivial, hence we can assume that $n \geq 4$. The equations

$$\sum_{i \in \overline{n-1}} x_i = -2,\ x_n = 1$$

resp.

$$\sum_{i \in \overline{n-1}} x_i = 2,\ x_n = 1$$

define two (n-2)-dimensional hyperplanes $\varkappa_{n,1}$, resp. $\varkappa_{n,2}$ which intersect C_+^{n-1} (cf. (2)). (See Fig. 8.) We can shift $\varkappa_{n,2}$ in the direction of the n-edges. In this way we obtain the (n-2)-dimensional hyperplane $\varkappa'_{n,2}$, the equations of $\varkappa'_{n,2}$ being

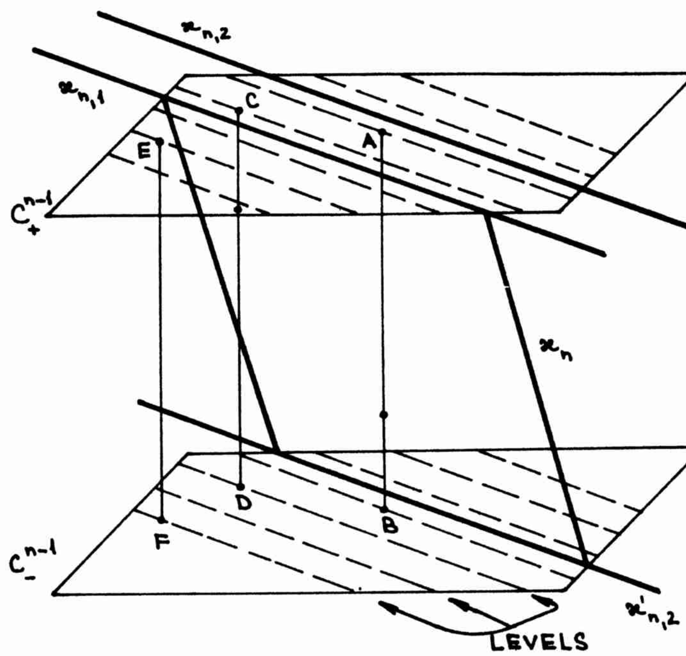

FIG.8

$$\sum_{i \le n-1} x_i = 2, \quad x_n = -1.$$

$\varkappa_{n,1}$ and $\varkappa'_{n,2}$ are contained in $\varkappa_n$, see (49). The strip between $\varkappa_{n,1}$, and $\varkappa_{n,2}$ in C_+^{n-1} contains all the vertices of the $(n/2-1)$-th and the $n/2$-th level of C_+^{n-1}.

Let us calculate $d(\varkappa_n)$. One of the half-spaces of R^n w.r.t. $\varkappa_n$ contains the whole j-th levels of C^n for $0 \le j \le n/2-1$ plus all the vertices of the $n/2$-th level which are in C_+^{n-1}, i.e., the $n/2$-th level of C_+^{n-1}. Thus

$$d(\varkappa_n) = \left| \sum_{j=0}^{n/2-1} (-1)^j \binom{n}{j} + (-1)^{n/2} \binom{n-1}{n/2} \right| =$$

$$= \left| (-1)^{n/2-1} \binom{n-1}{n/2-1} + (-1)^{n/2} \binom{n-1}{n/2} \right|.$$

But

$$\binom{n-1}{n/2-1}=\binom{n-1}{n/2},$$

hence

$$(50) \qquad d(\varkappa_n)=0.$$

Let us calculate $k(\varkappa_n)$. According to the calculations in the proof of Lemma 4, $\varkappa_{n,1}$ intersects $\binom{n-2}{n/2}$ edges of every of the first n-1 types in C_+^{n-1}, hence

$$(51) \qquad k_i(\varkappa_{n,1})=\binom{n-2}{n/2} \text{ for } i\in\overline{n-1}.$$

Similarly

$$(52) \qquad k_i(\varkappa'_{n,2})=k_i(\varkappa_{n,2})=\binom{n-2}{n/2-2}=\binom{n-2}{n/2} \text{ for } i\in\overline{n-1}.$$

But all the i-edges for $i\in\overline{n-1}$ are contained either in C_+^{n-1} or in C_-^{n-1}, $\varkappa_n\cap C_+^{n-1}=\varkappa_{n,1}$, $\varkappa_n\cap C_-^{n-1}=\varkappa'_{n,2}$. Hence,

$$k_i(\varkappa_n)=k_i(\varkappa_{n,1})+k_i(\varkappa'_{n,2}) \text{ for } i\in\overline{n-1}$$

and (51), (52) imply that

$$(53) \qquad k_i(\varkappa_n)=2\binom{n-2}{n/2} \text{ for } i\in\overline{n-1}.$$

It remains to calculate $k_n(\varkappa_n)$. An n-edge is intersected by $\varkappa_n$ if and only if one of its end-points is between $\varkappa_{n,1}$ and $\varkappa_{n,2}$ in C_+^{n-1}, i.e., if it belongs either to the n/2-th or to the (n/2-1)-th level of C_+^{n-1}. Hence

$$(54) \qquad k_n(\varkappa_n)=\binom{n-1}{n/2-1}+\binom{n-1}{n/2}=2\binom{n-1}{n/2}=\binom{n}{n/2}.$$

Because

$$\binom{n}{n/2}>2\binom{n-2}{n/2},$$

we get from (53) and (54)

$$k(\varkappa_n)=2\binom{n-2}{n/2}.$$

This equation together with (50) implies the lemma.

Theorem 4. There exist (n,0,k)-hyperplanes for every even positive integer n and for every even integer k such that

$$0\leq k\leq 2\binom{n-2}{n/2}.$$

Proof. The first n-1 coefficients in (49) can be subjected to an arbitrary sufficiently small change. The resulting hyperplane $\tilde{\varkappa}_n$ will have all the properties of $\varkappa_n$. Let

$$\sum_{i\in\overline{n-1}} b_{n,i}x_i+2x_n=0,$$

where

(55) $$|b_{n,i}-1|<\varepsilon \quad \text{for all } i\in\overline{n-1}$$

and $\varepsilon>0$ is sufficiently small, be the equation of $\tilde{\varkappa}_n$.

All pairs of vertices C_{ω_1}, $C_{\omega_2}\in C_+^{n-1}$ define finitely many directions and the pairs C_{ω_3}, $C_{\omega_4}\in C_-^{n-1}$ define just the same directions. The vectors of all these directions have the last coordinate equal to 0, hence we can choose $b_{n,i}$ satisfying (55) so that none of the hyperplanes $\varkappa_n(t)$

(56) $$\sum_{i\in\overline{n-1}} b_{n,i}x_i+tx_n=0,\ t\in[2,+\infty)$$

is parallel to any of these directions. Thus (56) cannot be satisfied for any t by the coordinates of two or more vertices in C_+^{n-1}, resp. C_-^{n-1}. On the other hand, if the coordinates of C_ω satisfy (56), then the coordinates of the opposite vertex $C_{\bar{n}-\omega}$ satisfy it, too. So for some values

(57) $$t_1<t_2<t_3<\ldots<t_\nu,$$

(ν is a suitable integer) in the interval $[2,+\infty)$, the hyperplane $\varkappa_n(t)$ contains just two opposite vertices in C^n, for all other values of $t\in[2,+\infty)$, there is no vertex of C^n in $\varkappa_n(t)$.

If t increases from 2 to $+\infty$, then in the values (57) always one of the vertices of C^n passes through $\varkappa_n(t)$ from the plus into the minus half-space w.r.t. $\varkappa_n(t)$, the opposite vertex passes simultaneously from the minus into the plus half-space, because $\varkappa_n(t)$ always contains the centre 0 of C^n and opposite vertices must be contained in opposite half-spaces. But n is even, hence the indices of C_ω and $C_{\bar{n}-\omega}$ are the same for every $\omega\subset\overline{n}$ and $d(\varkappa_n(t))$ remains unchanged, when t passes through some of the values (57). Thus

$$d(\varkappa_n(t))=d(\varkappa_n(2))=d(\tilde{\varkappa}_n)=d(\varkappa_n)=0$$

according to Lemma 6 for all $t\in[2,+\infty)$ different from the values (57).

In order to be able to control the values k_i, we will make an additional assumption concerning the coefficients $b_{n,i}$, namely

(58) $$b_{n,n-1}-1<-\sum_{i\in\overline{n-2}}|b_{n,i}-1|.$$

Let us recall that $\tilde{\varkappa}_n$ intersects just the edges of C^n which are intersected by $\varkappa_n$, $\varkappa_n$ intersects C_+^{n-1} in $\varkappa_{n,1}$ and $\varkappa_{n,1}$ intersects C_+^{n-1} between the n/2-th and the (n/2+1)-th level. C_ω is in the (n/2+1)-th level of C_+^{n-1} if and only if card $\omega = n/2+1$ and $n \notin \omega$. Such a C_ω is connected with the n/2-th level by an (n-1)-edge if and only if $n-1 \in \omega$ and it is in $\varkappa_n(t)$ for $t = -\sum_{i \in \overline{n-1}} b_{n,i} x_i$ according to (56). But for such a C_ω

$$-\sum_{i \in \overline{n-1}} b_{n,i} x_i = -\sum_{i \in \overline{n-1}} (b_{n,i}-1)x_i - \sum_{i \in \overline{n-1}} x_i =$$

$$= -\sum_{i \in \overline{n-1}} (b_{n,i}-1)x_i - (n/2-2)+(n/2+1) = -\sum_{i \in \overline{n-1}} (b_{n,i}-1)x_i + b_{n,n-1} - 1 + 3 \leq$$

$$\leq \sum |b_{n,i}-1| + b_{n,n-1} - 1 + 3 < 3$$

according to (58).

On the other hand, if C_ω is in the (n/2+1)-th level of C_+^{n-1} and it is not connected with the n/2-th level by an (n-1)-edge, then it is in $\varkappa_n(t)$ for a value $t > 3$.

As in the proof of Theorem 2 one can show that if C_ω is connected by an (n-1)-edge with the n/2-th level and it passes with the growing t through $\varkappa_n(t)$, then $k_i(\varkappa_n(t))$ for every $i \in \overline{n}$ either increases or decreases. but $k_{n-1}(\varkappa_n(t))$ always decreases. If we take into account that together with C_ω the opposite vertex $C_{\overline{n}-\omega}$ passes through $\varkappa_n(t)$ too, we see that $k_i(\varkappa_n(t))$ for every $i \in \overline{n}$ either increases or decreases by 2 and $k_{n-1}(\varkappa_n(t))$ always decreases by 2.

Let t increase from 2 to 3. We have seen that all the values of (57) which are contained in [2, 3], correspond to such pairs of vertices and vice-versa. Hence $k_{n-1}(\varkappa_n(t))$ drops by 2 in every such value t_s. For t=2

$$k_{n-1}(\varkappa_n(2)) = 2\binom{n-2}{n/2}$$

and is minimal among all the values $k_i(\varkappa_n(2))$. W.r.t. the above written facts one can easily see that it remains minimal for all the values $t \in [2,3]$ except the values (57), for which $k_i(\varkappa_n(t))$ is not defined. Hence,

$$k_{n-1}(\varkappa_n(t)) = k(\varkappa_n(t))$$

and we need to show that $k(\varkappa_n(t))$ really reaches the value 0 for t=3. It follows from the fact that there are in C^n $\binom{n-2}{n/2}$ vertices C_ω such that $n \notin \omega$ $n-1 \in \omega$ and card $\omega = n/2+1$.

An attentive reader may object that among the values (57) which correspond to the vertices of the (n/2+1)-th level, there could be mixed some values which correspond to other vertices of C^n. But an even more attentive reader may have noticed that the values t_s which correspond to the vertices in the $(n/2+1)$-th level, are all contained in a small neighbourhood of the value 3, if ε in (55) is sufficiently small. It follows from the calculations in (59) and from (55). Similarly one can show that the values t_s which correspomd to the vertices in $(n/2+2)$-th level, are close to 5, the values of the $(n/2+3)$-th level are close to 7, etc.

The proof of the consistency of our assumptions about $b_{n,i}$ is left to the reader.

Theorem 4 and Lemma 1 imply the existence of the $(n,0,k)$-hyperplanes for every odd integer $n \geq 3$ and every even integer k such that

$$0 \leq k \leq 2\binom{n-3}{(n-1)/2}.$$

Hence we have

Theorem 5. There exist $(n,0,k)$-hyperplanes for every $n \in \mathbf{N}$, $n \geq 2$ and every even integer k such that

$$0 \leq k \leq 2\binom{2[n/2-1]}{[n/2]}.$$

Remark 6. The $(4,0,2)$-hyperplane in Example 6 of Section 2 is obviously a special case of (49), hence Lemma 6 generalizes this example. The existence of $(4,3,3)$-hyperplanes, which is stated in Section 2, follows from Lemma 5 as well. The existence of all the other hyperplanes with $n \leq 4$, which is asserted in Section 2, follows from Theorem 3 and Theorem 5. But in Section 2 we also assert that no other hyperplanes for $n \leq 4$ exist. Of course, this is not true for a general n.

Remark 7. If n is even, then

$$\binom{n-1}{[(n-1)/2]} = \frac{n-1}{n-2} \cdot 2\binom{n-1}{n/2},$$

so the values of k in Lemma 5 and Lemma 6 are relatively very near for large n.

Remark 8. According to Stirling formula

$$\binom{n-1}{[(n-1)/2]} \sim 2^{n-1}\sqrt{2/n\pi} \tag{60}$$

and for n even

$$2\binom{n-2}{n/2} \sim 2^{n-1}\sqrt{2/n\pi}, \tag{61}$$

too.

Section 4. Concluding remarks. It is not hard to prove the following results:

Lemma 7. Let $S_{\lambda,\mu}: R^n \longrightarrow R^n$ be any operator with jumping nonlinearity. Then for almost every $f \in R^n$ (in the sense of the n-dimensional Lebesgue measure)

$$k(S_{\lambda,\mu}, f) \leq 2^n.$$

Proof can be found in [3].

Theorem 6. Let $S_{\lambda,\mu}: R^n \longrightarrow R^n$ be any operator with jumping nonlinearity. Then $k(S_{\lambda,\mu}) \leq 2^{n-1}$ and $d(S_{\lambda,\mu}) \leq 2^{n-1}$, if it is defined.

Proof can be done in the spirit of the proof of Lemma 7, and will be published elsewhere.

Now the main results of this article can be summarized in

Theorem 7. For any operator with jumping nonlinearity $S_{\lambda,\mu}: R^n \longrightarrow R^n$ and almost every $f \in R^n$

$$k(S_{\lambda,\mu}, f) \leq 2^n,$$
$$k(S_{\lambda,\mu}) \leq 2^{n-1}$$

and

$$d(S_{\lambda,\mu}) \leq 2^{n-1},$$

whenever $d(S_{\lambda,\mu})$ is defined. On the other hand, for every positive integer n and every positive integer d such that

$$0 \leq d \leq \binom{n-1}{[(n-1)/2]}$$

there exists $S_{\lambda,\mu}: R^n \longrightarrow R^n$ such that

$$d(S_{\lambda,\mu}) = k(S_{\lambda,\mu}) = d.$$

Also, for every positive integer $n \geq 2$ and every even integer k such that

$$0 \leq k \leq 2\binom{2[n/2-1]}{[n/2]},$$

there exists an operator with jumping nonlinearity such that

$$d(S_{\lambda,\mu}) = 0,$$

nevertheless

$$k(S_{\lambda,\mu}) = k.$$

The asymptotics in (60) and (61) implies that the last theorem cannot be substantially improved. One can also prove that for any existing (n,d,k)-

hyperplane the inequality

$$k \leq \binom{n-1}{[(n-1)/2]}$$

holds. This result was published in [4].

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