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MINIMAL MONADS

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Abstract: In the paper we present an analogue of some properties of classes of indiscernibles for minimal monads. Further we prove the existence of a minimal monad of a special character and using it we show six equivalents of the property of natural numbers "to be very far one from the other".

Key words: Alternative set theory, minimal monad, infinite natural number, Rudin-Keisler's ordering, definability with a parameter.

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In this paper we are interested in a relation between relative definability and distance of natural numbers. This relationship is important especially when one supposes the existence of infinitely large definable natural numbers. For our investigations there are very convenient minimal monads which have some properties analogous to the properties of classes of indiscernibles.

We introduce and examine the notion $\alpha <<_{FN} \beta$ of great distance between α, β which expresses the fact that β cannot be reached from α by any definable function transforming FN into FN. Six equivalents of this feature are given. The most interesting are the following ones:

There are an endomorphic universe A with standard extension and $\gamma \not\leq \beta$ such that $\alpha, \gamma \in Ex_A(FN)$ and $\alpha \in Ex_{A[\gamma]}(FN)$.

There is an endomorphic universe A such that for each function $f \in A$ we have

$$f''FN \subsetneq FN \Rightarrow f(\alpha) < \beta .$$

Let us remind now several facts from [Č-K].

Let μ_1, μ_2 be monads in $\doteq_{\{c\}}$ (i.e. classes of decomposition of V according to $\doteq_{\{c\}}$). We say that $\mu_1 \mathrel{\dot{\leq}}_{\{c\}} \mu_2$ iff there is a function $F \in Sd_{\{c\}}$

such that $F''\mu_2 = \mu_1$.

The ordering $\underset{\{c\}}{\preceq}$ on monads is similar to Rudin-Keisler´s ordering on ultrafilters (monads correspond with ultrafilters on the ring of $Sd_{\{c\}}$ classes by a one-one correspondence).

A monad μ (in $\underset{\{c\}}{\doteq}$) is minimal in $\underset{\{c\}}{\preceq}$ iff each function $F \in Sd_{\{c\}}$ is either constant or one-one mapping on μ.

In further considerations we shall limit ourselves to infinite monads. Moreover, we shall assume that they are subclasses of N, where we have natural ordering " $\leq$ ". This restriction is not substantial since there exists $F \in Sd_0$ such that F is a one-one mapping of N onto V (see [V]).

For an easier typing we shall, through the whole paper, use for elements of N also small Latin letters, e.g. $x \in Def$ is an abbreviation for $x \in Def \cap N$. This convention does not refer to the notation of set-definable functions and subsets of N.

When writing $F: A \rightarrow B$ we bear in mind that F is a function such that $F''A \subseteq B$ (i.e. we do not ask for $dom(F) = A$).

§ 1. At first we prove that monads cannot be divided into two parts by any definable element, i.e. that all elements of a monad have "the same position" with respect to the elements of Def.

Definition. Let X be a class. We shall call the class $\{\alpha; (\exists \beta, \gamma \in X)\ \beta \leq \alpha \leq \gamma\}$ a <u>convex hull</u> of X and denote $Cnh(X)$.

Lemma 1. Let $\mu (\subseteq N)$ be a monad in $\underset{\{c\}}{\doteq}$. Then $Cnh(\mu) \cap Def_{\{c\}} = \emptyset$.

Proof. Let $x \in Cnh(\mu) \cap Def_{\{c\}}$. Then there are $t, u \in \mu$ such that $t < x < u$ - a contradiction with the fact that the elements of a monad fulfil the same formulas.

Theorem 1. Let μ be a monad in $\underset{\{c\}}{\doteq}$. Then

(a) $(\forall x)[x < Cnh(\mu) \Rightarrow (\exists a \in Def_{\{c\}})\, x < a < Cnh(\mu)]$,

(b) $(\forall x)[x > Cnh(\mu) \Rightarrow (\exists a \in Def_{\{c\}})\, Cnh(\mu) < a < x]$.

Proof: We shall prove (a), the assertion (b) can be proved analogously. Take $x < Cnh(\mu)$.

Let $\mu = \bigcap\{X_n; n \in FN\}$, where $\{X_n; n \in FN\}$ is a descending sequence (in $\subseteq$) of classes from Sd_0 (see [V]). For X_n we shall ask moreover: $min(X_{n+1}) > min(X_n)$. (If $min(X_{n+1}) = min(X_n)$, we put $\overline{X}_{n+1} = X_{n+1} - \{min(X_{n+1})\}$ and

$\mu=\bigcap\{\overline{X}_n;n\in FN\}$.) We would like to prove:

(1) $$(\exists n\in FN)\min(X_n)>x;$$

i.e. we show that $\{X_n;n\in FN\}$ is cofinal with μ. Denote $a_n=\min(X_n)$ for every $n\in FN$. Let us prolong the sequence $\{a_n;n\in FN\}$. Then we can easily prove that

(2) $$(\exists\beta)(\forall\gamma)[FN<\gamma<\beta\Rightarrow a_\gamma\notin\mu].$$

To this end we shall construct a descending sequence $\{\beta_n;n\in FN\}$ such that $\beta_i>FN$ for every $i\in FN$ and

$$(\forall\delta)(i<\delta<\beta_i\Rightarrow a_\delta\in X_i).$$

Obviously, it is enough to choose β_i like this: Let β_i be the maximal element such that for each γ

$$i<\gamma<\beta_i\Rightarrow a_\gamma\in X_i.$$

But then there is such β^* that $FN<\beta^*<\beta_i$ for all $i\in FN$. Hence (2) is valid.

Thus, for each γ, it is true that $a_\gamma\notin\mu$. Let a_ξ be the smallest element of the prolonged sequence for which $a_\xi>x$ holds. For proving (1) it is sufficient to show now that there exists $k\in FN$ such that $\xi=k$ (i.e. that the index of a_ξ is finite). From the construction of a_ξ we know, however, that $a_{\xi-1}<x$ which implies $\xi-1\in Fin$ (in the opposite case we obtain $a_{\xi-1}\in\mu$ which contradicts (2)); therefore $\xi\in Fin$. Thus a_k is the required element from $Def_{\{c\}}$ fulfilling (a).

For further considerations it is useful to take the following notation:

$$Def\ N=Def\cap N,\quad {}^{\infty}Def\ N=Def\ N-FN$$

and analogously for $Def_{\{c\}}$.

If we suppose that infinitely large definable numbers exist, then they can separate monads. It is therefore useful to examine elements which are very far (in natural ordering) one from the other. A characterization of "great distance" gives the following definition.

Definition. Let $a,b<{}^{\infty}Def\ N$. We say that b *is much greater than* a *with respect to* FN (notation $a<<_{FN}b$) iff

(3) $$(\forall F\in Sd_0)(F:FN\to FN\Rightarrow F(a)<b).$$

Notice that, for ${}^{\infty}Def\ N=\emptyset$, $a<<_{FN}b$

describes the distance determined by all definable functions.

Very often, when working with functions from FN into FN, it is convenient to suppose that they are non-descending. Such a "trick" justifies the next lemma.

Lemma 2. There is a sequence of non-descending Sd_0 functions $\{F_i; i \in FN\}$ such that

1) $(\forall i \in FN)(dom(F_i)=N \,\&\, F_i : FN \to FN)$.

2) For each Sd_0 function $G:FN \to FN$ there is $F_k \in \{F_i; i \in FN\}$ such that $G(\alpha) \leq F_k(\alpha)$ for every $\alpha \in dom(G)$.

Proof. Let us enumerate all Sd_0 functions from FN into FN by $\{G_i; i \in FN\}$. Put, for every $\alpha \in N$,

(4) $F_i(\alpha) = \max(\bigcup_{j \leq i} G_j''(\alpha+1))$.

Then $\{F_i; i \in FN\}$ is the required sequence.

The next theorem asserts that each monad which comes into consideration intersects segments determined by the points which are very far one from the other.

Theorem 2. Let $a,b < {}^{\infty}Def\ N$, $a <<_{FN} b$ and let μ be such a monad that $\mu \cap (\bigcap {}^{\infty}Def\ N) \neq \emptyset$. Then $\mu \cap [a,b] \neq \emptyset$ (where $[a,b] = \{x; a \leq x \leq b\}$).

Proof. Let A be a set-definable class, $\mu \subseteq A \subseteq N$. We shall define a function G on N as follows:

(5) $(\forall t \in N) G(t) = \min\{A - [[\min(A-(t+1))] + 1]\}$.

Realize that for setting G(t) we find, roughly speaking, the "first" element of A which is over t and then G(t) is the "second one with the same property".

Obviously $G:FN \to FN$, $G \in Sd_0$. From Lemma 2 it follows that there is $i \in FN$ such that $F_i(t) > G(t)$ for every $t \in FN$. Furthermore, for all $n \in FN$, we have

(6) $[n, F_i(n)] \cap A \neq \emptyset$.

Let γ_0 be the largest element such that every less element fulfils (6); i.e.

$$(\forall \gamma \leq \gamma_0)\ [\gamma, F_i(\gamma)] \cap A \neq \emptyset.$$

Then $\gamma_0 \in Def$ and it is infinite. Therefore $a < \gamma_0$ (since $a < {}^{\infty}Def\ N$) and (6) for a holds. Thus $[a, F_i(a)] \cap A \neq \emptyset$. Suppose $\mu = \bigcap \{A_n; n \in FN\}$, where $\{A_n; n \in FN\}$ is a descending sequence of Sd_0 classes. Then (note that $\mu \subseteq A$)

for each $i \in FN$ we have $[a,b] \cap A_i \neq \emptyset$, which - due to the prolongation (see [V]) - completes the proof.

Remarks. Definition of $<<_{FN}$, Lemma 2 and the previous theorem can be reformulated into "parametric version" and proved analogously.

It follows from compactness of equivalences of indiscernibility that there are monads whose intersection with segments determined by points which are not very distant, is non-empty; hence the large distance is not a necessary condition.

In the next paragraph we show that each two different points of minimal monads are very far from each other.

Theorem 2 can be generalized also in another direction, as will be shown later. Before this we introduce a new notion and prove several assertions.

Definition. Let $x \notin Def_{\{c\}}$. Then we shall denote the class

$N-[Cnh(Def_{\{c\}} \cap x) \cup Cnh(Def_{\{c\}} \cap (N-x)]$

$Int_{\{c\}}(x)$ and call it an <u>interval of</u> x (<u>with respect to the parameter</u> c).

Notice that $Int_{\{c\}}(x)$ is the class of all natural numbers which have "the same position" w.r.t. $Def_{\{c\}}$ as x has.

From the definition it follows immediately:

Lemma 3. Let $x,y \notin Def_{\{c\}}$, then the following holds:

a) $x \in Int_{\{c\}}(x)$,

b) $Int_{\{c\}}(x) \neq Int_{\{c\}}(y) \Rightarrow x \neq y$,

c) $x \doteq_{\{c\}} y \Rightarrow Int_{\{c\}}(x) = Int_{\{c\}}(y)$,

d) $Int_{\{c\}}(x) = Cnh(Int_{\{c\}}(x)) = Cnh(\mu_{\{c\}}(x))$.

Further we shall deal with a generalization of the minimal cut (i.e. FN).

Definition. Let $X \subseteq Def_{\{c\}}$ such that $(\forall t \in Def_{\{c\}})$ $t \in X \equiv (\exists u \in X)\ u \geq t$.

Then X is called a <u>cut</u>.

We shall work only with cuts which have not the largest element - we denote them $\wp$.

Lemma 4. For each cut $\wp$ there is $\alpha \in N$ such that $\wp = Def_{\{c\}} \cap \alpha$; such a cut we denote $\wp(\alpha)$.

Proof. Since $\wp \subseteq Def_{\{c\}}$, $\wp$ is a countable class and therefore there is an increasing sequence $\{x_i \in Def_{\{c\}}; i \in FN\}$ which is cofinal with $\wp$.

We prolong this sequence in such a way that the monotony will be preserved. Let $\{y_i; i \in FN\}$ be an enumeration of all elements from $(Def_{\{c\}} - \wp)$. Let β_i be the largest index of the prolonged sequence such that $x_{\beta_i} < y_i$ (for all $i \in FN$). Then $\{\beta_i; i \in FN\}$ is a non-increasing sequence. Because $\{\beta_i; i \in FN\}$ is countable, there exists $\beta \notin FN$ such that $\beta < \beta_i$ for every $i \in FN$. Put $\alpha = x_\beta$, obviously $\wp(\alpha) = Def_{\{c\}} \cap \alpha$.

Further we shall take the following notation: Let $x \notin Def_{\{c\}}$, then we put

$$\wp_{\{c\}}(x) = Def_{\{c\}} \cap x,$$

$$\bar{\wp}_{\{c\}}(x) = Def_{\{c\}} \cap (N-x).$$

Note that now

$$Int_{\{c\}}(x) = N - [Cnh(\wp_{\{c\}}(x)) \cup Cnh(\bar{\wp}_{\{c\}}(x))].$$

For an easier typing we shall use - when there is no danger of misunderstanding - only $\wp$ instead of $\wp_{\{c\}}(x)$ and analogously $\bar{\wp}$ instead of $\bar{\wp}_{\{c\}}(x)$.

Lemma 5. Let $x \notin Def_{\{c\}}$, $X \in Sd_{\{c\}}$. Then the following properties are equivalent:

1) X is cofinal with $\wp_{\{c\}}(x)$,

2) X is coincial with $\bar{\wp}_{\{c\}}(x)$,

3) $X \cap Int_{\{c\}}(x) \neq \emptyset$.

Proof. We prove only 1) $\equiv$ 3); for 2) $\equiv$ 3) it suffices to modify this proof for the inverse ordering.

Suppose 1) holds. Since $\wp, \bar{\wp}$ are countable, we can enumerate them. Let $\wp = \{a_i; i \in FN\}$, $\bar{\wp} = \{b_i; i \in FN\}$. From 1) it follows that

$$(\forall j \in FN)(\bigcap_{i<j} [a_i, b_i]) \cap X \neq \emptyset.$$

But $\bigcap_{i \in FN} [a_i, b_i] = Int_{\{c\}}(x)$; hence 3) is valid.

For proving 3) $\Rightarrow$ 1) assume that $a \in X \cap Int_{\{c\}}(x)$ and X is not cofinal with $\wp$. Then there is $\bar{a} \in \wp$ such that there is no $b \in (X \cap \wp) - \bar{a}$. Put $d = \min(X - (\bar{a}+1))$. Then $d \in X$ and $d \in \wp$ (since $d \in Def_{\{c\}}$). Moreover, $\bar{a} < d \leq a (< \bar{\wp})$ - which contradicts $\neg$ 1).

The notion of the property of points "to be very distant" introduced above will be established now for more general cuts, namely $\wp$, and an analogue to Theorem 2 will be proved.

Definition. Let $x \notin \mathrm{Def}_{\{c\}}$, $a,b \in \mathrm{Int}_{\{c\}}(x)$. We say that b <u>is much greater than</u> a <u>with respect to</u> $\wp_{\{c\}}(x)$ iff

(7) $(\forall F \in \mathrm{Sd}_{\{c\}})(F: \wp_{\{c\}}(x) \rightarrow \wp_{\{c\}}(x) \Rightarrow F(a) < b)$;

notation $a \ll \wp_{\{c\}}(x) b$ or only $a \ll_{\wp} b$, when there is no danger of confusion.

For proving the next theorem, we shall modify firstly Lemma 2.

Lemma 6. Let $\wp$ be a non-trivial cut on $\mathrm{Def}_{\{c\}}$. Then there is a sequence of non-descending $\mathrm{Sd}_{\{c\}}$ functions $\{F_i; i \in FN\}$ such that

1) $(\forall i \in FN)\ (\mathrm{dom}(F_i)=N \ \&\ F_i: \wp \rightarrow \wp)$.

2) For each $\mathrm{Sd}_{\{c\}}$ function $G: \wp \rightarrow \wp$ there is $F_k \in \{F_i; i \in FN\}$ such that $G(\alpha) \leq F_k(\alpha)$ for every $\alpha \in \mathrm{dom}(G)$.

Proof. Let $\{G_i; i \in FN\}$ be an enumeration of all $\mathrm{Sd}_{\{c\}}$ functions which transform $\wp$ into $\wp$. Put

$$F_i(t)=\max\left(\bigcup_{j \leq i} G_j''(t)\right).$$

When we prove that $F_i: \wp \rightarrow \wp$ for each $i \in FN$, then $\{F_i; i \in FN\}$ will be the required sequence. Suppose, by contradiction, that there exists $F \in \{F_i; i \in FN\}$ such that $F''\wp \nsubseteq \wp$. Let $a \in \wp$ and $F(a) \notin \wp$. Then (note that $F(a) \in \mathrm{Def}_{\{c\}}$) there is $b \in \overline{\wp}$ such that $F(a) > b$. From our definition of $\{F_i; i \in FN\}$ it follows that there are $j \in FN$ and $t \in a$ such that $G_j(t) > b$. Denote $\bar{t}$ the smallest of elements for which $G_j(t) > b$. Then $\bar{t} \in \wp$ (since $\bar{t} \in \mathrm{Def}_{\{c\}}$), but $G_j(\bar{t}) \notin \wp$ (because $G_j(\bar{t}) > b$ and $b \in \overline{\wp}$) - a contradiction with $G_j: \wp \rightarrow \wp$.

Theorem 3. Let $a,b \in \mathrm{Int}(x)$, $a \ll_{\wp(x)} b$ and let μ be such a monad that $\mu \cap \mathrm{Int}(x) \neq \emptyset$. Then $\mu \cap [a,b] \neq \emptyset$.

Proof. For proving this assertion it suffices to use Lemma 6 and modify the proof of Theorem 2. Analogously to (5) we choose the class A and define the function G.

We have, however, to show that $G: \wp \rightarrow \wp$ (an analogous fact for G in the proof of Theorem 2 was obvious). Let $u \in \wp$; then there exists $v > u$ such that $v \in A$. Take the smallest of such elements - denote it v_1. Then $v_1 \in \wp$ ($A \cap \mathrm{Int}(x) \neq \emptyset$ since $\mu \subseteq A$; but this implies that A is cofinal with $\wp$ - see Lemma 5). Repeat now this consideration for "starting point" v_1; it brings us to $w_1 \in \wp$. But $G(u)=w_1$ and consequently $G: \wp \rightarrow \wp$.

When "checking" furthermore the proof of Theorem 2, we obtain here instead of (6)

(8) $(\forall p \in \wp)(\forall t \in p)\ [t;F_i(t)] \cap A \neq \emptyset$.

If γ_0 is the greatest element for which (8) holds, we have that $\gamma_0 \in$ Def-$\wp$. Hence $a < \gamma_0$ (since $a < \tilde{\wp}$) and thus $[a,F_i(a)] \cap A \neq \emptyset$. For completing this proof it is enough to use the same arguments as in the proof of Theorem 2.

Remark. We know that if ${}^{\infty}$Def $N=\emptyset$, then FN=Def$\cap$N. When, however, ${}^{\infty}$Def $N \neq \emptyset$, a natural question arises: How do all elements under ${}^{\infty}$Def N and reasonably definable look like? Such sort of considerations has brought us to the following definition.

Definition. Let $a \notin \mathrm{Def}_{\{c\}}$. We put

$\mathrm{Def}_{\{c\}}^{\wp \to \wp}(\{a\}) = \{x;(\exists F)(F \in \mathrm{Sd}_{\{c\}}\ \&\ F:\wp \longrightarrow \wp\ \&\ x=F(a))\}$;

we write here briefly only $\wp$ instead of $\wp_{\{c\}}(a)$.

When we compare this definition with the one of $a <<_{\wp} b$, we obtain immediately that

$$a <<_{\wp} b \equiv \mathrm{Def}_{\{c\}}^{\wp \to \wp}(\{a\}) \subseteq b.$$

The next theorem asserts that under the same assumptions as in Theorem 3 there exists even an infinite set inside $\mu \cap [a,b]$. At first we prove, however, an auxiliary result.

Lemma 7. Let $a <<_{\wp} b$, $d \in [a,b]$. Then either $a <<_{\wp} d$ or $d <<_{\wp} b$.

Proof. Evidently it suffices to verify that

$d < \mathrm{Def}_{\{c\}}^{\wp \to \wp}(\{a\}) \Rightarrow \mathrm{Def}_{\{c\}}^{\wp \to \wp}(\{d\}) \leq \mathrm{Def}_{\{c\}}^{\wp \to \wp}(\{a\})$.

Let $F \in \mathrm{Sd}_{\{c\}}$, $F:\wp \longrightarrow \wp$. We have to show that there exists $G \in \mathrm{Sd}_{\{c\}}$, $G:$ $:\wp \longrightarrow \wp$ such that $G(a) \geq F(d)$. It follows from Lemma 6 that there are non-descending $\mathrm{Sd}_{\{c\}}$ functions G_1, G_2 such that $d \leq G_1(a)$ and $F(d) \leq G_2(d)$. Then, however, $G_2(d) \leq G_2(G_1(a))$. Put now $G=G_2 \circ G_1$.

Theorem 4. Let $a,b \in \mathrm{Int}(x)$, $a <<_{\wp(x)} b$ and μ be such a monad that $\mu \cap \mathrm{Int}(x) \neq \emptyset$ (hence $\mu \subseteq \mathrm{Int}(x)$). Then there is $m \notin \mathrm{Fin}$ such that $m \subseteq \mu \cap [a,b]$.

Proof. The monad μ is a π-class and therefore it is revealed. Since $[a,b]$ is a set, $\mu \cap [a,b]$ is revealed, too. Hence it is sufficient to prove that there is an infinite countable class C such that $C \subseteq \mu \cap [a,b]$. For making the proof shorter we show only that $C \subseteq \mu \cap \{x;a<x<b\}$. We shall construct the class C by means of Lemma 7 and Theorem 3.

Let $a_1 \in \mu \cap \{x; a<x<b\}$; we have that $a <<_{\wp(x)} a_1$ or $a_1 <<_{\wp(x)} b$. Suppose e.g. that $a <<_{\wp(x)} a_1$ and apply again the above mentioned assertion, etc. Thus we obtain a countable sequence $\{a_i; i \in FN\}$ of elements from $\mu \cap \{x; a< x<b\}$. Put $C = \{a_i; i \in FN\}$.

Remarks. When substituting $\wp(x)$ by FN in Theorem 4, we have an analogous generalization of Theorem 2.

Theorems 3 and 4 can be similarly proved also for their parametrical version.

The following theorem speaks about monads generally, but its main significance will not become evident before studying monads of indiscernibles. In the forthcoming paper we show that if Ind is a monad of indiscernibles for the language L and $t \in$ Ind then $t \cap$ Ind is a monad of indiscernibles for $L_{\{t\}}$. Theorem 5 implies that the assumption $t \in$ Ind is a substantial one.

Theorem 5. Let μ be a monad. Then there are c, x such that $\mu \cap Int_{\{c\}}(x)$ is not a monad in $\overset{\circ}{=}_{\{c\}}$.

Proof. Since $\mu \notin$ Fin, there is $c \subseteq \mu$ which is infinite, too. Then ($\overset{\circ}{=}_{\{c\}}$ is compact) there are $x, y \in c$, $x \neq y$ such that $x \overset{\circ}{=}_{\{c\}} y$. Suppose $x<y$. For proving Theorem 5 it suffices to find $t \in \mu \cap Int_{\{c\}}(x)$ for which $\neg(t \overset{\circ}{=}_{\{c\}} x)$ holds. Denote the smallest element of c which is greater (in $<$) than x by z. Obviously $z \in \mu$ (since $z \in c$). Moreover, $z \in Int_{\{c\}}(x)$; for this realize that $x, y \in Int_{\{c\}}(x) = Cnh(Int_{\{c\}}(x))$ - see Lemma 3. Hence $z \in \mu \cap Int_{\{c\}}(x)$.

We show that $\neg(z \overset{\circ}{=}_{\{c\}} (x)$. Suppose that $z \overset{\circ}{=}_{\{c\}} x$. From the construction of z it follows that z is definable from x and c. This fact implies the existence of $Sd_{\{c\}}$ function F such that $F(x)=z$. Then, however, $F(x) \overset{\circ}{=}_{\{c\}} x$ which resulted in $F(x)=x$ (see Theorem 1 from [Č-K]) and hence $z=x$ - a contradiction with the definition of z. For completing the proof it is enough to put $z=t$.

§ 2. In the first theorem of this paragraph we prove that there exists a minimal monad close over FN which, moreover, possesses the following property: each set-definable function (with parameter c) either maps FN into FN or transfers this monad over an infinitely large definable element.

At first we shall formulate an assertion which is an immediate consequence of Lemma 5.

Lemma 8. Let $x \in Def_{\{c\}}$ be not cofinal with FN. Then there is $k \in$ FN such that $(x-k)\cap FN=\emptyset$.

Theorem 6. There exists a minimal monad μ in the class $X=\{\alpha; FN<\alpha<{}^{\infty}Def_{\{c\}}N\}$ such that for every $F\in Sd_{\{c\}}$, $F:N\to N$, there is $\mathfrak{X}\in Sd_{\{c\}}$, $\mathfrak{X}\supseteq\mu$, such that either (i) or (ii) takes place, where

(i) $F\upharpoonright\mathfrak{X}:FN\to FN$,

(ii) $F''\mathfrak{X}\cap FN=\emptyset$.

Proof. For ${}^{\infty}Def_{\{c\}}N=\emptyset$ we have $X=N-FN$ and $c\in Def$. Then the assertion is valid - see [Č-K], Theorem 11, and realize that (i) is true for each $F\in Sd_0$.

Assume further ${}^{\infty}Def_{\{c\}}N\neq\emptyset$. We shall construct a descending sequence $\{x_n; n\in FN\}$ with the following properties:

1) $(\forall n\in FN)x_n\in Def_{\{c\}}$;

2) $(\forall n\in FN)x_n$ is cofinal with FN;

3) $(\forall Y\in Sd_{\{c\}})[Y\subseteq N\Rightarrow(\exists n\in FN)(x_n\subseteq Y\vee x_n\subseteq N-Y)]$;

4) Let $\{F_n; n\in FN\}$ be an enumeration of all $Sd_{\{c\}}$ functions. Then F_n is either constant or a one-one mapping on x_{n+1} (for all $n\in FN$);

5) If $\{y_n; n\in FN\}$ is an enumeration of ${}^{\infty}Def_{\{c\}}N$, then $x_{n+1}\subseteq y_n$ (for all $n\in FN$);

6) For each $F_k\in\{F_n; n\in FN\}$ either (i) or (ii) is true when we put $\mathfrak{X}=x_{k+1}$.

The sets x_n will be constructed by induction based on n.

At first take, arbitrarily, $x_1\in{}^{\infty}Def_{\{c\}}N$; denote $\bar{x}_1=x_1\cap y_1$, where y_1 is the first element of our enumeration of ${}^{\infty}Def_{\{c\}}N$. We shall investigate $\hat{x}_1=\bar{x}_1\cap dom(F_1)$, where F_1 is the first element of $\{F_n; n\in FN\}$.

If $\hat{x}_1$ is not cofinal with FN, put $x_2=\bar{x}_1-dom(F_1)$. Obviously x_2 possesses each required property except 3), which will be examined later for all possible cases all at once.

For $\hat{x}_1$ being cofinal with FN we shall examine the system of classes of decomposition of $\hat{x}_1$ according to the equivalence $a\approx b\equiv F_1(a)=F_1(b)$. There are two possibilities:

a) $(\exists t\in rng(F_1))F_1^{-1}''\{t\}\cap\hat{x}_1$ is cofinal with FN,

b) the negation of a) holds.

In the case a) take $t\in rng(F_1)$ such that $F_1^{-1}''\{t\}\cap\hat{x}_1$ is cofinal with FN. Then evidently $t\in Def_{\{c\}}$ (realize for it that $F_1\in Sd_{\{c\}}$ and $F_1''FN\subseteq Def_{\{c\}}\neq\emptyset$). Put $\tilde{x}_1=F_1^{-1}''\{t\}\cap\hat{x}_1$; except 3) and 6) all conditions are valid.

Assume further b). Then we can take $\tilde{x}_1$ as the set of all smallest elements from classes of decomposition of $\hat{x}_1$ (according to F_1). Obviously 1), 4) and 5) hold. We prove now 2) for $\tilde{x}_1$. Suppose the contrary. Then there is $k\in FN$

such that $(\tilde{x}_1-k)\cap FN=\emptyset$ - see Lemma 8. Put $\alpha=\min(\tilde{x}_1-k)$; then $\alpha\notin FN$. Let further w contain all classes of decomposition of $\hat{x}_1$ (over F_1) such that their smallest elements belong to $\tilde{x}_1-k$. Then for each $z\in w$ we have $z\geq\alpha$ and hence w is not cofinal with FN. The class $\hat{x}_1-w$ has, however, only a finite number of elements from the decomposition. Thus at least one of them must be cofinal with FN - a contradiction (we suppose that none is).

Now we shall construct x_2. If F is constant on $\tilde{x}_1$, it suffices to put $x_2=\tilde{x}_1$ and all conditions except 3) are valid for x_2. Assume further that F_1 is one-one on $\tilde{x}_1$. If $F_1''\tilde{x}_1$ is not cofinal with FN, there is $k\in FN$ such that $(F_1''\tilde{x}_1-k)\cap FN=\emptyset$ - see Lemma 8. Put $x_2=\tilde{x}_1-F_1^{-1}{}''k$; it is evident that 1), 2), 4), 5) are true. For 6) realize that now (ii) takes place.

Suppose $F_1''\tilde{x}_1$ is cofinal with FN. Without loss of generality we may think that $F_1(t)>t$ (otherwise we shall take F_1+1). The set x_2 will be constructed by induction; we demand, at the same time, $x_2\subseteq\tilde{x}_1$. Let t_1 be such an element that $F_1(t_1)$ is minimal in $F_1''\tilde{x}_1$. Obviously $t_1\in FN$ (cofinality condition) and $F_1''(\tilde{x}_1-(t_1+1))$ is cofinal with FN. Let t_2 be such an element that $F_1(t_2)$ is minimal in $F_1''(\tilde{x}_1-(t_1+1))$, etc. This set-definable construction will go until a certain number $\eta\in N$. When we put now $x_2=\{t_\gamma;\gamma\in\eta\}$, we have that 1), 2), 4) and 5) are fulfilled. Moreover from our construction of x_2, it follows that (i) comes (if $t_i\in x_2\cap FN$, then $i\in FN$ and therefore $F_1(t_i)\in FN$). Hence 6) is also true for x_2.

Analogously we can construct x_3 from x_2, x_4 from x_3, etc. The obtained sequence $\{x_n;\ n\in FN\}$ possesses all required properties except 3), which we prove now. Let $Y\in Sd_{\{c\}}$, $Y\subseteq N$ and let χ_Y be its characteristic function. Then $\chi_Y=F_k$, where F_k is the function from our enumeration which is on x_{k+1} either one-one or constant. Since $dom(\chi_Y)\notin Fin$ and $rng(\chi_Y)=\{0,1\}$, χ_Y has to be constant on x_{k+1}. If for each $t\in x_{k+1}$ we have $\chi_Y(t)=0$, then $x_{k+1}\subseteq N-Y$; in the case $\chi_Y(t)=1$ we obtain $x_{k+1}\subseteq Y$. The sequence $\{x_n;n\in FN\}$ fulfils therefore 3) and hence $\bigcap\{x_n;n\in FN\}$ is a monad - let us denote it μ. Owing to 4) μ is a minimal monad.

For proving $\mu\subseteq X$ it suffices to show that $X\cap\mu\neq\emptyset$. Assume the contrary; let $u\in X$, $t\in\mu$. Then $Int_{\{c\}}(u)\cap Int_{\{c\}}(t)=\emptyset$. Construct $\wp_{\{c\}}(u)$ and $\wp_{\{c\}}(t)$. Evidently $\wp_{\{c\}}(u)=FN$ and $\wp_{\{c\}}(t)\neq FN$, which implies the existence of $\alpha\in{}^{\infty}Def_{\{c\}}N$ such that $\alpha\in\wp_{\{c\}}(t)$. Then $\alpha<t$, but $\alpha<\mu$, therefore there is $k\in FN$ for which $\alpha<x_k$ - a contradiction with 2).

For completing the proof it remains to show that for each $F\in Sd_{\{c\}}$, $F:N\rightarrow N$, there is $\mathfrak{X}\in Sd_{\{c\}}$, $\mathfrak{X}\supseteq\mu$, such that either (i) or (ii) takes place. Obviously it is enough to find $F_k \in \{F_n; n \in FN\}$ for which $F=F_k$ and put $\mathfrak{X} = x_{k+1}$.

Remark. In a forthcoming paper we prove the consistency of the existence of a minimal monad μ in X such that for a certain Sd_0 function there is no $\mathfrak{X} \in Sd_0$, $\mathfrak{X} \supseteq \mu$, with properties (i) or (ii).

There is a question: For what t there exists a minimal monad inside $Int_{\{c\}}(t)$. A partial solution gives the following theorem, which is a simple consequence of Theorem 6.

Theorem 7. Let $t \notin Def_{\{c\}}$. Let there exist $\alpha \in X = \{\gamma; FN < \gamma < {}^{\infty}Def_{\{c\}}N\}$ and a non-descending $Sd_{\{c\}}$ function $F: N \rightarrow N$ such that $F(\alpha) \in Int_{\{c\}}(t)$. Then there is a minimal monad $\nu \subseteq Int_{\{c\}}(t)$.

Proof. Firstly we show that we can assume dom(F)=N. If dom(F)=m, denote $\sigma = \max(m)$ and for all $t > \sigma$ put $F(t)=F(\sigma)$. Let further dom(F) be a proper class. If F is not defined in some σ, put $F(\sigma)=F(\beta)$, where β is the smallest of all elements from dom(F) which are larger than σ.

Suppose now dom(F)=N. Evidently, F remains to be non-descending. This, however, implies that $F'' Int_{\{c\}}(\alpha) = Int_{\{c\}}(F(d))$. Let μ be a minimal monad in X - it exists due to Theorem 6. Denote $\nu = F''\mu$. Then $\nu \subseteq Int_{\{c\}}(t)$ (if $\beta \in \mu$ then $Int_{\{c\}}(\beta) = Int_{\{c\}}(\alpha) = X$ and $F(\beta) \in Int_{\{c\}}(t)$). The monad ν cannot be a trivial one, since $t \notin Def_{\{c\}}$ and hence $Int_{\{c\}}(t)$ cannot be a singleton. As μ is minimal, F has to be one-one on μ and hence ν is minimal, too.

Next four theorems show an analogue of one property of classes of indiscernibles for minimal monads.

Theorem 8. Let μ be a minimal monad in $\overset{\circ}{=}_{\{c\}}$ which is a semiset. Let $F \in Sd_{\{c\}}$ be such a function that $F(t) < t$ for some (and hence every) $t \in \mu$. Then either 1) or 2) is valid, where

1) $(\forall u \in \mu)(u < t \Rightarrow u < F(t))$;

2) $(\exists d \in Def_{\{c\}})(\forall t \in \mu)\ F(d) < t < d$.

Proof. Since F is defined on the minimal monad μ, F is either constant or one-one on μ.

Suppose, at first, that F is constant on μ. Denote $F(t)=e$ for $t \in \mu$; evidently $e \in Def_{\{c\}}$. Construct $F^{-1}"\{e\}$. Obviously $F^{-1}"\{e\} \in Sd_{\{c\}}$ and $\mu \subseteq F^{-1}"\{e\}$. Because μ is a semiset, there exists $\bar{e} \in Def_{\{c\}}$ such that $\bar{e} > \mu$. Put $d = \max((F^{-1}"\{e\}) \cap \bar{e})$. Since $e, \bar{e} \in Def_{\{c\}}$, $F \in Sd_{\{c\}}$ and $F(d)=e$, we have $d \in Def_{\{c\}}$. Moreover, $\mu \subseteq \bar{e}$ (since $\bar{e} > \mu$), $\mu \subseteq F^{-1}"\{e\}$, therefore $d > \mu$. But $F(d)=e=F(t)<t$ for each $t \in \mu$. Hence $F(d)<t<d$ and 2) is fulfilled.

Let further F be one-one on μ . We shall assume $\neg 1) \& \neg 2)$. It follows from $\neg 2)$ that there exists $t \in \mu$ such that $F^{-1}" \wp_{\{c\}}(t) \subseteq \wp_{\{c\}}(t)$. Let us fix such t. Put $k=\min(dom(F^{-1}))$ and define for each $u \geq k$ a function H as follows:

$$H(u)=\max(F^{-1}"(u+1) \cup \{u+1\}).$$

Obviously $H \in Sd_{\{c\}}$ is a non-descending function and $H(u) > u$ for each $u \geq k$. Define further a function G as follows:

$$G(0)=k \,\&\, G(\gamma+1)=H(G(\gamma)),$$

i.e. $G(\gamma)$ is the γ-th iteration of H when starting from k. Finally, put for each $u \geq k$

$$\bar{F}(u)=\max(\{\gamma ; G(\gamma)<u\}).$$

Then $\bar{F} \in Sd_{\{c\}}$ and $\bar{F}$ is also a non-descending function. Denote $\alpha = \bar{F}(t)$. We would like to prove that $\alpha \in Def_{\{c\}}$. In accordance with our assumption $\neg 1)$ we have that there is $v \in \mu$ such that $v<t$ and $F(t) \leq v$. If we prove that $\bar{F}(t)=\bar{F}(v)$, we shall know that also $\bar{F}(v)=\alpha$ and since μ is a minimal monad, this will imply that $\bar{F}$ is constant on μ and therefore $\bar{F}(t)=\alpha \in Def_{\{c\}}$.

Let us prove $\bar{F}(t)=\bar{F}(v)$. Since $\bar{F}$ is a non-descending function, the inequality $v<t$ implies $\bar{F}(v) \leq \bar{F}(t)=\alpha$. For proving the converse inequality we shall show firstly that $G(\alpha)<v$. Suppose $G(\alpha) \geq v$. From the construction of α it follows that $G(\alpha)<t$ and $G(\alpha+1) \geq t$. Denote $\bar{F}(v)=\beta$. Then $G(\alpha+1) \geq t$ implies $G(\beta+1) \geq v$. But $G(\alpha-1)<v$ (since $G(\alpha-1)<H^{-1}(t)$). Thus $\alpha > \beta \geq \alpha-1$ and hence $\beta = \alpha-1$. This is, however, a contradiction, since t, v belong to the same monad. Therefore $G(\alpha)<v$, which implies $\bar{F}(v) \geq \alpha = \bar{F}(t)$. Hence $\bar{F}(v)=\bar{F}(t)=\alpha$.

Since $\alpha \in Def_{\{c\}}$, also $G(\alpha) \in Def_{\{c\}}$ and at the same time $G(\alpha)<t$. Obviously $G(\alpha+1) \in Def_{\{c\}}$, too, and from the definition of $\bar{F}$ we have $G(\alpha+1)>t$. Remember now that $G(\alpha+1)=H(G(\alpha))$ and the fact that for H it is true $H"\wp_{\{c\}}(t) \subseteq \wp_{\{c\}}(t)$ - see Lemma 6. We have proved here, however, that H does not fulfil this inclusion - a contradiction.

Remark. Note that for the classes of indiscernibles 2) from the previous theorem may be strengthened to the assertion "F is constant". In the forthcoming paper we prove that it is consistent with AST that there exists a minimal monad μ for which 2) cannot be substituted by "F is constant on μ".

Now we shall present several modifications of Theorem 8.

Theorem 9. Let μ be a minimal monad in $\stackrel{\bullet}{=}_{\{c\}}$ which is a proper class. Let $F \in Sd_{\{c\}}$ be such a function that $F(t) < t$ for every $t \in \mu$. Then either 1) or 2) is valid, where

1) $(\forall u \in \mu)(u < t \Rightarrow u < F(t))$;

2) F is constant on μ.

Proof. Suppose that F is one-one on μ. In this case realize that the condition 2) from Theorem 8 cannot come (since $Def_{\{c\}} < \mu$ and thus for each $F \in Sd_{\{c\}}$ and each $t \in \mu$ $F''\varphi_{\{c\}}(t) \leq \varphi_{\{c\}}(t)$ holds - see Lemma 6) and proceed analogously as in the proof of this theorem.

Theorem 10. Let μ be a minimal monad in $\stackrel{\bullet}{=}_{\{c\}}$ which is a semiset. Let $F \in Sd_{\{c\}}$ be such a function that $F(t) > t$ for one (and hence every) $t \in \mu$. Then either 1) or 2) is valid, where

1) $(\forall v \in \mu)(v > t \Rightarrow v > F(t))$;

2) $(\exists d \in Def_{\{c\}})(\forall t \in \mu)\ d < t < F(d)$.

Proof. Let F be constant on μ. Denote $F(t) = e$ for each $t \in \mu$. Construct $F^{-1}{}''\{e\}$; then this class is a non-empty $Sd_{\{c\}}$ class (since $e \in Def_{\{c\}}$ and $F \in Sd_{\{c\}}$). Put $d = \min(F^{-1}{}''\{e\})$. Then $d \in Def_{\{c\}}$, $d < t$ and $F(d) = F(t) > t$ - 2) is valid.

Suppose further that F is one-one on μ. Recall that also F^{-1} is one-one on μ and $\nu = F''\mu$ is a minimal monad in $\stackrel{\bullet}{=}_{\{c\}}$. Moreover, $Cnh(\mu) = Cnh(\nu)$, since $F(t)$ and t are "likewise" situated with respect to $Def_{\{c\}}$; if $Int_{\{c\}}(F(t)) \neq Int_{\{c\}}(t)$, then 2) holds. Now apply Theorem 8 on ν, F^{-1} and $F(t)$.

Theorem 11. Let μ be a minimal monad in $\stackrel{\bullet}{=}_{\{c\}}$ which is a proper class. Let $F \in Sd_{\{c\}}$ be such a function that $F(t) > t$ for every $t \in \mu$. Then

$$(\forall v \in \mu)(v > t \Rightarrow v > F(t)).$$

Proof. Note, at first, that F cannot be constant on μ (in the opposite case we have $F(t) \in Def_{\{c\}}$ for $t \in \mu$, which is in contradiction with $t > Def_{\{c\}}$ and $F(t) > t$).

Hence F is one-one on μ. Put $G = F^{-1}$. Then G is also one-one on μ and $G(t) < t$ for each $t \in \mu$. Denote $\nu = F''\mu$; then ν is a minimal monad which is a proper class. Now apply Theorem 9 on G and ν.

Especially, let us stress one consequence of the previous theorems for minimal monads lying close behind FN.

Theorem 12. Let $\mu < {}^{\infty}\mathrm{Def}\ N$ be a minimal monad, $x,y \in \mu$ and $x < y$. Then $\mathrm{Def}_{\{c\}}^{FN\to FN}(\{x\}) < y$.

Proof. Suppose, at first, that ${}^{\infty}\mathrm{Def}_{\{c\}}N=\emptyset$. Then $c \in \mathrm{Def}$ and therefore μ is a proper class. In this case it is sufficient to use Theorem 11.

If ${}^{\infty}\mathrm{Def}_{\{c\}}\ N \neq \emptyset$, we shall apply Theorem 10. Since we work with a function which maps FN into FN, the case 2) from this theorem cannot set in, the case 1) is exactly what we want to prove here.

The last theorem which can be also reformulated into a parametric version, and its corollary describe several equivalent expressions of the property "to be very far one from the other".

Theorem 13. Let $\alpha \in X=\{\eta, FN<\eta<{}^{\infty}\mathrm{Def}\ N\}$. Then for each γ the following are equivalent:

(i) $\alpha <<_{FN} \gamma$;

(ii) $(\exists \beta \in X)(\beta < \gamma\ \&\ \alpha < {}^{\infty}\mathrm{Def}_{\{\beta\}}\ N)$.

Proof. (i) $\Rightarrow$ (ii). Let μ be the monad from Theorem 6. In accordance with Theorem 4 there are $x,y \in \mu$ such that $\alpha < x < y < \gamma$ (take $x,y \in m$). We would like to prove that for each $F \in Sd_0$, $F:N \to N$, we obtain $F(y) > x$. From the construction of μ we know that there is $\mathfrak{X} \in Sd_0$ such that $\mu \subseteq \mathfrak{X}$ and $F''\mathfrak{X} \cap FN=\emptyset$ or $F \restriction \mathfrak{X} : FN \to FN$. In the first case we have that $\min(F''\mathfrak{X}) > FN$ and $\min(F''\mathfrak{X}) \in \mathrm{Def}$. This implies $\min(F''\mathfrak{X}) > X$ and therefore $F''\mathfrak{X} > X$. Since $y \in \mu \geq X$ and $F(y) > X$, we obtain $F(y) > x$. If $F \restriction \mathfrak{X}:FN \to FN$, we apply Theorem 8. The assertion 2) from this theorem is now excluded, hence 1) has to take place. Put now $\beta = y$; we have then ${}^{\infty}\mathrm{Def}_{\{\beta\}} > x > \alpha$.

(ii) $\Rightarrow$ (i). We have to prove now that for each $F \in Sd_0$, $F:FN \to FN$ we have $F(\alpha) < \gamma$. In accordance with Lemma 6 we can suppose that F is a non-descending function. Put $G(t)=\min(F^{-1}{}''\{u\})$, where u is the largest element from rng(F) which is smaller or equal to t. Then G is also non-descending. We have $G(\beta) \in \mathrm{Def}_{\{\beta\}}$, and, since $\beta \in X$, at the same time $G(\beta) \notin FN$ (otherwise F is constant). Hence $G(\beta) > \alpha$ and $F(\alpha) \leq \beta < \gamma$.

Corollary. For $\alpha < \beta$, $\alpha \in X=\{\eta; FN<\eta<{}^{\infty}\mathrm{Def}\ N\}$, the following are equivalent:

1) $\alpha <<_{FN} \beta$;

2) $(\exists$ e.u.A$)(\alpha \in E_A(FN) < \beta)$, where e.u.A means an endomorphic universe A;

3) $(\exists$ e.u.s.A$)(\alpha \in Ex_A(FN) < \beta)$, where e.u.s.A means e.u. with standard extension;

4) $(\exists$ e.u.A$)(\exists \gamma \leq \beta)(\alpha, \gamma \in E_A(FN) \,\&\, \alpha \in E_{A[\gamma]}(FN))$;

5) $(\exists$ e.u.s.A$)(\exists \gamma \leq \beta)(\alpha, \gamma \in Ex_A(FN) \,\&\, \alpha \in E_{A[\gamma]}(FN))$;

6) $(\exists \gamma)(\alpha < \gamma \leq \beta \,\&\, \alpha < {}^{\infty}Def_{\{\gamma\}}\, N)$;

7) $(\exists$ e.u.A$)(\forall f \in A)(f''FN \subseteq FN \Rightarrow f(\alpha) < \beta)$.

Proof - see [Č-T].

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