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A GENERALIZED VERSION OF THE GALE-NIKAIDO-DEBREU THEOREM

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Abstract. In this paper we use a fixed point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem to prove a generalized version of the classical Gale-Nikaido-Debreu theorem.

Key words. Fixed point, price simplex, excess demand correspondence.

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1. Introduction. In recent years, several infinite-dimensional generalizations of the classical Gale-Nikaido-Debreu theorem have been proved by Bojan [1], Florenzano [3], Mehta and Tarafdar [6], Toussaint [8] and Yannelis [9]. In these papers, the commodity space is either a Banach space or a locally convex linear topological space E and it is assumed that the positive cone P has an interior point e. The role of this assumption is to ensure, via the Alaoglu-Bourbaki theorem that the "price simplex" Δ is a weak*-compact an d convex subset of the dual cone P^* of P relative to the dual system $\langle E,E'\rangle$, where E' is the topological dual of E. The compactness of the "price simplex" is needed to apply the standard fixed-point theorems.

It should be observed that the interior point assumption holds for the Banach space C(S) of continuous functions on a compact Hausdorff space. In particular, it holds for the space L_∞ of essentially bounded measurable functions on a σ-finite positive measure space (see Toussaint [8]). However, it is not satisfied for the Lebesgue spaces L_p, $1\leq p<\infty$ and the space M(K) of finite signed Baire measures on a compact metric space (see Yannelis and Zame [10]).

The object of this paper is to prove an infinite-dimensional version of the Gale-Nikaido-Debreu theorem without assuming that the positive cone of the commodity space has a non-empty interior. The proof of this result is based on a recent fixed-point theorem of Mehta [5] and Tarafdar [7]. This theorem is equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem [2] on the coverings of simplexes (see Tarafdar [7] for a proof).

Since we do not assume that the positive cone P has an interior point, the domain of the "excess demand correspondence" cannot be guaranteed to be compact in the weak*-topology of the dual space. The advantage of our approach is that we do not have to assume that the domain of the "excess-demand correspondence" is weak*-compact.

2. The Gale-Nikaido-Debreu theorem. The following theorem which has recently been proved by Mehta [5] and Tarafdar [7] is equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem [2, Theorem 4].

Theorem 1. Let X be a nonempty convex subset of a real Hausdorff topological vector space.
Let $f:X \to 2^X$ be a set-valued mapping such that

(i) for each $x \in X$, $f(x)$ is a nonempty convex subset of X;

(ii) for each $y \in X$, $f^{-1}(y)=\{x \in X : y \in f(x)\}$ contains a relatively open subset O_y of X;

(iii) $\bigcup_{x \in X} O_x = X$;

(iv) there exists a nonempty $X_0 \subset X$ such that X_0 is contained in a compact convex subset X_1 of X and the set $D=\bigcap_{x \in X_0} O_x^c$ is compact.

Then there exists a point $x_0 \in X$ such that $x_0 \in f(x_0)$.

As an application of this fixed point theorem we now prove an infinite-dimensional version of the Gale-Nikaido-Debreu theorem without an interior point assumption. Note that this can also be done by applying the Fan-Knaster-Kuratowski-Mazurkiewicz theorem.

Theorem 2. Let (X,t) be a real Hausdorff locally convex space, C a closed convex cone of X such that the dual cone $C^* = \{p \in X : p(x) \leq 0 \text{ for all } x \in C\} \neq \{0\}$. Let $T:C^* \to 2^X$ be a correspondence such that

(i) for each $\bar{p} \in C^*$ with $\{\bar{q} \in C^* : q(x) > 0 \text{ for all } x \in T(\bar{p})\} \neq \emptyset$, there is a $q \in C^*$ such that $\bar{p} \in$ weak*-interior of $\{p \in C^* : q(x) > 0 \text{ for all } x \in T(p)\} = O_q$,

(ii) for each $p \in C^*$, $T(p)$ is convex and t-compact;

(iii) for each $p \in C^*$, there exists $x \in T(p)$ such that $p(x) \leq 0$;

(iv) there exists a nonempty $D_0 \subset C^*$ such that D_0 is contained in a compact convex subset D_1 of C^* and the set $\bigcap_{p \in D_0} O_p^c$ is compact.

Then there exists $p^* \in C^*$ such that $T(p^*) \cap C \neq \emptyset$.

Proof. Suppose that the theorem is false. Then $T(p) \cap C = \emptyset$ for all $p \in C^*$. Then since $T(p)$ is compact and C is convex, the Hahn-Banach separation theorem implies that there exists a non-zero continuous linear functional r such that $\sup_{x\in C} r(x) < b < \inf_{x\in T(p)} r(x)$, where b is a real number. Since C is a closed cone, $b>0$ and $r\in C^*$. Now define a map $F:C^*\to 2^{C^*}$ by $F(p)=\{q\in C^*:q(x)>o$ for all $x\in T(p)\}$. By the argument given above $F(p)\neq\emptyset$ for each $p\in C^*$. It is easily verified that F(p) is a convex set for each $p\in C^*$.

Condition (i) implies that conditions (ii) and (iii) of Theorem 1 are satisfied. Hence Theorem 1 implies that there exists a point $p_o\in C^*$ such that $p_o\in F(p_o)$ and this contradicts condition (iii). The contradiction proves the theorem.

q.e.d.

Corollary 1. Let (X,t) be a real Hausdorff locally convex space, C a closed convex cone of X such that the dual cone $C^*=\{p\in X':p(x)\leq o$ for all $x\in C\}\neq\{o\}$. Let $T:C^*\to 2^X$ be a correspondence such that

(i) for each $\bar{p}\in C^*$ with $\{\bar{q}\in C^*:\bar{q}(x)>o$ for all $x\in T(\bar{p})\}\neq\emptyset$ there is a $q\in C^*$ such that $\bar{p}\in$ weak*-interior of $\{p\in C^*:q(x)>o$ for all $x\in T(p)\}=0_q$;

(ii) for each $p\in C^*$, T(p) is convex and t-compact;

(iii) for each $p\in C^*$, there exists $x\in T(p)$ such that $p(x)\leq o$;

(iv) for each $p\in C^*\setminus D_1$ there exists $q\in D_o$ such that $p\in 0_q$, where $D_o\subset D_1\subset C^*$, $D_o\neq\emptyset$ and D_1 is compact and convex.

Then there exists $p^*\in C^*$ such that $T(p^*)\cap C\neq\emptyset$.

Proof. It only needs to be proved that condition (iv) of Theorem 2 is satisfied. Now condition (iv) of the corollary implies that for each $p\in C^*\setminus D_1$ there exists $q\in D_o$ such that $p\notin 0^c_q$.

Consequently, $\bigcap_{p\in D_o} 0^c_p\subset D_1$. Each set 0^c_p is closed by hypothesis and D_1 is compact. Hence, $\bigcap_{p\in D_o} 0^c_p$ is compact and the proof of the corollary is finished.

q.e.d.

Suppose now that the cone C has an interior point. Then under the condition of Theorem 2 it is well-known (Jameson [4 , p. 123] and Florenzano [3]) that the set $\Delta=\{p\in C^*:p(e)=-1$, e an interior point of $C\}$ is weak*-compact and convex. We now prove the following result about locally convex spaces ordered by a cone having an interior point.

Corollary 2. Let (X,t) be a real Hausdorff locally convex linear topological space, C a closed convex cone of X having an interior point e, $C^*=\{p\in X':p(x)\leq o$ for all $x\in C\}\neq\{o\}$ the dual cone of C, and $\Delta=\{p\in C^*:p(e)=-1\}$. Let $T:\Delta\to 2^X$ be a correspondence such that:

(i)′ for each $\bar{p} \in \Delta$ with $\{\bar{q} \in \Delta : \bar{q}(x) > o$ for all $x \in T(\bar{p})\} \neq \emptyset$ there is a $q \in \Delta$ such that $\bar{p} \in$ weak*-interior of $\{p \in \Delta : q(x) > o$ for all $x \in T(p)\} = 0_q$ (relative to C^*);

(ii)′ $T(p)$ is convex and t-compact for all $p \in \Delta$;

(iii)′ for all $p \in \Delta$, there exists $x \in T(p)$ such that $p(x) \not> o$.

Then there exists $\bar{p} \in \Delta$ such that $T(\bar{p}) \cap C \neq \emptyset$.

Proof. We first extend the map $T: \Delta \longrightarrow 2^X$ to a map $T^*: C^* \longrightarrow 2^X$. To do this, one observes that C^* is equal to the cone generated by Δ , i.e. $C^* = \bigcup_{\lambda > o} \lambda \Delta$ so that each element $\hat{p}$ of C^* has a unique expression of the form $\hat{p} = \lambda p$ for $p \in \Delta$ (Jameson [4, p. 123] and Florenzano [3]).

Define $T^*: C^* \longrightarrow 2^X$ by

$$T^*(p) = \begin{cases} T(p) \text{ if } p \in \Delta \\ \lambda T(q) \text{ if } p \in C^* \setminus \Delta, \text{ and } p = \lambda q \text{ for } q \in \Delta \text{ with } \lambda > o. \end{cases}$$

In view of conditions (ii)′ and (iii)′ it is clear that conditions (ii) and (iii) of Theorem 2 are satisfied for the map T^*, since in any topological vector space the function which takes x to λx, where λ is a non-zero scalar, is a homeomorphism.

To prove that condition (i) of Theorem 2 is satisfied, let $\hat{p}$ belong to C^*. Then $\hat{p} = \lambda \bar{p}$ for some $\bar{p} \in \Delta$. By condition (i)′ there exists $q \in \Delta$ such that $\bar{p} \in$ weak*-interior of the set $\{p \in \Delta : q(x) > o$ for all $x \in T(p)\} = 0_q$.

Observe that weak*-interior of $\{p \in \Delta : q(x) > o$ for all $x \in T(p)\}$ =

= weak*-interior of $p \in \Delta : q(x) > o$ for all $x \in \lambda T(p)\}$, since $\lambda > o$.

Hence $\hat{p} = \lambda \bar{p} \in$ weak*-interior of $\{\lambda p \in C^* : q(x) > o$ for all $x \in \lambda T(p)\}$ =

= weak*-interior of $\{\lambda p \in C^* : q(x) > o$ for all $x \in T^*(\lambda p)\}$.

This proves that condition (i) of Theorem 2 holds.

Finally, to prove that condition (iv) of Theorem 2 is satisfied, let $D_o = D_1 = \Delta$. Now for each $p \in \Delta$, there is by condition (iii)′ a $q \in \Delta$ such that $p \notin 0_q^c$ where the complement is taken in Δ . Since 0_q^c is a closed subset of Δ by hypothesis, it follows that $\bigcap_{q \in \Delta} 0_q^c$ is compact.

Consequently, Theorem 2 implies that there exists a point $\bar{p} \in C^*$ such that $T(\bar{p}) \cap C \neq \emptyset$. It remains to be proved that $\bar{p} \in \Delta$. To this end, let $z \in T(\bar{p}) \cap C$. Then for any $q \in C^*$, $q(z) \not> o$, since $z \in C$ and q is a positive linear functional. This implies that $\bar{p} \notin \{p \in C^* : q(x) > o$ for all $x \in T^*(p)\}$.

A fortiori, $\bar{p} \in 0_q$. Consequently,

$$\bar{p} \in \bigcap_{q \in C^*} 0_q^c \subset \bigcap_{q \in \Delta} 0_q^c \subset D_1 = \Delta .$$

q.e.d.

Remark. For other applications of Theorem 1 and the equivalent Fan-Knaster-Kuratowski-Mazurkiewicz theorem the reader is referred to Mehta [5].

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