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ON SATURATED ALMOST DISJOINT FAMILIES

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Abstract: An almost disjoint family $\mathcal{A}\subset[X]^{\omega}$ is called <u>saturated</u> if every subset of X not covered by finitely many elements of $\mathcal{A}$ contains some member of $\mathcal{A}$. We show that in the model obtained by iteratively adding ω_1 dominating reals to V the following statement is true: On every infinite set there is a saturated almost disjoint family. The question whether this statement is true in ZFC, or even in L, remains open.

Key words: Almost disjoint family, saturated family.

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Given a set X and a collection $\mathcal{A}$ of subsets of X we denote by $I_{\mathcal{A}}$ the ideal on X generated by $\mathcal{A}\cup[X]^1$, i.e. the members of $I_{\mathcal{A}}$ are the sets that can be almost covered by finitely many elements of $\mathcal{A}$. As is usual, we write

$$I^+_{\mathcal{A}}=P(X)\setminus I_{\mathcal{A}}.$$

Definition. An almost disjoint family $\mathcal{A}\subset[X]^{\omega}$ is called saturated if $\mathcal{A}$ refines $I^+_{\mathcal{A}}$, i.e. if for every set $H\in I^+_{\mathcal{A}}$ there is some $A\in\mathcal{A}$ with $A\subset H$.

The main result of this note may now be formulated as follows:

Theorem. If P is the partial order that adds iteratively ω_1 dominating reals to V, then the following statement (✱) holds in V^P:

(✱) For every infinite set X there is a saturated almost disjoint family $\mathcal{A}\subset[X]^{\omega}$.

The proof of this result is based on several lemmas to be given below. We shall use D to denote the standard notion of forcing that adds a dominating real, i.e. a function $r:\omega\to\omega$ such that $r(n)>f(n)$ for all but finitely many $n\in\omega$ whenever $f\in{}^{\omega}\omega\cap V$, cf. [3].

Lemma 1. Let $\mathcal{A}\subset[X]^{\omega}$ be almost disjoint and $H\in I^+_{\mathcal{A}}$, $\mathcal{A},H\in V$. Then in V^D, there is a set $S\in[H]^{\omega}$ such that $|S\cap A|<\omega$ for each $A\in\mathcal{A}$, i.e. $\mathcal{A}\cup\{S\}$

is almost disjoint.

Proof. If there are only finitely many $A \in \mathcal{A}$ with $|A \cap H| = \omega$, say $A_0, \dots, A_n$, then clearly every set $S \in [H \setminus \bigcup_{i=0}^{n} A_i]^{\omega}$ works, even in V.

Otherwise let $\{A_n : n \in \omega\}$ be distinct members of $\mathcal{A}$ such that $|A_n \cap H| = \omega$ for all $n \in \omega$. Since the A_n's are almost disjoint, the sets

$$B_n = A_n \cap H \setminus \bigcup\{A_i : i < n\}$$

are disjoint and infinite. Let us write

$$B_n = \{a_{n,i} : i \in \omega\}$$

for each $n \in \omega$.

All this was done in V, but now we claim that the set

$$S = \{a_{n,r(n)} : n \in \omega\} \in [H]^{\omega}$$

defined in V^D is as required. Indeed, for each $m \in \omega$ we clearly have

$$|S \cap A_m| \leq m < \omega$$

since $A_m \cap B_n = \emptyset$ whenever $n > m$. If, on the other hand, $A \in \mathcal{A} \setminus \{A_n : n \in \omega\}$ then let us consider the function $f_A \in {}^{\omega}\omega \cap V$ defined as follows:

$$f_A(n) = \max\{i \in \omega : a_{n,i} \in A\},$$

that is well-defined because $|A \cap B_n| < \omega$. But r dominates f_A, hence we clearly have $|A \cap S| < \omega$. ⊣

Lemma 2. (Cf. [6] or [7], Lemma 5.) If W is an extension of V that contains a new real then in W there is an almost disjoint family $\mathcal{B} \subset [\omega]^{\omega}$ which refines $[\omega]^{\omega} \cap V$.

Actually, we only need this result in the case where $W = V^D$. In order to make this note self-contained we give a proof for this special case. First recall that D consists of pairs $\langle p, f \rangle$ where p is a strictly increasing map of a natural number into ω and $f \in {}^{\omega}\omega$. $\langle p, f \rangle \leq \langle q, h \rangle$ iff $p \supset q$, $f(n) \geq h(n)$ for each natural number, n, and for each $k \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ we have $p(k) > h(k)$. The generic dominating function will be denoted by r. Next we fix a partition $\{A_n : n < \omega\}$ of ω into ω-many infinite pieces in V.

We choose in V a bijection g between $[\omega]^{<\omega}$ and ω. Then for each $X \in [\omega]^{\omega} \cap V$ let us consider the set X^* defined as follows:

$$X^* = \{\min(X \cap r''A_{g(X \cap n)}) : n < \omega\}.$$

We claim that

$\mathcal{B} = \{X^* : X \in [\omega]^\omega \cap V\}$ is as required.

A standard density argument shows that whenever $X, A \in [\omega]^\omega \cap V$ we have $X \cap r''A \neq \emptyset$. Thus X^* is an infinite subset of X. To show that $\mathcal{B}$ is almost disjoint it is sufficient to observe that $X \cap n \neq Y \cap n$ implies $|X^* \cap Y^*| \leq n$ for each $X, Y \in [\omega]^\omega \cap V$. This completes the proof of the special case.

Let us denote by D_2 the notion of forcing that adds, iteratively, two dominating reals to V.
(Formally, $D_2 = D * \dot{D}$, where $\dot{D}$ names in V the poset in V^D that adds a dominating real.) Lemmas 1 and 2 then easily imply the next result.

Lemma 3. Let $\mathcal{A} \subset [X]^\omega$ be almost disjoint, then in V^{D_2} there exists a family $\mathcal{B} \subset [X]^\omega$ such that

(i) $\mathcal{A} \cup \mathcal{B}$ is almost disjoint,

(ii) $\mathcal{B}$ refines $V \cap I^+_{\mathcal{A}}$.

Proof. First, by Lemma 1, we choose in V^D for each $H \in V \cap I^+_{\mathcal{A}}$ a set $S_H \in [H]^\omega$ for which $\mathcal{A} \cup \{S_H\}$ is almost disjoint and put $\mathcal{S} = \{S_H : H \in V \cap I^+_{\mathcal{A}}\}$. Let $\mathcal{C}$ be a maximal almost disjoint subcollection of $\mathcal{S}$. Then, for each $S \in \mathcal{C}$ we may apply Lemma 2 (with V^D instead of V, V^{D_2} instead of W and S instead of ω) to obtain in V^{D_2} an almost disjoint collection $\mathcal{B}(S) \subset [S]^\omega$ refining $V^D \cap [S]^\omega$. We claim that

$$\mathcal{B} = \bigcup\{\mathcal{B}(S) : S \in \mathcal{C}\}$$

is as required. That (i) holds is obvious from the choice of $\mathcal{S}$ and $\mathcal{C}$.

To show (ii), consider any $H \in V \cap I^+_{\mathcal{A}}$. By the maximality of $\mathcal{C}$ there is some $S \in \mathcal{C}$ with $|S \cap S_H| = \omega$, but then we have a set $B \in \mathcal{B}(S)$ with

$$B \subset S \cap S_H \subset H,$$

which was to be shown.

We are now ready to give the proof of our main result.

Proof of the theorem. We may clearly consider $P = P_{\omega_1}$ as given by the finite support iteration

$$\langle P_\alpha : \alpha \leq \omega_1, Q_\alpha : \alpha < \omega_1 \rangle, \text{ where } V^{P_\alpha} \vdash Q_\alpha = D_2$$

for each $\alpha < \omega_1$.

To prove that $(*)$ holds in V^P it will clearly suffice to show it for

for $X \in V$. Now, given such an X, we define almost disjoint families $\mathcal{A}_\alpha \subset [X]^\omega$ with $\mathcal{A}_\alpha \in V^{P_\alpha}$ by induction on $\alpha \in \omega_1$ as follows.

We set $\mathcal{A}_0 = \emptyset$ and for every limit α we put $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta : \beta \in \alpha\}$. Standard tricks (cf. e.g. [5] p.281) concerning the choices made in the successor steps will insure that $\mathcal{A}_\alpha \in V^{P_\alpha}$.

Now, if $\alpha = \beta + 1$ and $\mathcal{A}_\beta$ has already been defined then we can apply Lemma 3 to get a collection $\mathcal{B}_\beta \in V^{P_\alpha} = V^{P_\beta * D_2}$ such that $\mathcal{A}_\beta \cup \mathcal{B}_\beta \subset [X]^\omega$ is almost disjoint and $\mathcal{B}_\beta$ refines $V^{P_\beta} \cap I^+_{\mathcal{A}_\beta}$. We then put $\mathcal{A}_\alpha = \mathcal{A}_\beta \cup \mathcal{B}_\beta$.

Since Lemma 3 involves choices (e.g. of the family $\mathcal{C} \subset \mathcal{S}$), the tricks we referred to above consist in making these choices "uniform" by fixing a large enough cardinal κ and a well-ordering $\prec$ of $V(\kappa)$ before we start our induction so that all the relevant sets we have to choose from, or rather names for them, already occur in $V(\kappa)$, and then every choice we have to make will be the $\prec$-least one.

If someone is not convinced by this argument, there is another way to get around this difficulty that makes use of the fact that each P_α is CCC. This makes sure that when $\langle \mathcal{A}_\beta : \beta \in \alpha \rangle$ has been defined for a limit $\alpha \in \omega_1$ with $\mathcal{A}_\beta \in V^{P_{\gamma(\beta)}}$ and $\gamma(\beta) < \omega_1$ for each $\beta \in \alpha$ then there is a $\gamma(\alpha) \in \omega_1$ such that $\langle \mathcal{A}_\beta : \beta \in \alpha \rangle \in V^{P_{\gamma(\alpha)}}$, and in this case we may define

$$\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta : \beta \in \alpha\} \text{ in } V^{P_{\gamma(\alpha)}}.$$

Having completed the induction, we set $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \alpha \in \omega_1\}$ and claim that $\mathcal{A}$ is as required, i.e. it refines $I^+_{\mathcal{A}}$.

Thus let $H \in I^+_{\mathcal{A}}$ and note first that there is a $K \in [H]^\omega_{\mathcal{A}}$ with $K \in I^+_{\mathcal{A}}$ as well. Indeed, if

$$\mathcal{H} = \{A \in \mathcal{A} : |A \cap H| = \omega\}$$

is finite then K may be chosen as any element of $[H \setminus \bigcup \mathcal{H}]^\omega$. Otherwise, let $\{A_n : n \in \omega\}$ be distinct members of $\mathcal{H}$, clearly then

$$K = \bigcup\{A_n \cap H : n \in \omega\}$$

is as required.

Next, since P is CCC, there is some $\alpha \in \omega_1$ with $K \in V^{P_\alpha}$. Obviously, we have then $K \in V^{P_\alpha} \cap I^+_{\mathcal{A}_\alpha}$ as well. But then, by our construction, there is some $A \subset \mathcal{A}_{\alpha+1} \subset \mathcal{A}$ with $A \subset K \subset H$, and our proof is complete.

In [2] the following problem was raised: For what cardinals κ is there an almost disjoint family $\mathcal{A}\subset[\kappa]^{\omega}$ that refines $[\kappa]^{\omega_1}$?

Since, trivially, $[\kappa]^{\omega_1}\subset I^+_{\mathcal{A}}$, we immediately get that every saturated family has this property, and in our V^P a saturated family exists for each κ . In [4] it was shown that an almost disjoint $\mathcal{A}\subset[\kappa]^{\omega}$ refining $[\kappa]^{\omega_1}$ exists for $\kappa=2^{\omega}$ in ZFC and for every $\kappa<\omega_{\omega_1}$ in L. In [1] it was shown that an almost disjoint $\mathcal{A}\subset[\kappa]^{\omega}$ refining $\{X\subset\kappa:\mathrm{tip}(X)\geq\omega^2\}$ exists for $\kappa=(2^{\omega})^{+n}$, $n\in\omega$, in ZFC. Several similar problems are also discussed in [1]. On the other hand it is still unknown whether a saturated $\mathcal{A}\subset[\kappa]^{\omega}$ exists in ZFC.

To conclude, we note that our notion of forcing P is CCC with $|P|=2^{\omega}$, hence V^P has the same cardinal arithmetic as V, moreover P is "mild" and thus will not effect large cardinals. Thus the problem of producing a model in which there is no saturated almost disjoint family on some set X looks very hard.

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