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ON WEAKLY UNIFORMLY ROTUND SPACES

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Abstract: If X is a Banach space whose dual is weakly uniformly rotund and $(V_n)_{n=1}^{\infty}$ is a sequence of subspaces of X^* whose characters tend to one, then $\bigcap_{n=1}^{\infty} V_n=\{0\}$. A weakly sequentially complete Banach space whose dual is weakly uniformly rotund is reflexive.

Key words: Weakly uniformly rotund spaces, character of subspace.

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1. Introduction. We shall consider weakly uniformly rotund spaces (in symbol WUR). I. Singer proved in [5] that if X^{**} is not smooth, then X^* contains no closed proper subspace of the character one. We obtain an analogous result (Theorem 2.1) for a space X whose dual is WUR. We shall also deal with the reflexivity of the WUR space.

Let us recall some notions and basic results. We consider only real Banach spaces. Let X be a Banach space. Its topological dual and its second topological dual are denoted by X^* and X^{**} respectively. The symbols B(X), S(X) mean the closed unit ball and the unit sphere around the origin in X. The canonical embedding of X into X^{**} is denoted by Q. The value of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x^*, x\rangle$. Let $f \in S(X^*)$. The function $\delta(X,f)$: $:\langle 0,2) \rightarrow \langle 0,1)$ defined by

$$\delta(X,f)(\varepsilon)=\inf\{1-\frac{\|x+y\|}{2};\ \|x\|=1, \|y\|=1,\ |f(x-y)|\geq \varepsilon\}$$

is called the modulus of weak convexity in the direction f. A Banach space X, respectively its dual X^*, is said to be weakly uniformly rotund (in short WUR), respectively weakly* uniformly rotund (in short W*UR) if for every $\varepsilon \in (0,2)$ the following holds:

$\delta(X,f)(\varepsilon)>0$ for every $f\in S(X^*)$,

respectively $\delta(X^*,Q(x))(\varepsilon)>0$ for every $x\in S(X)$.

Given a $y\in X$, the function $\rho(X,y):\langle 0,\infty\rangle\to\langle 0,\infty\rangle$ defined by

$$\rho(X,y)(\tau)=\sup\left\{\frac{\|x+z\|}{2}+\frac{\|x-z\|}{2}-1;\ x\in S(X),\ z=\tau y\right\}$$

is called the modulus of smoothness in the direction y. A Banach space X is said to be uniformly Gâteaux smooth if

$$\lim_{\tau\to 0+}\frac{\rho(X,y)(\tau)}{\tau}=0 \text{ for every } y\in S(X).$$

Let us recall the following well known results.

Theorem 1.1. If X is WUR, then every $x^{**}\in X^{**}$ is a sequential weak* limit of elements of $Q(X)$.

Theorem 1.2 ([2]). Let X be a Banach space. The following conditions are equivalent:

(i) X is WUR

(ii) X^* is uniformly Gâteaux smooth

(iii) X^{**} is W*UR

Theorem 1.3 ([2]). Let X be a Banach space. The following conditions are equivalent:

(i) X is uniformly Gâteaux smooth

(ii) X^* is W*UR

2. Banach spaces whose dual is WUR. We introduce the following notions (see e.g. [4]). Let X be a Banach space and let V be a linear subspace of X^*. The number $r(V)=\sup\{\alpha\geq 0; B(V)$ is weakly* dense in $\alpha B(X)\}$ is called the character of V. Let us suppose that V is weakly* dense in X and let P be the projection on $Q(X)+V^{\perp}$ defined by $P(z+y)=z$ for every $z\in Q(X)$, $y\in V^{\perp}$. Then ([4]) $r(V)=\frac{1}{\|P\|}$.

The following theorem has been motivated by a result of Singer (see [5]).

Theorem 2.1. Let X be a Banach space whose dual is WUR. Let $(V_n)_{n=1}^{\infty}$ be a sequence of subspaces of X^* such that $\lim_{n\to\infty} r(V_n)=1$. Then $\bigcap_{n=1}^{\infty} V_n^{\perp}=\{0\}$.

Proof: Let us assume that the converse holds. We shall derive a contradiction. Let us suppose that there is a sequence of subspaces $(V_n)_{n=1}^{\infty}$ of X^* satisfying the following conditions:

$$\lim_{n\to\infty} r(V_n)=1 \text{ and there is } \Phi\in(\bigcap_{n=1}^{\infty} V_n^{\perp})\cap S(X^{**}).$$

For each $n\in N$ we find $f_n\in S(X^*)$, $x_n\in S(X)$, such that $\Phi(f_n)>\frac{1}{2}$, $f_n(x_n)>1-\frac{1}{n}$, $f_n\in X^*\setminus V_n$. As $r(V_n)$ converges to one, we shall suppose that $r(V_n)>0$ for all $n\in N$. It means that V_n is weakly* dense in X^* for any $n\in N$ and we can put $W_n=Q(X)\oplus V_n^{\perp}$. Let $P_n:$ $:W_n\to W_n$ be the projection defined by

$$P_n(z+y)=z \text{ for every } z\in Q(X),\ y\in V_n^{\perp}.$$

Denote the canonical embedding of X^* into X^{***} by Q_{X^*} and put

$$g_n=Q_{X^*}(f_n) \qquad \bar{h}_n=r(V_n)(Q^{-1}P_n)^* f_n \qquad n=1,2,\dots$$

Let h_n denote the norm preserving extension of $\bar{h}_n$ from W_n on the whole X^{**}.

We have $\|g_n\| = \|f_n\| =1$, because Q_{X^*} is an isometry and

$$\|h_n\|\le r(V_n)\|(Q^{-1})^*\|\cdot\|P_n^*\|\cdot\|f_n\|=1, \text{ because}$$

$r(V_n)=\frac{1}{\|P\|}=\frac{1}{\|P^*\|}$. We obtain that

$\langle Q(x_n),g_n\rangle=\langle Q(x_n),Q_{X^*}(f_n)\rangle=\langle f_n,Q(x_n)\rangle=\langle x_n,f_n\rangle>1-\frac{1}{n}$

and $\langle Q(x_n),h_n\rangle=r(V_n)\langle Q(x_n),(Q^{-1}P_n)^*f_n\rangle=r(V_n)\langle(Q^{-1}P_nQ)(x_n),f_n\rangle=$ $=r(V_n)\langle x_n,f_n\rangle=r(V_n)(1-\frac{1}{n})$.

These inequalities imply that $\|g_n+h_n\|\xrightarrow[n\to\infty]{}2$. Obviously,

$$\langle\Phi,g_n\rangle=\langle f_n,\Phi\rangle>\frac{1}{2} \text{ and } \langle\Phi,h_n\rangle=r(V_n)\langle Q^{-1}P_n\Phi,f_n\rangle=$$

$=r(V_n)\langle 0,f_n\rangle=0$.

Therefore $(g_n-h_n)(\Phi)>\frac{1}{2}$ for all $n\in N$. Hence X^{***} is not W^*UR and by Theorem 1.2 X^* is not WUR. This is a contradiction.

The assumption of the just proved theorem can be weakened a little. Namely, it suffices for X to be a subspace of another Banach space Y whose dual is WUR. Indeed, it is a routine matter that in this case X^* has a dual WUR norm, too. Further, $X^{\perp\perp}$ is linearly isometric with $(Y^*/X^{\perp})^*$ and $Y^*/X^{\perp}$ with X^*. Therefore X^{**} is linearly isometric with $X^{\perp\perp}$. Consequently X is uniformly Gâteaux smooth and so X^* is WUR.

By the end of this note we shall deal with reflexivity of WUR spaces.

Lemma 2.2. Let X be a Banach space whose dual is WUR. Then X has no subspace which is isomorphic to ℓ_1.

Proof: Assuming the converse, we derive a contradiction. Let Y be a subspace of X which is isomorphic to ℓ_1. Then Y^* is isomorphic to ℓ_∞ and thus it has no equivalent WUR norm (see [2 p.120]). Hence there are $(f_n)_{n=1}^\infty \subset S(Y^*), (g_n)_{n=1}^\infty \subset S(Y^*)$, $F \in Y^{**}$ and $\varepsilon > 0$ so that $\lim_{n\to\infty} \|f_n+g_n\|=2$, $|F(f_n-g_n)| > \varepsilon$ for all $n \in N$. Let $\bar{f}_n$, $\bar{g}_n$ be norm-preserving extensions of the functionals f_n, g_n from Y on the whole X and let I denote the embedding of Y into X. Then

$$\langle \bar{f}_n-\bar{g}_n, I^{**}(F)\rangle = \langle I^*(\bar{f}_n-\bar{g}_n), F\rangle = \langle f_n-g_n, F\rangle$$

for all $n \in N$.

Since X^* is WUR $\lim_{n\to\infty} F(f_n-g_n)=0$, which is a contradiction.

Theorem 2.3. Let X be a weakly sequentially complete Banach space. Then X is reflexive if either X or X^* is WUR.

Proof: Let X be WUR and let $x^{**} \in S(X^{**})$. Using Theorem 1.1 we can find a sequence $(x_n)_{n=1}^\infty \subset X$ such that $\lim_{n\to\infty} Q(x_n)=x^{**}$ in the weak* topology. Then $(x_n)_{n=1}^\infty$ is a weak Cauchy sequence and hence it converges weakly in X. Therefore $x^{**} \in Q(X)$ and so X is reflexive. Let X^* be WUR. Since the reflexivity of Banach spaces is separably determined we can assume that X is separable. According to Lemma 2.2 X has no subspace which is isomorphic to ℓ_1. By [3], for every $x^{**} \in X^{**}$ there is a sequence $(x_n)_{n=1}^\infty \subset X$ such that $\lim_{n\to\infty} Q(x_n)=x^{**}$ in the weak* topology. Then, again, by the weak sequential completeness of X, $x^{**} \in Q(X)$, which completes the proof.

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