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LINEAR COMPLEMENTARITY PROBLEM AND EXTREMAL HYPERPLANES

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Abstract: We prove that certain (n-1)-dimensional hyperplanes in R^n have an extremality property w.r.t. the linear complementarity problem. Some other results about general hyperplanes in R^n are also contained in this article. The problem is related to the investigation of certain types of nonlinear differential equations and variational inequalities.

Key words: Linear complementarity, hyperplanes, n-dimensional cube.

Classification: 90C33, 47H15, 05A05

Introduction. This article is motivated by the investigation of the linear complementarity problem (LCP), which can be formulated as follows:

Let A be a given nxn-matrix. Let $f \in R^n$ be a given vector. We want to find a vector $u \in R^n$ such that

$$u^+ - Au^- = f$$

where u^+ and u^- are the positive and the negative part of u, respectively. I.e., for $u=(u_i)_{i\in n} \in R^n$ we define $u^+=(u_i^+)_{i\in\overline{n}} \in R^n$ and $u^-=(u_i^-)_{i\in\overline{n}} \in R^n$ by means of the formulae

$$u_i^+=\max\{u_i,0\},\ u_i^-=\max\{-u_i,0\}$$

for all $i \in \overline{n}$ (see below for the notation $\overline{n}$).

There exists a vast literature about the LCP. From the many articles about the subject, let us notice, e.g., [1],[2] and [3]. We do not discuss them here, because we are concerned by the LCP from another point of view, than the authors of the above mentioned papers.

The pioneering work of Ambrosetti and Prodi in the theory of a class of abstract nonlinear equations (see [4] and [5]) was generalized in the paper [6] of Fučík, Kučera and Nečas and in various subsequent papers. It has been shown that the problem of the solvability of certain differential equations can be reduced to a finite dimensional problem and that the LCP is a typical example of such problems. Many references can be found in [7]. From the recent related papers let us mention [8].

There is also the paper [9] of Fučík and Milota, which shows that the solution of an appropriate LCP can substantially simplify the solution of certain variational inequalities. For instance, this way we can find the solution to the problem

$$y''=\varphi \quad \text{in }]0,1[- \{x_1,x_2,\dots,x_n\},$$
$$y(0)=y(1)=0,$$
$$y \text{ is continuous in } [0,1],$$
$$y(x_i)\geq 0 \text{ and } y_+'(x_i)-y_-'(x_i)\leq 0 \quad \text{for all } i\in\bar{n},$$
$$(y_+'(x_i)-y_-'(x_i))\, y(x_i)=0 \text{ for all } i\in\bar{n},$$

where φ is a given function and x_i, $i\in\bar{n}$ are given points of the interval $]0,1[$. This is a mathematical model of a loaded string over some one-point obstacles. (Cf. [10].) In this context let us mention also the paper [11].

In [12] it has been shown that the LCP is related to some sort of classification of hyperplanes in R^n in the sense that the existence of various types of hyperplanes in R^n implies the existence of various classes of LCP's. Hence, it is interesting to investigate, which types of hyperplanes $\wp\subset R^n$ do exist. Some partial results are contained in [13], another result is formulated in Theorem 4 of this article.

From this point of view Theorem 4 is our main result, but its proof is rather simple after having proved Theorem 1, which seems to be our most complicated result.

Section 1. Definitions and auxiliary results

Notation. (i) $\bar{n}=\{1,2,\dots,n\}$.

(ii) Let $\wp\subset R^n$ be any $(n-1)$-dimensional hyperplane which does not contain the straight line

$$x_1=x_2=x_3=\dots=x_n.$$

Then $\wp^+$ is the open half-space of R^n w.r.t. $\wp$ which contains the points $(a,a,a,\dots,a)$ for all sufficiently big values of a, $\wp^-$ is the opposite open half-space of R^n.

(iii) $[a]$ denotes the integer part of a, $[a,b]$ denotes a closed interval.

Definition 1. For any $\omega\subset\bar{n}$ let us define the point $C_\omega=(c_i^\omega)_{i\in\bar{n}}\in R^n$ by means of the formulae

(1) $$c_i^\omega=-1, \text{ if } i\in\omega,$$
$$c_i^\omega=1, \text{ if } i\in\bar{n}-\omega.$$

All the points C_ω, $\omega\subset\bar{n}$ are the vertices of the n-dimensional cube $C^n\subset R^n$.

Definition 2. i-edges are all the (1-dimensional) edges of C^n which are parallel to the x_i-coordinate axis in R^n.

Definition 3. Let $\wp \subset R^n$ be an (n-1)-dimensional hyperplane which does not contain any vertex $C_\omega \in C^n$. For such a hyperplane and any $i \in \bar{n}$ we can define $k_i(\wp)$ as the number of all the i-edges which are intersected by $\wp$. Further we define

(2) $$k(\wp)=\min\{k_i(\wp)\,|\,i\in\bar{n}\}.$$

Lemma 1. Let

(3) $$\sum_{i\in\bar{n}} a_i x_i = b$$

be the equation of a hyperplane $\wp \subset R^n$. Let $k_i(\wp)$, $i\in\bar{n}$ be defined and let for some $j,\ m\in\bar{n}$

(4) $$|a_j| \leq |a_m|.$$

Then

(5) $$k_j(\wp)\leq k_m(\wp).$$

Proof. If $a_j=0$, then $\wp$ is parallel to the x_j-coordinate axis and cannot intersect any j-edge. Hence $k_j(\wp)=0$ and (5) holds.

Let $a_j \neq 0$, then $a_m \neq 0$ according to (4). Let us look at the 2-dimensional faces C^2_ξ of C^n which are contained in the parallel planes $\wp_\xi$. The equations of $\wp_\xi$ are

(6) $$\begin{aligned} &x_i = -1, \text{ if } i \in \xi, \\ &x_i = 1, \text{ if } i\in \bar{n}-\xi-\{j,m\}, \\ &\xi \subset \bar{n}-\{j,m\}. \end{aligned}$$

Because $a_j a_m \neq 0$,

$$\wp_\xi \cap \wp = p_\xi$$

is a straight line in $\wp_\xi$ and we can define $k_j(p_\xi)$ and $k_m(p_\xi)$ as the number of the j-edges and the m-edges of C^2_ξ which are intersected by p_ξ. These numbers are well-defined, because $C_\omega \in p_\xi \Rightarrow C_\omega \in \wp$, hence in the opposite case $k_i(\wp)$ would not be defined. Further

(7) $$k_j(\wp)=\sum_{\xi\subset\bar{n}-\{j,m\}} k_j(p_\xi),$$

(8) $$k_m(\wp)=\sum_{\xi\subset\bar{n}-\{j,m\}} k_m(p_\xi),$$

hence it is sufficient to prove that for every $\xi \subset \bar{n}-\{j,m\}$

(9) $$k_j(p_\xi)\leq k_m(p_\xi),$$

(5) then follows from (7) and (8).

The equations of p_ξ are (6) and (3), (3) can be rewritten as

$$a_jx_j+a_mx_m=b-\sum_{i\in \bar{m}-\{j,m\}}a_ix_i .$$

Using (6) we have

$$b-\sum_{i\in\bar{m}-\{j,m\}} a_ix_i=b+\sum_{i\in\xi} a_i-\sum_{i\in\bar{m}-\xi-\{j,m\}}a_i=b_\xi ,$$

thus the equations of p_ξ are (6) and

(10) $$a_jx_j+a_mx_m=b_\xi .$$

Let p_ξ intersect a j-edge of C_ξ^2 . Then p_ξ must intersect some other edge of C_ξ^2 . If it were the other j-edge, there would exist two numbers x_j^1 and x_j^2 such that (see (10))

(11) $$|x_j^1|<1,\ |x_j^2|<1,$$

(12) $$a_jx_j^1+a_m=b_\xi ,$$

(13) $$a_jx_j^2-a_m=b_\xi .$$

Subtracting the equation (13) from (12) we obtain

$$2a_m=a_j(x_j^2-x_j^1),$$

hence

$$2|a_m|=|a_j||x_j^2-x_j^1|\leq|a_j|(|x_j^2|+|x_j^1|)<2|a_j|$$

according to (11). This is a contradiction, because we suppose (4). Hence p_ξ cannot intersect two j-edges of C_ξ^2 and if it intersects a j-edge, it must also intersect an m-edge of C_ξ^2 . This implies (9).

Lemma 2. Let $\wp(t)\subset R^n$ be the hyperplane

(14) $$\sum_{i\in\bar{n}} x_i=t.$$

(i) Let $p\in\bar{n}\cup\{0\}$ and $t=n-2p$. Then $k_i(\wp(t))$, $i\in\bar{n}$ are not defined.

(ii) Let $p\in\bar{n}$ and $t\in]n-2p,n-2p+2[$. Then

$$k_i(\wp(t))=\binom{n-1}{p-1} \text{ for all } i\in\bar{n},$$

hence

$$k(\wp(t))=\binom{n-1}{p-1}.$$

(iii) Let $t\in]-a,-n[\cup]n,+\infty[$. Then

$$k_i(\wp(t))=0 \text{ for all } i\in\bar{n},$$

hence

$$k(\wp(t))=0.$$

Proof. For any vertex $C_\omega=(c_i^\omega)_{i\in\bar n}\in C^n$ we have

(15) $$\sum_{i\in\bar n} c_i^\omega=\sum_{i\in\omega} c_i^\omega+\sum_{i\in\bar n-\omega} c_i^\omega=(-1)\,\mathrm{card}\,\omega+1\,(n-\mathrm{card}\,\omega)=n-2\,\mathrm{card}\,\omega$$

according to (1). $\mathrm{card}\,\omega\in\bar n\cup\{0\}$, hence $k_i(\wp(t))$ are not defined iff $t=n-2p$ and $p\in\bar n\cup\{0\}$. This is (i).

(15) implies:

(a) If $t\in]-\infty,-n[$, then $\sum_{i\in\bar n} c_i^\omega>t$ for any $C_\omega\in C^n$.

Thus $C_\omega\in\wp(t)^+$ and $C^n\subset\wp(t)^i$.

(b) If $t\in]n,+\infty[$, then $\sum_{i\in\bar n} c_i^\omega<t$ for any $C_\omega\in C^n$.

Thus $C_\omega\in\wp(t)^-$ and $C^n\subset\wp(t)^-$.

(iii) follows from (a) and (b) (using the convexity of C^n).

Let

$$t\in]n-2p,\ n-2p+2[\ ,\ p\in\bar n.$$

According to (15)

(16) $$C_\omega\in\wp(t)^+,\ \text{if card}\,\omega\le p-1,$$

(17) $$C_\omega\in\wp(t)^-,\ \text{if card}\,\omega\ge p$$

and $k_i(\wp(t))$ is defined. Two points C_ω, $C_\xi\in C^n$ with $\mathrm{card}\,\omega\le\mathrm{card}\,\xi$ are the end-points of an i-edge of C^n iff $i\notin\omega$ and $\xi=\omega\cup\{i\}$. This i-edge is intersected by $\wp(t)$, iff $C_\omega\in\wp(t)^+$ and $C_\xi\in\wp(t)^-$. Combining the last facts with (16) and (17), we see that

$$k_i(\wp(t))=\mathrm{card}\{(\omega,\xi)\,|\,\omega\subset\bar n,\ \xi\subset\bar n,\ i\notin\omega,\ \xi=\omega\cup\{i\},\ \mathrm{card}\,\omega\le p-1,$$
$$\mathrm{card}\,\xi\ge p\}=\mathrm{card}\{\omega\,|\,\omega\subset\bar n,\ \mathrm{card}\,\omega=p-1,\ i\notin\omega\}=\binom{n-1}{p-1},$$

which is (ii).

For the convenience let us formulate a simple consequence of Lemma 2.

Lemma 3. Let $\wp(t)\subset R^n$ be the hyperplane (14). Then $k_i(\wp(t))=k(\wp(t))$ for any $i\in\bar n$. If $|t_1|\le|t_2|$ and $k(\wp(t_1))$, $k(\wp(t_2))$ are defined, then

(18) $$k(\wp(t_1))\ge k(\wp(t_2)).$$

The maximal value of $k(\wp(t))$, $t\in R$ is

$$\binom{n-1}{\left[\frac{n-1}{2}\right]}$$

which is attained in the interval $]-1,1[$, if n is odd, and in the set $]-2,0[\cup]0,2[$, if n is even.

Proof. The combinatorial identity

$$\binom{r}{s}=\binom{r}{r-s}$$

and Lemma 2 imply

(19) $$k(\wp(t))=k(\wp(-t)).$$

(19) can be alternatively proved using the central symmetry of C^n w.r.t. 0.

Another well-known fact is the inequality

$$\binom{r}{s}\geq\binom{r}{s-1}, \text{ whenever } s\in\overline{[r/2]}.$$

From this inequality and Lemma 2 (ii) follows (18) for $0\leq t_1\leq t_2$. Recalling (19) we have (18) in general. The last assertion of Lemma 3 follows also from Lemma 2 using the fact that

$$\max\left\{\binom{r}{s}\,\middle|\, s\in\bar{r}\cup\{0\}\right\}=\binom{r}{[r/2]}.$$

Lemma 4. Let $\wp\subset R^n$ be the hyperplane (3). Let $k(\wp)$ be defined. Let

$$\alpha=\min\{|a_i|\,|\,i\in\bar{n}\}.$$

Let $\tilde{\wp}\subset R^n$ be the hyperplane

(20) $$\sum_{i\in\bar{n}-\{j\}}a_ix_i+bx_j=a_j.$$

Then $k(\tilde{\wp})$ is defined and

(i) if $|b|>\alpha$ and $|a_j|=\alpha$, then $k(\tilde{\wp})\geq k(\wp)$;

(ii) if $|b|>\alpha$ and $|a_j|>\alpha$, then $k(\tilde{\wp})=k(\wp)$;

(iii) if $|b|=\alpha$, then $k(\tilde{\wp})=k(\wp)$;

(iv) if $|b|<\alpha$, then $k(\tilde{\wp})\leq k(\wp)$.

Proof. We shall prove only (i), the proof of the other assertions of Lemma 4 is very similar. C^n can be identified with the (n-dimensional) face

$$C^n_-=\{x\in C^{n+1}\,|\,x_{n+1}=-1\}$$

of the cube C^{n+1}. Then $\wp$ and $\tilde{\wp}$ will be identified with the ((n-1)-dimensional) hyperplanes

(21) $$x_{n+1}=-1,\ \sum_{i\in\bar{n}}a_ix_i=b$$

and

(22) $$x_{n+1}=-1,\ \sum_{i\in\bar{n}-\{j\}}a_ix_i+bx_j=a_j,$$

respectively. (Cf. (3) and (20)). $\wp$ and $\tilde{\wp}$ are contained in the hyperplane $\sigma=\{x\in R^{n+1}\,|\,x_{n+1}=-1\}\subset R^{n+1}$. Let $\wp'$ and $\tilde{\wp}'$ be the hyperplanes in R^{n+1} which are spanned by 0 and $\wp$ and by 0 and $\tilde{\wp}$, respectively. Their equations will be

(23) $$\sum_{i\in\bar{n}}a_ix_i+bx_{n+1}=0$$

and

$$(24)\qquad \sum_{i\in\overline{m}-\{j\}} a_i x_i + b x_j + a_j x_{n+1} = 0,$$

respectively. (Cf. (21) and (22).)

$0\in\varphi'$, hence φ' is invariant w.r.t. the central symmetry of C^{n+1}. Thus φ' does not contain any vertex of C^{n+1}. Else φ would contain a vertex of C^n_- and $k(\varphi)$ would not be defined. For $i\in\overline{n}$ any i-edge of C^{n+1} is contained either in C^n_- or in $C^n_+=\{x\in C^{n+1}\,|\,x_{n+1}=1\}$. φ' intersects just all the i-edges in C^n_- which are intersected by φ , and all the i-edges in C^n_+ which can be obtained from them by means of the above mentioned symmetry of C^{n+1}. Hence

$$(25)\qquad k_i(\varphi')=2k_i(\varphi) \text{ for all } i\in\overline{n}.$$

Using the same argument, we can prove that

$$(26)\qquad k_i(\tilde{\varphi}')=2k_i(\tilde{\varphi}) \text{ for all } i\in\overline{n},$$

whenever one side of this formula makes sense.

Now we shall use Lemma 1. From the assumptions $|b|>\alpha$, $|a_j|=\alpha$ together with (3), (2) and (23) it follows that

$$(27)\qquad k(\varphi)=k_j(\varphi),$$

$$(28)\qquad \min\{k_i(\varphi')\,|\,i\in\overline{n}\}=k_j(\varphi')\leq k_{n+1}(\varphi').$$

Let us interchange the variables x_j and x_{n+1}. C^{n+1} is invariant w.r.t. this change of coordinates, the equation (23) is transformed onto the equation (24) and vice-versa, hence φ' is mapped onto $\tilde{\varphi}'$ and vice-versa. k_j and k_{n+1} will be also interchanged, the other k_i's remain unchanged. Thus

$$(29)\qquad \begin{aligned} &k_i(\varphi')=k_i(\tilde{\varphi}') \text{ for all } i\in\overline{n}-\{j\},\\ &k_j(\varphi')=k_{n+1}(\tilde{\varphi}'),\\ &k_{n+1}(\varphi')=k_j(\tilde{\varphi}'). \end{aligned}$$

(As well we see that $\tilde{\varphi}'$ and thus also $\tilde{\varphi}$ cannot contain any vertex of C^{n+1}, else φ' would contain some vertex of C^{n+1}.)

From (29) and (28) it follows that

$$\min\{k_i(\tilde{\varphi}')\,|\,i\in\overline{n}\}=\min\{k_i(\varphi')\,|\,i\in\overline{n+1}-\{j\}\}\geq k_j(\varphi')=\min\{k_i(\varphi')\,|\,i\in\overline{n}\},$$

hence according to (2), (25) and (26)

$$\begin{aligned} 2k(\varphi)&=2\min\{k_i(\varphi)\,|\,i\in\overline{n}\}=\min\{k_i(\varphi')\,|\,i\in\overline{n}\}\leq\min\{k_i(\tilde{\varphi}')\,|\,i\in\overline{n}\}=\\ &=2\min\{k_i(\tilde{\varphi})\,|\,i\in\overline{n}\}=2k(\tilde{\varphi}) \end{aligned}$$

and we have proved (i).

Section 2. The theorems

Theorem 1. Let $p \in \overline{n-1}$. Let $\wp_0 \subset R^n$ be the hyperplane

(30) $$a_p \sum_{i\in\bar{p}} x_i + \sum_{i\in\bar{n}-\bar{p}} a_i x_i = b,$$

where

(31) $$a_{p+1} > a_p > 0,$$

(32) $$a_i \geq a_{p+1} \text{ for all } i \in \bar{n}-\bar{p}.$$

Let $k(\wp_0)$ be defined. Let $N \in [-1,1]$, if p is odd, and $N \in [-2,2]$, if p is even. Let $j \in \bar{n}-\bar{p}$. Let $T>0$ be the maximal value, for which

(33) $$a_p + T \leq a_i \text{ for all } i \in \bar{n}-\bar{p}-\{j\},$$

(34) $$a_p + T \leq a_j + NT.$$

Let $\wp(t)$, $t \in [0,T]$ be the hyperplane

(35) $$(a_p+t) \sum_{i\in\bar{p}} x_i + \sum_{i\in\bar{n}-\bar{p}-\{j\}} a_i x_i + (a_j+NT)x_j = b.$$

Let

$$M = \{ t \in [0,T] \mid k(\wp(t)) \text{ is not defined} \}.$$

Then

(i) $0 \notin M$,

(ii) M is finite,

(iii) $k(\wp(t))$ is a nondecreasing function on $[0,T]-M$.

Proof. $\wp(0) = \wp_0$ and we assume that $k(\wp_0)$ is defined. Hence $0 \notin M$.

$t_0 \in M$ iff $\wp(t_0)$ contains some $C_\omega \in C^n$. That means

(36) $$a_p \sum_{i\in\bar{p}} c_i^\omega + \sum_{i\in\bar{n}-\bar{p}-\{j\}} a_i c_2^\omega + a_j c_j^\omega - b = -t_0\Big(\sum_{i\in\bar{p}} c_i^\omega + N c_j^\omega\Big)$$

(cf. 35)). Because of our assumption $C_\omega \notin \wp_0$ and (30) implies that the left-hand side of (36) is not equal to 0. But then (36) has at most one solution t_0. Thus for every C_ω there exists at most one value $t_0 \in [0,T]$, for which $C_\omega \in \wp(t_0)$ and $k(\wp(t_0))$ is not defined. Hence M is finite.

Let $t \in [0,T]-M$ be fixed. For any $\tau \in [0,T]$ sufficiently close to t the hyperplanes $\wp(t)$ and $\wp(\tau)$ intersect exactly the same edges of C^n, thus $k(\wp(\tau)) = k(\wp(t))$. Hence $k(\wp(t))$ can change its value only in the points of M.

Let t_0 be such a point and let $\wp(t_0)$ contain $C_\omega \in C^n$. The equation of $\wp(t)$ can be written in the form

$$\sum_{i\in\bar{p}} x_i = \Big(b - \sum_{i\in\bar{n}-\bar{p}-\{j\}} a_i x_i - a_j x_j - tNx_j\Big)/(a_p+t)$$

and for C_ω we obtain the equation

(37) $$\sum_{i\in\bar{p}} c_i^{\omega}=(b-\sum_{i\in\bar{n}-\bar{p}-\{j\}} a_i c_i^{\omega}-a_j c_j^{\omega}-t_0 N c_j^{\omega})/(a_p+t_0).$$

Let $\xi=\omega\cap(\bar{n}-\bar{p}-\{j\})$ and $\wp_\xi$ be the (p+1)-dimensional hyperplane

(38) $$x_i=c_i^{\omega},\ i\in\bar{n}-\bar{p}-\{j\}.$$

Then $\wp_\xi\cap C^n=C_\xi^{p+1}$ is a (p+1)-dimensional face of C^n and $C_\omega\in C_\xi^{p+1}$. $\wp_\xi\cap\wp(t)=$ $=\wp_\xi(t)$ is a p-dimensional hyperplane in $\wp_\xi$. $C_\omega\in\wp_\xi(t_0)$ and $\wp_\xi(t)$ does not contain any C_ω for $t\neq t_0$ sufficiently close to t_0, because M is finite. For such t we can define $k_1(\wp_\xi(t))$ as the number of 1-edges contained in C_ξ^{p+1} and intersected by $\wp_\xi(t)$. Of course, $k_1(\wp_\xi(t_0))$ is not defined. We want to find conditions which ensure that $k_1(\wp_\xi(t))$ is nondecreasing, when t passes through t_0.

Let $c_j^{\omega}=1$. From (37) we obtain

(39) $$\sum_{i\in\bar{p}} c_i^{\omega}=(\beta_\xi-a_j-t_0 N)/(a_p+t_0),$$

where

(40) $$\beta_\xi=b-\sum_{i\in\bar{n}-\bar{p}-\{j\}} a_i c_i^{\omega}$$

is a value which is constant on C_ξ^{p+1} according to (38). Clearly,

(41) $$\sum_{i\in\bar{p}} c_i^{\omega}=p-2q,$$

where q is the number of negative coordinates in the ordered p-tuple $(c_i^{\omega})_{i\in\bar{p}}$, because $|c_i^{\omega}|=1$ for every $i\in\bar{n}$ and $\omega\subset\bar{n}$. Hence for some $q\in\bar{p}\cup\{0\}$

(42) $$(p-2q)(a_p+t_0)=\beta_\xi-a_j-t_0 N$$

According to (39) and (41), $\wp_\xi(t_0)$ is given by (38) and

$$\sum_{i\in\bar{p}} x_i=(\beta_\xi-a_j-t_0 N)/(a_p+t_0)$$

and w.r.t. Lemma 3 $k_1(\wp(t))$ is increasing in t_0, if the function $|\varphi(t)|$ is decreasing in t_0, where

$$\varphi(t)=(\beta_\xi-a_j-tN)/(a_p+t).$$

Thus we only need to find conditions which ensure that

(43) $$\varphi(t_0)\,\varphi'(t_0)<0.$$

Using (42) we obtain

$$\begin{aligned}\varphi(t_0)\,\varphi'(t_0)&=(\beta_\xi-a_j-t_0N)(-Na_p-\beta_\xi+a_j)/(a_p+t_0)^3=\\ &=(p-2q)(a_p+t_0)(-N(a_p+t_0)-(p-2q)(a_p+t_0))/(a_p+t_0)^3=\\ &=-(p-2q)(p-2q+N)/(a_p+t_0).\end{aligned}$$

Because $a_p+t_o>0$ according to our assumptions, the condition (43) is fulfilled, if

(44) $$(p-2q)(p-2q+N)>0.$$

If $c_j^{\omega}=-1$, we can proceed similarly as in the previous case and we obtain instead of (44) the condition

(45) $$(p-2q)(p-2q-N)>0.$$

Let p be odd. Then p-2q is an odd integer, i.e. $|p-2q|\geq 1$. Hence p-2q has the same sign as p-2q+N and p-2q-N (and (44) and (45) are fulfilled), if

(46) $$|N|<1.$$

If N=1, then (44) is not fulfilled only if p-2q= -1. But then (42) implies that

$$(p-2q)a_p-t_o=\beta_\xi-a_j-t_o,$$
$$(p-2q)a_p+a_j=\beta_\xi,$$

hence according to (40)

(47) $$(p-2q)a_p+a_j+\sum_{i\in\bar{n}-\bar{p}-\{j\}}a_i c_i^{\omega}=b.$$

But $c_j^{\omega}=1$, $a_j=a_j c_j^{\omega}$ and (47) together with (41) implies that

$$a_p\sum_{i\in\bar{p}}c_i^{\omega}+a_j c_j^{\omega}+\sum_{i\in\bar{n}-\bar{p}-\{j\}}a_i c_i^{\omega}=b,$$

i.e., $C_\omega\in\wp_o$, which is a contradiction.

Similarly we can show that for N= -1 (44) is always fulfilled and that for $|N|=1$ (45) also holds. Hence (44), (45) hold for every $N\in[-1,1]$.

Let p be even. Then p-2q is an even integer. If p-2q=0, then according to Lemma 3 (applied to C_ξ^{p+1}) the value $k_1(\wp_\xi(t))$ remains unchanged, if t passes through t_o. Hence the behaviour of $|\wp(t)|$ in the neighbourhood of t_o is not important in this case. If $p-2q\neq 0$, then $|p-2q|\geq 2$ and we can repeat the above written argument (for p odd) with the value 2 instead of 1. (Of course, instead of (46) we obtain the condition $|N|<2$ etc.)

Now we only need to notice that

(48) $$k_1(\wp(t))=\sum_{\xi\in\bar{n}-\bar{p}-\{j\}}k_1(\wp(t)),$$

because every 1-edge belongs to just one C_ξ^{p+1}, $\xi\in\bar{n}-\bar{p}-\{j\}$, and that a sum of nondecreasing functions is a nondecreasing function. Hence $k_1(\wp(t))$ is nondecreasing on $[0,T]-M$.

The assumptions (31), (32), (33) and (34) imply that for every $t\in[0,T]$ and all $i\in\bar{n}-\bar{p}-\{j\}$

$$a_p+t\leq a_j+Nt \text{ and } a_p+t\leq a_i.$$

Thus (2) and Lemma 1 imply that

$$k(\wp(t)=k_1(\wp(t)) \tag{49}$$

for all $t\in[0,T]-M$. (iii) follows from (48) and (49).

Remark 1. Let us drop the assumption

$$k(\wp_0) \text{ is defined} \tag{50}$$

of Theorem 1. Then (i) and (ii) is not necessarily true, because then the equation (36) can have infinitely many solutions. But this can happen only if

$$a_p \sum_{i\in\bar{p}} c_i^{\omega} + \sum_{i\in\bar{m}-\bar{p}} a_i c_i^{\omega} + a_j c_j^{\omega} - b = 0$$

and

$$\sum_{i\in\bar{p}} c_i^{\omega} + N c_j^{\omega} = 0.$$

Then every $t_0\in[0,T]$ solves (36), $C_\omega \in \wp(t_0)$ for every $t_0\in[0,T]$ and $M = [0,T]$. So if we assume only that

$$k(\wp(t)) \text{ is defined for some } t\in[0,T]$$

instead of (50), we cannot prove (i), but (ii) and (iii) remain to be true.

Theorem 2. Let $p\in\overline{n-1}$. Let $\wp_0\subset R^n$ be the hyperplane (30), let (31) and (32) hold and let $k(\wp_0)$ be defined. Let $N\in[-1,1]$, if p is odd, and $N\in[-2,2]$ if p is even. Let $j\in\bar{n}-\bar{p}$. Let $T>0$ be the maximal value, for which (33) and (34) hold. Let $\varepsilon>0$.

Then there exists a number $\tilde{b}$, a continuous path $\wp(t)$, $t\in[0,1]$ in the space of $(n-1)$-dimensional hyperplanes of R^n and a set $M\subset[0,1]$ such that

(i) $|b-\tilde{b}|<\varepsilon$,

(ii) $\wp(0)=\wp_0$,

(iii) $\wp(1)$ is the hyperplane

$$(a_p+T)\sum_{i\in\bar{p}} x_i + \sum_{i\in\bar{m}-\bar{p}-\{j\}} a_i x_i + (a_j+NT)x_j = \tilde{b}, \tag{51}$$

(iv) $0\notin M$,

(v) $1\notin M$,

(vi) M is finite,

(vii) $k(\wp(t))$ is not defined iff $t\in M$,

(viii) $k(\wp(t))$ is a nondecreasing function on $[0,1]-M$.

Proof. Let us define the continuous path $\wp(t)$ of the hyperplanes by means of the formulae

$$a_p \sum_{i\in\bar{p}} x_i + \sum_{i\in\bar{m}-\bar{p}} a_i x_i = b + 2t(\tilde{b}-b) \text{ for } t\in[0,1/2], \tag{52}$$

$$(a_p+(2t-1)T)\sum_{i\in\bar{p}} x_i + \sum_{i\in\bar{m}-\bar{p}-\{j\}} a_i x_i + (a_j+N(2t-1)T)x_j = \tilde{b}$$

for $t \in [1/2,1]$.

Then (ii) and (iii) hold. Let

$$M=\{t \in [0,1] \mid k(\wp(t)) \text{ is not defined}\}.$$

Then (vii) holds and because we assume that $k(\wp_0)$ is defined, (ii) implies (iv).

For $\omega \subset \bar{n}$ let us define

$$b_\omega^0 = a_p \sum_{i \in \bar{p}} c_i^\omega + \sum_{i \in \bar{n}-\bar{p}} a_i c_i^\omega .$$

Then for $t \in [0,1/2]$ (52) implies that

(54) $$C_\omega \in \wp(t) \Longleftrightarrow b+2t(\tilde{b}-b)=b_\omega^0 .$$

Because (iv) holds, $b_\omega^0 \neq b$ for every $\omega \subset \bar{n}$, hence

$$\min\{|b_\omega^0 - b| \mid \omega \subset \bar{n}\} = \delta > 0.$$

Let

(54) $$|\tilde{b}-b| < \delta.$$

Then $b+2t(\tilde{b}-b) \neq b_\omega^0$ for every $\omega \subset \bar{n}$ and every $t \in [0,1/2]$. That means according to (53) that $C_\omega \notin \wp(t)$ for any $t \in [0,1/2]$, hence $k(\wp(t))$ is defined for every $t \in [0,1/2]$ and $k(\wp(t))$ is constant (thus nondecreasing) on $[0,1/2]$. Especially

(55) $$k(\wp(1/2)) \text{ is defined and } M \subset (1/2,1].$$

Now we can use Theorem 1 with $\wp(1/2)$ instead of $\wp_0$ and $(2t-1)T(t \in [1/2,1])$ instead of $t(t \in [0,T])$. Taking into account also (55), we obtain (vi) and (viii).

Let

$$b_\omega^1 = (a_p+T) \sum_{i \in \bar{p}} c_i^\omega + \sum_{i \in \bar{n}-\bar{p}-\{j\}} a_i c_i^\omega + (a_j+NT)c_j^\omega .$$

W.r.t. (51) $k(\wp(1))$ is defined and $1 \notin M$, if

(56) $$\tilde{b} \neq b_\omega^1 \quad \text{for all } \omega \subset \bar{n}.$$

Hence (v) follows from (56).

Thus in order to prove Theorem 2 we only have to choose $\tilde{b}$ so that (54), (56) and (i) hold. But this is always possible.

Remark 2. In the generic case $b \neq b_\omega^1$ for all $\omega \subset \bar{n}$, hence we can choose $\tilde{b}=b$.

Remark 3. The choice of $\tilde{b}$ can be subjected to some other requirements, e.g., we can require that

$$|\tilde{b}| > |b|.$$

Theorem 3. Let $\tau \subset R^n$ be the hyperplane (3). Let $k(\tau)$ be defined. Let $\sigma \subset R^n$ be the hyperplane

$$\sum_{i \in \bar{n}} x_i = 1/2.$$

Let us assume that

(57) $\quad a_i \geq 0$ for all $i \in \bar{n}$,

(58) $\quad a_i \leq a_j$, whenever $i \leq j$, $i \in \bar{n}$, $j \in \bar{n}$.

Then there exists a continuous path $\varphi(t)$, $t \in [0,T]$ in the space of (n-1)-dimensional hyperplanes of R^n and a set $M \subset [0,T]$ such that

(i) $\varphi(0) = \tau$,

(ii) $\varphi(T) = \sigma$,

(iii) $k(\varphi(t))$ is defined iff $t \in [0,T]-M$,

(iv) $0 \notin M$, $T \notin M$,

(v) M is finite,

(vi) $k(\varphi(t))$ is a nondecreasing function on $[0,T]-M$.

Proof. We can apply Theorem 2 with τ instead of φ_0, N=0 and p=1. Denoting the value $\tilde{b}$ as b_1 and M as M_1, Theorem 2 ensures the existence of $\varphi(t)$, $t \in [0,1]$ s.t. (i) holds, $k(\varphi(t))$ is defined iff $t \in [0,1]-M_1$, M_1 is finite, $k(\varphi(t))$ is nondecreasing on $[0,1]-M$ and $\varphi(1)$ is the hyperplane

$$a_2(x_1+x_2)+\sum_{i \in \overline{n-2}} a_i x_i = b_1.$$

Because $1 \notin M$, $k(\varphi(1))$ is defined.

Now we can apply Theorem 2 with $\varphi(1)$ instead of φ_0, $t-1$ ($t \in [1,2]$) instead of t ($t \in [0,1]$), b_1 instead of b, M_2 instead of M, b_2 instead of $\tilde{b}$, N=0 and p=2. We obtain $\varphi(t)$ for $t \in [1,2]$, $\varphi(2)$ will be the hyperplane

$$a_3(x_1+x_2+x_3)+\sum_{i \in \overline{n-3}} a_i x_i = b_2$$

and $k(\varphi(2))$ will be defined, because we obtain $2 \notin M_2$.

This way, by means of the repeated use of Theorem 2 (for all $p \in \overline{n-1}$ in general) we can construct $\varphi(t)$, $t \in [0,n-1]$ and the set $\tilde{M} = \bigcup_{i \in \overline{n-1}} M_i \subset [0,n-1]$ so that all the assertions of Theorem 3 hold with $n-1$ instead of T, $\tilde{M}$ instead of M and the hyperplane $\varphi(n-1)$ with the equation

(59) $$a_n \sum_{i \in \bar{n}} x_i = b_{n-1}$$

instead of σ.

τ is a hyperplane, hence at least one of the coefficients a_n, $i \in \bar{n}$ is positive, (57), (58) then implies that $a_n > 0$ and (59) can be rewritten in the form

$$\sum_{i\in\bar{m}} x_i = b_{n-1}/a_n.$$

Let us define $\varphi(t)$ for $t\in[n-1,n]$ by means of the equation

$$\sum_{i\in\bar{m}} x_i = (n-t)b_{n-1}/a_n + (t-n+1)/2.$$

Lemma 3 then implies that $k(\varphi(t))$ is nondecreasing on $[n-1,n]-M_n$, M_n is finite etc. So we obtain Theorem 3 with $T=n$, $M=\widetilde{M}\cup M_n$.

Theorem 4. Let $\tau\subset R^n$ be a hyperplane, let $k(\tau)$ be defined. Then

$$k(\tau)\le\binom{n-1}{[(n-1)/2]}.$$

Proof. Let (3) be the equation of τ. We can assume (57). If it were not the case, we could use the reflections of R^n w.r.t. some coordinate hyperplanes in order to obtain the equation

$$\sum_{i\in\bar{m}} |a_i|\,\xi_i = b$$

in the new coordinates ξ. W. r.t. the symmetries of C^n such a transformation does not change the numbers $k_i(\tau)$.

We can also assume (58). In the opposite case a suitable permutation of the coordinates transforms (3) (satisfying (57)) so that (58) is fulfilled w.r.t. the new coordinates. On the other hand, a permutation of the coordinates can change only the order of the k_i's, the value $k(\tau)$ remains unchanged.

Now we can apply Theorem 3 and we see that

$$k(\tau)\le k(\sigma).$$

According to Lemma 3

$$k(\sigma)=\binom{n-1}{[(n-1)/2]},$$

which completes the proof.

Theorem 5. Let $\tau\subset R^n$ be the hyperplane (3), let us assume (57) and (58). Let $m\in\overline{n-1}$ be odd and

(60) $$\beta_m=\Big(\sum_{i\in\bar{m}} a_i - 2\sum_{i\in\overline{(m-1)/2}} a_{2i+1}\Big)/(n-m+1).$$

Let

$$a_m\le\beta_m\le a_{m+2}.$$

(We define $a_{n+1}=a_n$, if n is even.)

(i) If $|b/\beta_m|\in[n-2p,n-2p+2[$ for some $p\in\bar{n}$, then $k(\tau)\le\binom{n-1}{p-1}$.

(ii) If $|b/\beta_m|\ge n$, then $k(\tau)=0$.

Proof. The proof of Theorem 3 was based on Theorem 2 with N=0. But we can apply Theorem 2 or Theorem 1 with N= -1 for p odd, and N= -2 for p even.

Let $m \in \overline{n-1}$ be odd. Let $\beta \in [a_m, a_{m+1}]$ be such that

(61) $$(a_2-a_1)+2(a_3-a_2)+(a_4-a_3)+2(a_5-a_4)+\ldots+2(a_m-a_{m-1})+(\beta-a_m)=\sum_{i\in\overline{n}-\overline{m}}(a_i-\beta),$$

i.e.,

(62) $$\beta=\beta_m.$$

Let us apply Theorem 1 with p=1, N= -1, j=n and $\wp_0=\tau$. Then $\wp(t)$ are the hyperplanes

(63) $$(a_1+t)x_1+\sum_{i\in\overline{n-1}}a_ix_i+(a_n-t)x_n=b.$$

Now either a_1+t attains the value a_2 before a_n-t attains the value β or not. If not, we stop the path (63) in the value $\tilde{t}$, for which $a_n-\tilde{t}=\beta$ and then we apply Theorem 1 with p=1, N= -1 and j=n-1. (With $t-\tilde{t}$ instead of t.) a_1+t grows further and a_1+t either attains the value a_2 before $a_{n-1}-(t-\tilde{t})$ attains the value β or not. If not, we stop in the value $\tilde{\tilde{t}}$, for which $a_{n-1}-(\tilde{\tilde{t}}-\tilde{t})=\beta$ and then we apply Theorem 1 with p=1, N= -1 and j=n-2 etc.

During these continuous changes of coefficients in (3) the first coefficient grows by (a_2-a_1). N= -1, hence the coefficients a_i, $i\in\overline{n}-\overline{m}$ decrease altogether by the same value (a_2-a_1). In fact, (58) and (61) imply that

$$\sum_{i\in\overline{n}-\overline{m}}(a_i-\beta)\geq(a_2-a_1),$$

thus after some steps we obtain for some T_1, some $r_1\in\overline{n-1}$ and some α_1 as $\wp(T_1)$ the hyperplane

(64) $$a_2(x_1+x_2)+\sum_{i\in\overline{r_1}-\overline{2}}a_ix_i+\alpha_1x_{r_1+1}+\beta\sum_{i\in\overline{n}-\overline{r_1+1}}x_i=b_m$$

and $\wp(t)$ is defined for $t\in[0,T_1]$. (Cf. (63).)

Now we can apply Theorem 1 with p=2, N= -2, $j=r_1+1$, $\wp_0=\wp(T_1)$ and $t-T_1$ instead of t. We proceed in the construction of $\wp(t)$ as above, i.e., we begin with $j=r_1+1$, if α decreases to β before a_2+t-T_1 attains the value a_3, we continue with $j=r_1$ etc. This way a_2 grows by a_3-a_2. Because N= -2, the sum

(65) $$\sum_{i\in\overline{r_1}-\overline{m}}(a_i-\beta)+(\alpha_1-\beta),$$

simultaneously decreases by $2(a_3-a_2)$. In fact, (65) is equal to

$$\sum_{i\in\overline{n}-\overline{m}}(a_i-\beta)-(a_2-a_1)$$

and (58) and (61) imply that

$$\sum(a_i-\beta)-(a_2-a_1)\geq 2(a_3-a_2).$$

Thus after some steps we obtain for some T_2, some $r_2 \in \overline{r}_1$ and some α_2 as $\varphi(T_2)$ the hyperplane

(66) $$a_3(x_1+x_2+x_3)+\sum_{i \in \overline{r_2}-\overline{3}} a_i x_i+\alpha_2 x_{r_2+1}+\beta \sum_{i \in \overline{m}-\overline{r_2+1}} x_i=b$$

and $\varphi(t)$ is defined for $t \in [0,T_2]$.

Now, we can apply Theorem 1 with $p=3$, $N=-1$, $j=r_2+1$ etc. to the hyperplane (66).

We can easily see that after some such steps we obtain for some T as $\varphi(T)$ the hyperplane

(67) $$\beta \sum_{i \in \overline{m}} x_i=b.$$

In fact at first the coefficient a_1 grows by a_2-a_1. Then a_2 in (64) grows by a_3-a_2 etc., in the end a_m grows by $\beta-a_m$. This implies that simultaneously the sum

(68) $$\sum_{i \in \overline{n}-\overline{m}} (a_i-\beta)$$

decreases at first by (a_2-a_1), then by $2(a_3-a_2)$ etc., in the end by $(\beta-a_m)$. But the sum of all these values is just (68) according to (61). Hence, if all the coefficients a_i, $i \in \overline{m}$ attain the value β, the other coefficients a_i, $i \in \overline{n}-\overline{m}$ must attain the same value.

Let us remark that in some instants of this construction it can be necessary to change at first by a small value the value of b. Hence in general we obtain as $\varphi(T)$ instead of (67) the hyperplane σ, defined by means of the equation

(69) $$\beta \sum_{i \in \overline{m}} x_i=\tilde{b}$$

with some suitable $\tilde{b}$ arbitrarily close to b. (Cf. Theorem 2.)

This way we have constructed a continuous path $\varphi(t)$, $t \in [0,T]$, which begins in τ and ends in σ. This path consists of finitely many straight line segments. According to Theorem 1 (or Theorem 2) the function $k(\varphi(t))$ is defined for all points of these segments except of some finite set and it is a nondecreasing function on any of these segments. In the end-points of these segments $k(\varphi(t))$ is defined and continuous. Hence, $k(\varphi(t))$ is defined and nondecreasing on $[0,T]-M$, where M is a finite set, which contains neither 0, nor T. Thus

(70) $$k(\tau) \leq k(\sigma).$$

Inserting (62) into the equation (69) of σ, we see immediately that we have proved the following assertion.

A: Let $m \in \overline{n-1}$ be odd. Let

$$a_m \leq \beta_m \leq a_{m+1}.$$

Let σ be the hyperplane (69). Then (70) holds.

Similarly we can prove:

B: Let $m \in \overline{n-1}$ be even. Let

$$(71) \qquad \beta_m = \left(\sum_{i \in \overline{m}} a_i - 2 \sum_{i \in \overline{(m-2)/2}} a_{2i+1} \right) / (n-m+2),$$

let

$$a_m \leq \beta_m \leq a_{m+1}.$$

Let σ be the hyperplane (69). Then (70) holds.

If m is odd, then m+1 is even and we can insert the value m+1 instead of m into the assertion B. Comparing then (60) and (71), we see that β_{m+1} calculated according to (71) coincides with β_m of (60). Hence, the assertions A and B can be unified as follows:

Let $m \in \overline{n-1}$ be odd. Let us define β_m as in (68). Let $a_m \leq \beta_m \leq a_{m+2}$.
Let σ be the hyperplane

$$\sum x_i = \tilde{b} / \beta_m$$

(with $\tilde{b}$ arbitrarily close to b and such that $|\tilde{b}| > |b|$ - cf. Remark 3). Then (70) holds.

Now we can apply Lemma 2 in order to calculate $k(\sigma)$ and we obtain the assertion of Theorem 5 (we also use the inequalities $|b| < |\tilde{b}| < |b| + \varepsilon$).

Remark 4. Let τ be the hyperplane (3), satisfying (57) and (58). Then the assumptions of Theorem 5 are always satisfied for some odd integer $m \in \overline{n-1}$.

Remark 5. To any hyperplane τ with the equation (3) there always exists a symmetry of C^n which transforms (3) so, that the transformed equation satisfies (57) and (58). (See the proof of Theorem 4.)

Section 3. The examples. The continuous path $\varphi(t)$ which was constructed in the proof of Theorem 3, is rather complicated and one can seek for some more simple path with analogous properties. The most simple path joining τ with σ of Theorem 3 is the path $\varphi(t)$, $t \in [0,1]$ defined by means of the equation

$$(72) \qquad \sum_{i \in \overline{m}} (a_i + t(1-a_i)) x_i = b + t(1/2 - b).$$

We can formulate

Conjecture 1. Let $\tau \subset R^n$ be the hyperplane (3). Let us assume (57), (58) and $b > 0$. Let $\varphi(t)$ be the hyperplane (72). Let $M = \{ t \in [0,1] \mid k(\varphi(t))$ is not defined$\}$. Then $k(\varphi(t))$ is a nondecreasing function on $[0,1] - M$.

This conjecture can be proved for n=1 and n=2, but for $n \geq 3$ it is false as the following example shows.

Example 1. Let $\tau \subset R^3$ be the plane

(73) $$x_1+2x_2+3x_3=5.$$

Let $\alpha \in]0,1[$ (e.g., $\alpha=1/2$). Then (72) implies that $\varphi(t)$ is the plane

(74) $$x_1+(2-t)x_2+(3-2t)x_3=5+t(\alpha-5).$$

Theorem 5 implies that $k(\tau) \leq 1$, because $1 \leq (1+2+3)/3 \leq 3$ and $1 \leq 5/2 \leq 3$. On the other hand, the vector $(0,1,1)$ solves the equation (73), hence $k(\tau)=1$.

Let $t=(1+\eta)/(2-\alpha)$, where $\eta > 0$ is sufficiently small. Then $t \in [0,1]$ and (74) can be rewritten in the form

(75) $$(2-\alpha)x_1+(3-2\alpha-\eta)x_2+(4-3\alpha-2\eta)x_3=5-4\alpha+(\alpha-5)\eta .$$

If $x_2=1$, $x_3=1$, then (75) implies $x_1= -1-\eta$.
If $x_2=1$, $x_3= -1$, then (75) implies $x_1=(6-5\alpha+(\alpha-6)\eta)/(2-\alpha)$.
If $x_2= -1$, $x_3=1$, then (75) implies $x_1=(4-3\alpha+(\alpha-4)\eta)/(2-\alpha)$.
If $x_2= -1$, $x_3= -1$, then (75) implies $x_1=(12-9\alpha+(\alpha-8)\eta)/(2-\alpha)$.

In the first case $\eta>0$ implies that $x_1<-1$. Because $\alpha \in (0,1)$, we have

$$12-9\alpha > 6-5\alpha > 4-3\alpha > 2-\alpha > 0,$$

hence in the other 3 cases $x_1>1$, if $\eta > 0$ is sufficiently small. Thus for $t=(1+\eta)/(2-\alpha)$ with such an η we have proved that

$$k(\varphi(t))=0$$

and $k(\varphi(t))$ is not a nondecreasing function.

In the proof of Theorem 3 we have constructed a continuous path $\varphi(t)$ which passes through the hyperplanes φ_p defined by the equations

(76) $$\max\{a_i \mid i \in \overline{p}\} \sum_{i \in \overline{p}} x_i + \sum_{i \in \overline{n}-\overline{p}} a_i x_i = b_p',$$

where all the b_p' s are close to b. The path $\varphi(t)$ is "almost linear" between φ_p and φ_{p+1}, for all $p \in \overline{n-1}$.

Let σ_p be the hyperplanes

$$p^{-1} \sum_{i \in \overline{p}} a_i \sum_{i \in \overline{p}} x_i + \sum_{i \in \overline{n}-\overline{p}} a_i x_i = \tilde{b}_p,$$

where $\tilde{b}_p'$ s are close to b. We can ask, whether it is possible to define a continuous path $\sigma(t)$ which passes through all the hyperplanes σ_p and such that $k(\sigma(t))$ is a nondecreasing function of t on its domain.

Using Theorem 1 with p=1, j=2 and N= -1 we can construct the first part of $\sigma(t)$ namely the part between σ_1 and σ_2. Using Theorem 1 with p=2, j=3 and N= -2, we can then construct the second part of $\sigma(t)$ between σ_2 and σ_3. For the construction of the third part between σ_3 and σ_4 we would need

Theorem 1 with $j=4$ and $N=-3$. But such a theorem is false and in fact the construction of the third part of $\sigma(t)$ is impossible, if we require the monotonicity of $k(\sigma(t))$. Namely, it can happen that $k(\sigma_4) < k(\sigma_3)$ as the following example shows, thus we cannot substitute the arithmetic mean for the maximum in (76).

Example 2. Let $\sigma_3 \subset R^4$ be the hyperplane

$$x_1+x_2+x_3+5x_4=5. \tag{77}$$

Then σ_4 is the hyperplane

$$2(x_1+x_2+x_3+x_4)=5. \tag{78}$$

The vectors $(0,1,-1,1)$ and $(0,-1,1,1)$ satisfy the equation (77), hence $k_1(\sigma_3) \geq 2$ and $k(\sigma_3) \geq 2$. (In fact $k(\sigma_3)=k_1(\sigma_3)=2$.) From (78) we obtain according to Theorem 5 or Lemma 2 that $k(\sigma_4) \leq 1$. (In fact $k(\sigma_4)=1$.) Thus we see that $k(\sigma_4) < k(\sigma_3)$.

The last example illustrates the use of Lemma 4 and Theorem 5.

Example 3. Let $\tau \subset R^4$ be the hyperplane

$$x_1+3x_2+4x_3+12x_4=2.$$

Theorem 4 implies only the estimate

$$k(\tau) \leq 3.$$

Using Lemma 4, we obtain

$$k(\tau) \leq k(\rho_1), \tag{79}$$

where ρ_1 is the hyperplane

$$x_1+2x_3+3x_3+4x_4=12. \tag{80}$$

Proceeding as in the proof of Theorem 3 we can show that

$$k(\rho_1) \leq k(\rho_2), \tag{81}$$

where ρ_2 is the hyperplane

$$4(x_1+x_2+x_3+x_4)=12.$$

According to Lemma 2

$$k(\rho_2)=1.$$

Hence (79) and (81) imply that

$$k(\tau) \leq 1.$$

But we can also apply Theorem 5 to (80). In this case

$$\sum_{i \in 4} a_i/4=2.5, \quad a_1=1, \quad a_3=3,$$

hence

$$a_1 \leq \sum_{i \in \bar{4}} a_i/4 \leq a_3.$$

Further $12/2.5 \geq 4$, hence $k(\rho_1)=0$ and (79) implies that $k(\tau)=0$.

References

[1] COTTLE R., DANTZIG G.: Complementary pivot theory of mathematical programming, Lin. Alg. and its Appl. 1(1968), 103-125.

[2] MURTY K.G.: On the number of solutions to the complementarity problem and spanning properties of complementary cones, Lin . Alg. and its Appl. 5(1972), 65-108.

[3] ROHN J.: Solving interval linear systems, Freiburger Intervall-Berichte 84/7, 1-14.

[4] AMBROSETTI A., PRODI G.: On the inversion of some differentiable mappings with singularities between Banach spaces, Annali Mat. Pura Appl. 93(1972), 231-246.

[5] PRODI G., AMBROSETTI A.: Analisi non lineare, I quaderno, Pisa, 1973.

[6] FUČÍK S., KUČERA M., NEČAS J.: Ranges of nonlinear asymptotically linear operators, J. Diff. Equ. 17(1975), 375-394.

[7] FUČÍK S.: Solvability of nonlinear equations and BVP, Reidel-Dordrecht & Society of Czechoslovak Math. and Phys.-Prague, 1980.

[8] RUF B.: Multiplicity results for nonlinear elliptic equations (preprint).

[9] FUČÍK S., MILOTA J.: Linear and nonlinear variational inequalities on half-spaces, Comment. Math. Univ. Carolinae 16(1975), 663-682.

[10] ŠVARC R.: On the solutions of equations with jumping nonlinearities, Z. angew. Math. Mech. 67(1987),5.

[11] JOHN O., QUITTNER P., STARÁ J.: Spectral properties and solvability of variational inequalities (to be published).

[12] ŠVARC R.: The operators with jumping nonlinearities and combinatorics (to be published).

[13] ŠVARC R.: Some combinatorial results about the operators with jumping nonlinearities (to be published).

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