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ON THE NUMBER OF COMPACT SUBSETS IN TOPOLOGICAL GROUPS

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Abstract: Results on the number of compact subsets in topological groups are proved. Examples are provided.

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Notation and terminology. Let (G,τ) be a nondiscrete Hausdorff group, e be its neutral element and $\mathcal{K}$ denote the set of all compact subsets of G. For any set X, $|X|$ denotes the cardinality of X; and, for any topological space X, $\psi(X)$, $\chi(X)$, $w(X)$, $c(X)$, $wL(X)$ denote the pseudocharacter, character, weight, cellularity, weak Lindelöf number of X, respectively.

1. Number of compact subsets

Definition (due to I. Juhász). The boundedness number of (G,τ) - denoted by bo(G) - is the smallest infinite cardinal number α such that for any open neighborhood V of e, there is a subset A of G, with $|A| \le \alpha$, so that $V.A=G$.

Notice that this notion is different from total-β-boundedness introduced by Comfort in [3].

Theorem 1. The following inequalities hold $\psi(G) \le |G| \le |\mathcal{K}| \le bo(G)^{\psi(G)}$.

Proof. There is a collection of open symmetric neighborhoods of e, $\mathcal{V}$, such that $|\mathcal{V}| = \psi(G)$ and $\bigcap\{V.V \mid V \in \mathcal{V}\} = \{e\}$. For each $V \in \mathcal{V}$ fix a subset A_V of G such that $V.A_V=G$ and $|A_V| \le bo(G)$. Now the proof follows the one which appears in [1], since $\bigcap_{V\in\mathcal{V}} (\bigcup_{x\in A_V} Vx.Vx) = \Delta$, the diagonal of $G\times G$,

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Partially supported by CNPq.

$|G| \leq bo(G)^{\psi(G)}$ and any compact subset of G has density not bigger than $\psi(G)$.

Remarks. 1) As a matter of fact, the proof above shows that the set of all closed subsets of G whose densities do not exceed $\psi(G)$ has cardinality not bigger than $bo(G)^{\psi(G)}$.

2) It is easy to see that $bo(G) \leq wL(G) \leq c(G)$ (hence, $bo(G)^{\psi(G)} = wL(G)^{\psi(G)} = c(G)^{\psi(G)}$) and $w(G) = bo(G) \cdot \chi(G)$.

3) If bo(G) is either a successor cardinal or a singular cardinal, then $o(G)^{bo(G)} = o(G)$, where o(G) denotes the number of open sets in G.

Corollary 1. If $bo(G) \leq 2^{\psi(G)}$, then $\psi(G) \leq |\mathcal{K}| \leq 2^{\psi(G)}$.

Lemma. If K is a nonempty compact subset of G, then $\psi(K,G) \leq \psi(G)$.

Proof. Let $\mathcal{V}$ be a collection of symmetric open neighborhoods of e, closed under finite intersections. Furthermore we shall assume that $|\mathcal{V}| = \psi(G)$ and $\bigcap\{cl(V) | V \in \mathcal{V}\} = \{e\}$. Then $\bigcap\{V.K | V \in \mathcal{V}\} = K$; indeed, let $y \notin K$, then there is $V \in \mathcal{V}$ such that $Vy \cap K = \emptyset$ (otherwise, $Vy \cap K \neq \emptyset$, $\forall V \in \mathcal{V}$ and since K is compact, $\bigcap\{cl(Vy) \cap K | V \in \mathcal{V}\}$ would be nonempty, which is impossible). But if $Vy \cap K = \emptyset$, then $y \notin V.K$.

Corollary 2. If $bo(G) \leq 2^{\psi(G)}$ and there is a compact subset K of G such that $\psi(K,G) < \psi(G)$, then $|\mathcal{K}| = 2^{\psi(G)}$.

Proof. If there is a nonempty compact subset K of G such that $\psi(K,G) < \psi(G)$, then for each $x \in K$, $\psi(G) = \psi(x,G) \leq \psi(x,K) \cdot \psi(K,G)$. It follows from Čech-Pospíšil's theorem that $|K| \geq 2^{\psi(G)}$, hence $|\mathcal{K}| = 2^{\psi(G)}$.

Theorem 2. (GCH) If G is pseudocompact, then $|\mathcal{K}|^{\aleph_0} = |\mathcal{K}|$.

Proof. Since G is infinite and pseudocompact, then $|G| \geq 2^{\aleph_0}$. We may assume that if α is a cardinal number such that $\alpha \geq 2^{\aleph_0}$, $cf(\alpha) \neq \aleph_0$, then $\alpha^{\aleph_0} = \alpha$.

From Theorem 1 and since $bo(G) = \aleph_0$, either $|\mathcal{K}| = 2^{\psi(G)}$ or $|G| = |\mathcal{K}| = \psi(G)$. In the first case there is nothing to prove; let us consider that $|G| = |\mathcal{K}| = \psi(G)$. From van Douwen's theorem 1.1 ([4]) if $cf(|G|) = \aleph_0$, there is a cardinal $\mu < |G|$, such that $\psi(G) \leq w(G) \leq 2^{\mu}$. But $|G| \geq 2^{\mu}$, hence $|G| = |\mathcal{K}| = 2^{\mu}$ and the proof is completed.

Lemma. If V is an open symmetric neighborhood of e and $\mathcal{K}(cl(V))$ denotes the set of all compact subsets of cl(V), then $|\mathcal{K}| = bo(G) |\mathcal{K}(cl(V))|$.

Proof. It is immediate that $|\mathcal{K}| \geq bo(G)$ and $|\mathcal{K}| \geq |\mathcal{K}(cl(V))|$: On the other hand, let B be a subset of G such that $bo(G) \geq |B|$ and V.B=G. For each nonempty finite subset F of B let $\mathcal{K}_F$ denote the set of all compact subsets of G contained in V.F. The function from $\mathcal{K}_F$ into $\prod\{\mathcal{K}((cl\ V)y) | y \in F\}$ which assigns to each $K \in \mathcal{K}_F$ the point $(cl Vy \cap K)_{y \in F}$ is injective. But $\mathcal{K} = \bigcup\{\mathcal{K}_F | \emptyset \neq F \subset B,\ \text{finite}\}$ and $|\mathcal{K}(cl\ Vy)| = |\mathcal{K}(cl\ V)|$, hence $|\mathcal{K}| \leq bo(G)|\mathcal{K}(cl\ V)|$, which completes the proof.

Remark. The GCH cannot be avoided in Theorem 2, since I. Juhász, under CH and using forcing arguments, obtained an HFD subgroup of $\{0,1\}^{\omega_1}$, such that $|\mathcal{K}|^{\aleph_0} \neq |\mathcal{K}|$.

2. Examples

Example 1. ([5] or[2], page 1170.) Under $\aleph_1 = 2^{\aleph_0}$ and $\aleph_2 < 2^{\aleph_1}$ there is a hereditarily separable pseudocompact group G with $|G| = |\mathcal{K}| = \aleph_2$ (which is not a power of 2, but $\aleph_2^{\aleph_0} = \aleph_2$).

Example 2. Let G be the topological subgroup of $\{0,1\}^{\omega_1}$ whose members are the $(x_\alpha)_{\alpha \in \omega_1}$ such that $\{\alpha \in \omega_1 | x_\alpha = 1\}$ is countable. G is countably compact, $\psi(G) = \aleph_1$, $|G| = 2^{\aleph_0}$ and $|\mathcal{K}| = 2^{\aleph_1}$. (Notice that the set $\{(x_\alpha)_{\alpha \in \omega_1} \in G | x_\alpha = 1$ for at most one $\alpha \in \omega_1$ is compact and has just one accumulation point.)

Example 3. Let us consider $\{0,1\}^{\omega_1}$ with the G_δ-topology (each factor with the discrete topology). For each $\beta \in \omega_1$, let y_β be the point $(x_\alpha)_{\alpha \in \omega_1}$ such that $x_\alpha = 1$, $\forall \alpha < \beta$ and $x_\alpha = 0$, otherwise. Denote by G the topological subgroup generated by the y_β. Then $bo(G) = \aleph_0$ and $wL(G) = \aleph_1$. (Notice that no countable subcollection of $\{pr_\xi^{-1}(\{0\}) | \xi \in \omega_1\}$, where pr_ξ denotes the projection for each $\xi \in \omega_1$, has its union dense in G.)

Example 4. Let α be an infinite cardinal such that $\alpha^{\aleph_0} = \alpha$. Comfort proved that there is a dense countably compact subgroup G_* of $\{0,1\}^{2^\alpha}$ such that $|G_*| = \alpha$. Denoting by G the topological product group $\Sigma \times G_*$, where Σ denotes the subgroup of $\{0,1\}^\alpha$ such that its elements have at most countably many coordinates different from 0, we have that $|G| = \alpha$, $\psi(G) = \alpha$ and $w(G) = |\mathcal{K}| = 2^\alpha$. Notice that G is countably compact.

Example 5. Let α be an infinite cardinal number, whose cofinality is β and let $(\alpha_i)_{i\in\beta}$ be a strictly increasing family of cardinals such that $\alpha = \sup_{i\in\beta} \alpha_i$. For each $i \in \beta$ let G_i be a discrete topological group with $|G_i| = \alpha_i$. The topological product group $G = \prod_{i\in\beta} G_i$ has a boundedness number equal to α, $\psi(G) = \beta$ and $|\mathcal{K}| = \alpha^{\beta}$.

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