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ON THE REFLECTIVE HULL PROBLEM
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Abstract: It is well-known that if a subcategory of a sufficiently nice category \mathcal{A} has a cowell-powered epireflective hull, then it has a reflective hull in \mathcal{A} . A recent paper of R.E. Hoffmann shows that this amounts to characterize cowell-powered reflective subcategories in \mathcal{A} . We improve this result by dealing with a more general class of subcategories. The extension is shown to be particularly relevant in the category of topological spaces through a connection with the work of J.M. Harvey.

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Let \mathfrak{B} be a subcategory (always assumed to be full and isomorphism-closed) of a well-complete (i.e. well-powered and complete) and cowell-powered category \mathcal{A} . The following facts are well-known (see [2],[3],[4]):

- (1) Any subcategory of \mathcal{A} has an epireflective hull.
- (2) A subcategory \mathfrak{D} of \mathcal{A} with $\mathfrak{B} \subseteq \mathfrak{D} \subseteq \overline{\mathfrak{B}}$ (where $\overline{\mathfrak{B}}$ is the epireflective hull of \mathfrak{B} in \mathcal{A}) is reflective in \mathcal{A} if and only if it is epireflective in $\overline{\mathfrak{B}}$.
- (3) $\overline{\mathfrak{B}}$ is well-complete.
From this it follows that
- (4) If $\overline{\mathfrak{B}}$ is cowell-powered, then \mathfrak{B} has a reflective hull in \mathcal{A} .

This condition on $\overline{\mathfrak{B}}$ obviously calls for an improvement: it is too unstable (it depends too heavily on \mathfrak{B} itself) and it is suspected to be rather strong (for an example of an epireflective subcategory of \mathbf{Top} , the category of topological spaces which is not cowell-powered, see [2]). The first result of this paper gives a refinement of (1). This leads to a corresponding improvement of (4). The fact that this is a significant improvement is shown through a connection with a recent paper of J.M. Harvey ([1]). Note that J. Adámek and J. Rosický have found two reflective subcategories of a well and cowell-complete category having their intersection not reflective,

and hence having no reflective hull (see [5]).

First, we need a "local" version of cowell-poweredness. If \mathcal{B} is a subcategory of \mathcal{C} , we say that \mathcal{C} is \mathcal{B} -cowell-powered if any object of \mathcal{C} is the domain of a representative set of epimorphisms with codomains in \mathcal{B} . Denote by $Es(\mathcal{B})$ (respectively $P(\mathcal{B})$) the subcategory of \mathcal{C} having as objects the extremal subobjects (resp. the products) of those in \mathcal{B} . The "local form" of (1) says that if \mathcal{C} is an $Es(P(\mathcal{B}))$ -cowell-powered well-complete category, then \mathcal{B} has an epireflective hull (which has $Es(P(\mathcal{B}))$ as its class of objects). We improve this result by removing the "P" part:

Theorem. Let \mathcal{B} be a subcategory of a well-complete category \mathcal{C} . If \mathcal{C} is $Es(\mathcal{B})$ -cowell-powered, then \mathcal{B} has an epireflective hull (which has $Es(P(\mathcal{B}))$ as its class of objects).

Proof. Remark that \mathcal{C} is an (Epi, Extremal mono) category (see [3]). Any epireflective subcategory of \mathcal{C} being closed for products and extremal subobjects, we have only to show that $Es(P(\mathcal{B}))$ is epireflective.

For an object c of \mathcal{C} , consider a representative set

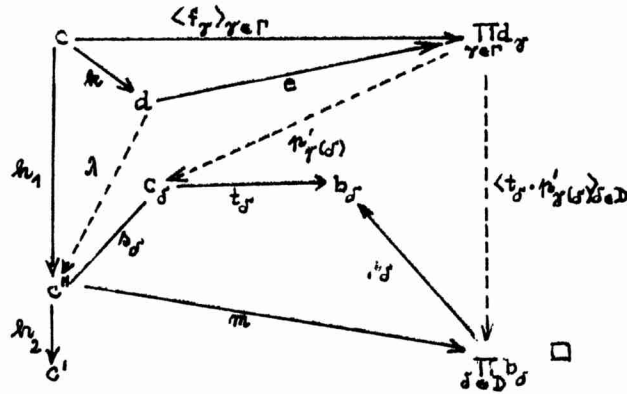
$\{f_\gamma: c \rightarrow d_\gamma \mid \gamma \in \Gamma\}$ of the epimorphisms from c to an object of $Es(\mathcal{B})$, and

$$\langle f_\gamma \rangle_{\gamma \in \Gamma} = ek: c \rightarrow d \rightarrow \prod_{\gamma \in \Gamma} d_\gamma$$

the (Epi, Extremal mono)-factorization of the induced morphism from c to the product $\prod_{\gamma \in \Gamma} d_\gamma$. We prove that k is the required reflection morphism.

Let $h: c \rightarrow c'$ be a morphism with c' in $Es(P(\mathcal{B}))$ and $h_2 h_1: c \rightarrow c'' \rightarrow c'$ its (Epi, Extremal mono)-factorization. c'' being also in $Es(P(\mathcal{B}))$, there exist a set D and an extremal mono $m: c'' \rightarrow \prod_{\sigma \in D} b_\sigma$ with each b_σ in \mathcal{B} . For each $\sigma \in D$, let $t_\sigma s_\sigma: c'' \rightarrow c_\sigma \rightarrow b_\sigma$ be the (Epi, Extremal mono)-factorization of $p_\sigma m$, where p_σ is the canonical projection $\prod_{\sigma \in D} b_\sigma \rightarrow b_\sigma$. Each $s_\sigma h_1$, being an epi with codomain in $Es(\mathcal{B})$, can be identified (via an isomorphism) with $f_{\gamma(\sigma)}$ for some $\gamma(\sigma) \in \Gamma$.

Then $p_{\gamma(\sigma)} \cdot \langle f_\gamma \rangle_{\gamma \in \Gamma} = s_\sigma h_1$ (where $p_{\gamma(\sigma)}$ is the canonical projection $\prod_{\gamma \in \Gamma} d_\gamma \rightarrow d_{\gamma(\sigma)} = c_\sigma$). k being an epi and m an extremal mono, the equality $\langle \langle t_\sigma \cdot p_{\gamma(\sigma)} \rangle_{\sigma \in D} \rangle_{\gamma \in \Gamma} \cdot ek = mh_1$ induces a unique $\lambda: d \rightarrow c''$ such that $\lambda k = h_1$ and $m \lambda = \langle \langle t_\sigma \cdot p_{\gamma(\sigma)} \rangle_{\sigma \in D} \rangle_{\gamma \in \Gamma} \cdot e$. Because k is epi, $h_2 \lambda$ is unique such that $h_2 \lambda k = h$.



Corollary. Let \mathcal{B} be a subcategory of a well-complete cowell-powered category \mathcal{A} such that $\overline{\mathcal{B}}$ (the epireflective hull of \mathcal{B}) is \mathcal{B} -cowell-powered. Then \mathcal{B} is cowell-powered and

- a) \mathcal{B} is closed for products and extremal subobjects in $\overline{\mathcal{B}}$ if and only if \mathcal{B} is reflective in \mathcal{A} . In this case $\overline{\mathcal{B}}$ is cowell-powered.
- b) If \mathcal{B} is the intersection of reflective subcategories of \mathcal{A} , then it is reflective in \mathcal{A} and $\overline{\mathcal{B}}$ is cowell-powered.
- c) If $\overline{\mathcal{B}}$ is $\text{Es}(\overline{\mathcal{B}})$ -cowell-powered, then \mathcal{B} has a reflective hull.

Proof. The inclusion of \mathcal{B} in $\overline{\mathcal{B}}$ preserves epimorphisms (see [4] for example), from which it follows that \mathcal{B} is cowell-powered.

a) (\leftarrow) This follows easily from the facts that $\overline{\mathcal{B}}$ is a (Epi, Extremal mono) category and \mathcal{B} is epireflective in $\overline{\mathcal{B}}$.

(\rightarrow) In Theorem 1, take $\mathcal{C} = \overline{\mathcal{B}}$. Then the equalities $\text{Es}(\overline{\mathcal{B}}) = \mathcal{B} = \text{Es}(P(\overline{\mathcal{B}}))$ imply the epireflectivity of \mathcal{B} in $\overline{\mathcal{B}}$, and hence the reflectivity of \mathcal{B} in \mathcal{A} . The cowell-poweredness of $\overline{\mathcal{B}}$ follows from [4].

b) Let $\mathcal{B} = \bigcap_{\gamma \in \Gamma} \mathcal{B}_\gamma$ with each \mathcal{B}_γ reflective in \mathcal{A} . Then $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}_\gamma}$ for each $\gamma \in \Gamma$, and this, with the fact that they are both reflective in \mathcal{A} , implies that an extremal mono of $\overline{\mathcal{B}}$ is an extremal mono of $\overline{\mathcal{B}_\gamma}$ for each $\gamma \in \Gamma$. \mathcal{B}_γ being closed for the extremal subobjects in $\overline{\mathcal{B}_\gamma}$, we conclude that \mathcal{B} is closed for the extremal subobjects of $\overline{\mathcal{B}}$. As it is clearly closed for products, we have the result by part a).

c) This follows from the theorem and points (2) and (3) above. \square

To make the connection with the terminology and results of [1], we

must recall some of its definitions: given a subcategory \mathcal{B} of \mathcal{A} , we will say that a morphism $f:Y \rightarrow Y$ in \mathcal{A} is \mathcal{B} -generating and that \mathcal{B} is f -generated if for any $r,s:Y \rightrightarrows Z$ in \mathcal{A} with Z in \mathcal{B} and such that $rf=sf$, we have $r=s$; a \mathcal{B} -generating morphism with co-domain in \mathcal{B} is a \mathcal{B} -epi; \mathcal{B} is cowell-powering (in \mathcal{A}) if each object in \mathcal{A} is the domain of a representative set of \mathcal{B} -epis; \mathcal{B}_0 (respectively \mathcal{B}_1) is the maximal (full) subcategory of \mathcal{A} which is f -generated for any \mathcal{B} -epi (resp. \mathcal{B} -generating) f . The following facts are obvious or follow immediately from the remarks at the beginning of [1]:

- i) \mathcal{B} is closed under extremal subobjects in $\overline{\mathcal{B}}$ if and only if it is closed under extremal subobjects in \mathcal{B}_1 (resp. \mathcal{B}_0).
- ii) \mathcal{B} is cowell-powering subcategory of \mathcal{A} if and only if $\overline{\mathcal{B}}$ (resp. $\mathcal{B}_1, \mathcal{B}_0$) is \mathcal{B} -cowell-powered.

With this in mind, we immediately obtain the proposition 1, the corollary and the proposition 3 of [1] respectively from parts a), b) and c) of our corollary.

There are certain advantages in considering $\overline{\mathcal{B}}$, because it is $\text{Es}(\mathcal{P}(\mathcal{B}))$ (in well-complete cowell-powered categories), but it is the relative insensitivity of \mathcal{B}_0 and \mathcal{B}_1 to changes in \mathcal{B} , pointed out by Harvey through several examples in Top , that shows that the weakening of the condition " \mathcal{B} is cowell-powered" to " $\overline{\mathcal{B}}$ is $\text{Es}(\mathcal{B})$ -cowell-powered" is a significant one.

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