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INFLATED MAPPINGS FOR SINGULARITIES  
OF CODIMENSION  $\leq 2$

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**Abstract:** Singularities of an imperfect bifurcation problem  $F(u, \lambda, \alpha) = 0$  of codimension  $\leq 2$  are related to simple roots of auxiliary operators (inflated mappings). All generic cases are discussed.

**Key words:** Imperfect bifurcation problems, organizing centre, classification, numerical approximation.

**Classification:** 47H15, 65J15, 58C27, 14B05

1. Introduction. Let  $U$  and  $Y$  be Banach spaces. We consider an operator  $F: U \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow Y$ . In a bifurcation context, the variable  $x$  of  $F = F(x)$  is a triple  $x = (u, \lambda, \alpha)$  where  $u$  is the state variable,  $\lambda$  is a control parameter and  $\alpha$  is the parameter of an imperfection.

A point  $x_0 = (u_0, \lambda_0, \alpha_0)$  is called the singular point of  $F$  if

$$(1.1) \quad F(x_0) = 0$$

$$(1.2) \quad \dim \text{Ker } F_u(x_0) = m \geq 1$$

where  $F_u$  is the partial Fréchet derivative of  $F$  (at  $x_0$ ) w.r.t. the variable  $u$ , and  $\text{Ker } F_u(x_0)$  is the kernel of  $F_u(x_0): U \rightarrow Y$ .

We assume  $F_u(x_0): U \rightarrow Y$  to be Fredholm operator with index zero and  $F \in C^\infty(X, Y)$  where  $X$  is a neighbourhood of  $x_0$ .

The singular point  $x_0$  satisfying (1.1), (1.2) is not isolated in general. There is an idea to seek for "the most singular" point  $x_0$  which is locally available. Such a point is called an organizing centre of the operator  $F$ . In fact, the knowledge of an organizing centre makes it possible to describe (at least qualitatively) the solution sets

$$S_\alpha = \{(u, \lambda): F(u, \lambda, \alpha) = 0, (u, \lambda) \text{ close to } (u_0, \lambda_0)\}$$

for all parameters  $\alpha$  close to  $\alpha_0$ .

The organizing centre could be linked with a simple root of an auxiliary operator  $\mathcal{F}$  which is defined by means of  $F$  and its partial derivatives.

It is used to call such an operator  $\mathcal{F}$  the inflated mapping of  $F$ .

The classification of organizing centres is developed in terms of singularities of the germs of smooth mappings  $g: \mathbb{R}_m \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow \mathbb{R}_m$ , see [1]. It is assumed that  $g$  is the operator of a bifurcation equation to the problem (1.1). In [2], we have suggested a very simple idea to transform the defining conditions for a particular singularity of  $g$  into defining conditions of the relevant organizing centre  $x_0$  (i.e. into a definition of  $\mathcal{F}$ ).

In this paper, we demonstrate the idea assuming that the particular singularity of the bifurcation equation has codimension less than or equal to two. This condition means a restriction on the complexity of the structure which is simulated by the equation  $F(u, \lambda, \alpha) = 0$ .

2. Reduction to a bifurcation equation. Let us recall basic ideas of Liapunov-Schmidt reduction of the equation (1.1).

First, we choose a linear bounded operator  $L: U \rightarrow \mathbb{R}_m$ , fulfilling the following property: If  $v \in \text{Ker } F_u(x_0)$  and  $Lv = 0$  then  $v = 0$ . Then we define a projection

$$\Pi: U \rightarrow \text{Ker } F_u(x_0)$$

satisfying the following implication: If  $u \in U$  then  $\Pi u = v \in \text{Ker } F_u(x_0)$  and  $Lv = Lu$ . Let  $\Pi^C$  be the complement of  $\Pi$ , i.e.  $\Pi^C = I - \Pi$  ( $I$  is the identity  $U \rightarrow U$ ). We set  $W = \Pi^C(U)$ , i.e.  $W = \{v \in U: Lv = 0\}$ .

Let  $\mathcal{R}(F_u(x_0))$  be the range of  $F_u(x_0)$ ; the range is closed. There exists a bounded projection

$$Q: Y \rightarrow \mathcal{R}(F_u(x_0)).$$

Let  $Q^C$  be its complement. Then we can split  $Y$  so that

$$Y = \mathcal{R}(F_u(x_0)) \oplus Q^C(Y)$$

where both components are closed and  $\dim Q^C(Y) = \dim \text{Ker } F_u(x_0) = m$ .

For each  $r \in Y$  there exists a unique  $z \in U$  such that  $F_u(x_0)z = Qr$ ,  $Lz = 0$ . By setting  $F_u^+(x_0)r = z$ , a linear bounded operator

$$(2.1) \quad F_u^+(x_0): Y \rightarrow W$$

is defined. Note that  $F_u^+(x_0)$  is an infinite dimensional analogue of the prescribed range/null space generalized inverse, see [6].

The condition (1.1) is reduced to a so called bifurcation equation, see the coming (2.3): Assuming  $(v, \lambda, \alpha) \in \text{Ker } F_u(x_0) \times \mathbb{R}_1 \times \mathbb{R}_k$ , we define  $w \in U$  to be the solution to

$$(2.2) \quad QF(w+v, \lambda, \alpha) = 0, \quad Lw = 0 \quad (\text{i.e. } w \in W).$$

The Implicit Function Theorem yields

$$w = w(v, \lambda, \alpha), \quad w \in C^\infty(\mathcal{V}, \mathcal{W})$$

where  $\mathcal{V} \subset \text{Ker } F_u(x_0) \times \mathbb{R}_1 \times \mathbb{R}_k$  and  $\mathcal{W} \subset \mathcal{W}$  are neighbourhoods of  $(\pi u_0, \lambda_0, \alpha_0)$  and  $\pi^C u_0$  respectively.

Define  $\mathcal{U} = \{(u, \lambda, \alpha) : (\pi u, \lambda, \alpha) \in \mathcal{V}, \pi^C u \in \mathcal{W}\}$ . It can be concluded that

$$F(u, \lambda, \alpha) = 0, \quad (u, \lambda, \alpha) \in \mathcal{U}$$

if and only if

$$(2.3) \quad g(v, \lambda, \alpha) = 0, \quad (v, \lambda, \alpha) \in \mathcal{V}$$

where

$$(2.4) \quad g(v, \lambda, \alpha) = Q^C F(v + w(v, \lambda, \alpha), \lambda, \alpha).$$

Since both  $\text{Ker } F_u(x_0)$  and  $Q^C(Y)$  can be identified with  $\mathbb{R}_m$ ,  $g$  could be understood as a germ of a  $C^\infty$ -mapping

$$g: \mathbb{R}_m \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow \mathbb{R}_m$$

centred at  $(v_0, \lambda_0, \alpha_0)$ , where  $v_0 = \pi u_0$ .

3. Classification by codimension. Let us define a germ  $h = h(v, \lambda)$  of  $C^\infty$ -mapping

$$h: \mathbb{R}_m \times \mathbb{R}_1 \rightarrow \mathbb{R}_m$$

centred at  $(v_0, \lambda_0)$  such that  $h(v, \lambda) = g(v, \lambda, \alpha_0)$ . Thus,  $g = g(v, \lambda, \alpha)$  is a k-parameter unfolding of the germ  $h$ . The unfolding parameter  $\alpha$  is assumed to be close to  $\alpha_0$ .

We recall the concept of codimension (notation:  $\text{codim}$ ) of the germ  $h$ . We refer to [1], p. 121, for the rigorous definition. Roughly speaking,  $\text{codim}$  equals the least number of the unfolding parameters which could describe all qualitatively substantial perturbations of  $h$ .

Codimension is related to the singular point  $(v_0, \lambda_0)$  and the germ  $h$  itself. Because of the link between  $h$  and  $g$ , we relate the same codimension to the singular point  $(v_0, \lambda_0, \alpha_0)$  of  $g$ .

Let us assume  $\text{codim} \leq 2$ . As shown in [1], Theorem 2.1, p. 400, this implies

- a)  $m=1$ , i.e.  $h: \mathbb{R}_1 \times \mathbb{R}_1 \rightarrow \mathbb{R}_1$  (nomenclature of the case: "bifurcation from a simple eigenvalue")
- b) there are just six classes of singularities  $(v_0, \lambda_0)$  of  $h: \mathbb{R}_2 \rightarrow \mathbb{R}_1$  having  $\text{codim} \leq 2$ , namely

Table 1: Classification of singularities (codim  $\leq 2$ )

| Case No | nomenclature                          | codim |
|---------|---------------------------------------|-------|
| (i)     | limit point                           | 0     |
| (ii)    | simple bifurcation or an isola centre | 1     |
| (iii)   | hyteresis point                       | 1     |
| (iv)    | pitchfork                             | 2     |
| (v)     | quartic fold                          | 2     |
| (vi)    | asymmetric cusp                       | 2     |

Each of the singularities listed above has to satisfy a set of defining conditions and some nondegeneracy conditions, see [1], Table 2.3, p. 198. These conditions can be viewed as definitions of  $(v_0, \lambda_0)$  and, because of the link between  $h$  and  $g$ , definitions of  $(v_0, \lambda_0, \alpha_0)$  in particular cases (i) - (vi):

Table 2: Defining conditions  $G(v_0, \lambda_0, \alpha_0)=0$  for singularities of codim  $\leq 2$

| Case No | operator $G: \mathbb{R}_1 \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow \mathbb{R}_\ell$ | $\ell$ |
|---------|--|--------|
| (i)     | $G \equiv (g, g_v)$  | 2      |
| (ii)    | $G \equiv (g, g_v, g_\lambda)$   | 3      |
| (iii)   | $G \equiv (g, g_v, g_{vv})$  | 3      |
| (iv)    | $G \equiv (g, g_v, g_\lambda, g_{vv})$   | 4      |
| (v)     | $G \equiv (g, g_v, g_{vv}, g_{vvv})$   | 4      |
| (vi)    | $G \equiv (g, g_v, g_\lambda, \det D^2 g)$   | 4      |

where

$$D^2 g = \begin{pmatrix} g_{vv} & g_{v\lambda} \\ g_{v\lambda} & g_{\lambda\lambda} \end{pmatrix}.$$

Table 3: Nondegeneracy conditions for singularities of codim  $\leq 2$

| Case No | nondegeneracy conditions at $(v_0, \lambda_0, \alpha_0)$                           |
|---------|--|
| (i)     | $g_{vv} \neq 0, g_\lambda \neq 0$  |
| (ii)    | $g_{vv} \neq 0, \det D^2 g \neq 0$   |
| (iii)   | $g_{vvv} \neq 0, g_\lambda \neq 0$   |
| (iv)    | $g_{vvv} \neq 0, g_{v\lambda} \neq 0$  |
| (v)     | $g_{vvvv} \neq 0, g_\lambda \neq 0$  |
| (vi)    | $g_{vv} \neq 0, \frac{d^3}{dt^3} g(v_0+bt, \lambda_0+t, \alpha_0) \neq 0$ at $t=0$ |

where  $b = -g_{v\lambda} / g_{vv}$

We resume that each particular singularity  $(v_0, \lambda_0, \alpha_0)$  is a root of some operator

$$G: \mathbb{R}_1 \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow \mathbb{R}_2.$$

One can check that if  $k = \text{codim}$  then  $\ell = k+2$ ; for, compare  $\ell$  and  $\text{codim}$  of Table 2 and Table 1.

Our aim is to have  $G$  regular at  $(v_0, \lambda_0, \alpha_0)$  in order to determine  $(v_0, \lambda_0, \alpha_0)$  locally uniquely. To this end, we shall assume  $g$  to be a universal unfolding of  $h$  (for  $\alpha$  being close to  $\alpha_0$ ). We refer to [1], p. 121, for the exact definition. The following remarks can elucidate the definition slightly:

- a)  $k = \text{codim}$
- b) the solutions  $(v, \lambda)$  to the equation

$$h(v, \lambda) + \text{a small perturbation } (v, \lambda) = 0$$

are "qualitatively the same" (in the sense of the so called contact equivalence) as the solutions  $(v, \lambda)$  to the equation  $g(v, \lambda, \alpha) = 0$  for an  $\alpha$  being close to  $\alpha_0$ ; note that only those  $(v, \lambda)$ 's are considered which are close enough to  $(v_0, \lambda_0)$ .

Proposition 1. Let  $g$  be a universal unfolding of  $h$ . Let us assume  $k \leq 2$  (i.e.,  $m=1$  and the singularity  $(v_0, \lambda_0)$  is classified by one of the cases (i) - (vi) in Table 1). Then the appropriate mapping

$$G: \mathbb{R}_1 \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow \mathbb{R}$$

see Table 2, is regular at  $(v_0, \lambda_0, \alpha_0)$ , i.e. the singular point  $(v_0, \lambda_0, \alpha_0)$  is defined locally uniquely as a simple root of  $G=0$ .

Proof. We are to prove that the Jacobian of  $G$  at  $(v_0, \lambda_0, \alpha_0)$  does not vanish.

In [1], Table 3.2, p. 204, there are listed the necessary and sufficient algebraic conditions upon  $(v_0, \lambda_0, \alpha_0)$  for  $\alpha$  to be the parameter of a universal unfolding  $g$  of  $h$ , provided that the singularity  $(v_0, \lambda_0)$  of  $h$  is classified by one of the cases (i) - (vi). (The authors of [1] call it "the recognition problem for universal unfoldings".) What we have to do is very simple: For each of the particular cases (i) - (vi), one has to interpret the adequate condition in [1], p. 204, as the nondegeneracy condition for the Jacobian of the mapping  $G$ .

For the sake of brevity, we shall treat only the case (vi). i.e., the asymmetric cusp, which seems to be the most troublesome: According to the quoted result of [1], the germ  $g=g(v, \lambda, \alpha)$  is a universal unfolding of  $h$  if and only if the  $3 \times 3$  matrix (mind that  $\alpha \in \mathbb{R}_2$ )

$$M = \begin{pmatrix} g_v, g_{vv}, g_{v\lambda} \\ g_\alpha, g_{v\alpha}, g_{\lambda\alpha} \end{pmatrix},$$

is nonsingular at  $(v_0, \lambda_0, \alpha_0)$ .

The gradient of  $G$  at  $(v_0, \lambda_0, \alpha_0)$  is the following  $4 \times 4$  matrix:

$$N = \begin{pmatrix} g_v & , & g_\lambda & , & g_\alpha \\ g_{vv} & , & g_{v\lambda} & , & g_{v\alpha} \\ g_{v\lambda} & , & g_{\lambda\lambda} & , & g_{\lambda\alpha} \\ (\det D^2 g)_v & , & (\det D^2 g)_\lambda & , & (\det D^2 g)_\alpha \end{pmatrix}$$

Let us remind that  $g_v = g_\lambda = 0$ , see Table 2. Moreover, since  $\det D^2 g = 0$  and  $g_{vv} \neq 0$ , we have

$$\begin{aligned} bg_{vv} + g_{v\lambda} &= 0 \\ bg_{v\lambda} + g_{\lambda\lambda} &= 0 \end{aligned}$$

where  $b = -g_{v\lambda}/g_{vv}$ ; it is understood that  $g$  and its partial derivatives are evaluated at  $(v_0, \lambda_0, \alpha_0)$ .

Multiply the first column of  $N$  by  $b$  and add it to the second column which results in the vector  $(0, 0, 0, (\det D^2 g)_\lambda + b(\det D^2 g)_v)^T$ . It can be easily verified that the last component of this vector is just

$$\begin{aligned} g_{vv} \frac{d^3}{dt^3} g(v_0 + bt, \lambda_0 + t, \alpha_0) \text{ at } t=0. \text{ Thus,} \\ \det N = g_{vv} \frac{d^3}{dt^3} g(v_0 + bt, \lambda_0 + t, \alpha_0) \Big|_{t=0} \det M \end{aligned}$$

which implies (see Table 3, (vi)) that  $\det N \neq 0$ .

It proves the regularity of  $G$  in the case (vi). The remaining cases can be treated similarly.

**Remark 1.** If  $g_{vv} \neq 0$  then the condition (vi) of Table 2 can be formulated as follows:

$$(3.1) \quad G(v_0, \lambda_0, \alpha_0, b) = 0, \quad b \in \mathbb{R}_1$$

for some  $b \in \mathbb{R}_1$ , where

$$(3.2) \quad G \equiv (g, g_v, g_\lambda, bg_{vv} + g_{v\lambda}, bg_{v\lambda} + g_{\lambda\lambda}).$$

Thus, both the condition (vi) of Table 3 and the validity of (3.1) are equivalent to (vi) of Table 2 and Table 3. Moreover, if the assumptions of Proposition 1 hold then the above mapping  $G: \mathbb{R}_1 \times \mathbb{R}_1 \times \mathbb{R}_2 \times \mathbb{R}_1 \rightarrow \mathbb{R}_5$  is regular at  $(v_0, \lambda_0, \alpha_0, b)$ , where  $b = -g_{v\lambda}/g_{vv}$  at  $(v_0, \lambda_0, \alpha_0)$ .

Remark 2. So far in this Section we have assumed  $g: \mathbb{R}_m \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow \mathbb{R}_m$  since we have identified both  $\text{Ker } F_U(x_0)$  and  $Q^C(Y)$  with  $\mathbb{R}_m$ , see Section 2. Naturally, all the conditions of Table 2 and Table 3 and Remark 1 can be formulated in terms of the original variable  $v \in \text{Ker } F_U(x_0)$  of the mapping  $g: \text{Ker } F_U(x_0) \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow Q^C(Y)$ :

Since  $m=1$ , we choose a fixed  $\eta \in F_U(x_0)$ ,  $\eta \neq 0$ , and replace

$$g_v := g_v \eta, \quad g_{v\lambda} := g_{v\lambda} \eta, \quad g_{vv} := g_{vv} \eta^2$$

(let us write  $g_{vv} \eta^2$  instead of  $g_{vv} \eta \eta$ );  $g$  and its partials depend on  $(v, \lambda, \alpha)$ . Then the operators  $G$  of Table 2 act as follows:  $G=G(v, \lambda, \alpha)$ ,  $G: \text{Ker } F_U(x_0) \times \mathbb{R}_1 \times \mathbb{R}_k \rightarrow [Q^C(Y)]^L$ . Similar changes should be done in Remark 1.

4. Inflated mappings corresponding to singularities of codim  $\leq 2$ . In this Section we are going to reformulate the defining conditions of Table 2 in terms of  $F$  and its partial derivatives. The idea of such a reformulation has already been mentioned in [2].

Convention. Any notion connected with a singular point  $(v_0, \lambda_0, \alpha_0)$  of  $g$  in Section 3 is naturally transferred to the relevant singular point  $(u_0, \lambda_0, \alpha_0)$  of  $F$ .

We assume those singular points  $x_0 = (u_0, \lambda_0, \alpha_0)$  of  $F$  which have  $\text{codim} \leq 2$ , see the above Convention. Recall that  $\text{codim} \leq 2$  implies  $m=1$ , i.e.  $L: U \rightarrow \mathbb{R}_1$  (see Section 2). Thus, by definition,  $x_0$  has to satisfy

$$(4.1) \quad F(x_0) = 0$$

and

$$(4.2) \quad \exists v \in U: F_U(x_0)v = 0, \quad Lv \neq 0.$$

We recall  $W = \{v \in U: Lv = 0\}$ . If a point  $\eta_0 \in U$ ,  $L\eta_0 \neq 0$  is given then Condition (4.2) is equivalent to

$$(4.3) \quad \exists \eta_1 \in W: F_U(x_0)\eta_1 + F_U(x_0)\eta_0 = 0$$

(namely,  $v = \eta_1 + \eta_0$ ).

Note that (4.1) and (4.3) imply

$$(4.4) \quad g = g_v \eta = 0, \quad \eta = \eta_1 + \eta_0$$

at  $(v_0, \lambda_0, \alpha_0)$ .

Proposition 2. A singular point  $x_0 = (u_0, \lambda_0, \alpha_0)$  of  $\text{codim} \leq 2$  satisfies the defining conditions (3.1) if and only if there exist  $b \in \mathbb{R}_1$ ,  $\eta_i \in W$ ,  $i=1,2,3,4$ , such that  $(u_0, \lambda_0, \alpha_0, b, \eta_1, \eta_2, \eta_3, \eta_4)$  is a root of an operator



$$\mathcal{F}: U \times \mathbb{R}_1 \times \mathbb{R}_k \times \mathbb{R}_1 \times [W]^4 \longrightarrow [Y]^2$$

which is defined as follows: Let  $\eta_0 \in U$ ,  $L\eta_0 \neq 0$  be given. Then

$$(4.5) \quad \mathcal{F}(u, \lambda, \alpha, c, \xi_1, \xi_2, \xi_3, \xi_4) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_{\lambda} \\ F_u \xi_3 + (cF_{uu}v + F_{uu} \xi_2 + F_{u\lambda})v \\ F_u \xi_4 + c(F_{uu} \xi_2 + F_{u\lambda})v + F_{uu} \xi_2^2 + 2F_{u\lambda} \xi_2 + F_{\lambda\lambda} \end{pmatrix}$$

where  $v := \xi_1 + \eta_0$ ; the values of  $F$ ,  $F_u$ ,  $F_{\lambda}$ ,  $F_{uu}$  are understood to be evaluated at  $(u, \lambda, \alpha)$ .

The remaining components  $b, \eta_1, \eta_2, \eta_3, \eta_4$  of the root have the following interpretations:

$$(4.6) \quad b = - \frac{g_{v\lambda} \eta}{g_{vv} \eta^2}$$

$$(4.7) \quad \eta_1 + \eta_0 \in \text{Ker } F_u(x_0), \quad L(\eta_1 + \eta_0) \neq 0$$

$$(4.8) \quad \eta_2 = w_{\lambda}$$

$$(4.9) \quad \eta_3 = (bw_{vv}\eta + w_{v\lambda})\eta$$

$$(4.10) \quad \eta_4 = bw_{v\lambda}\eta + w_{\lambda\lambda},$$

where  $\eta = \eta_1 + \eta_0$ ; the operator  $w$  and its partial derivatives are evaluated at  $(\Pi u_0, \lambda_0, \alpha_0)$ .

Proof. Let  $x_0 = (u_0, \lambda_0, \alpha_0)$  be a singular point of codim  $\leq 2$ . Then (4.1), (4.3) and (4.4) hold. Note that (4.1) and (4.3) represent two conditions that a root of  $\mathcal{F}$ , see (4.5), must fulfil.

Let us fix  $\eta \in \text{Ker } F_u(x_0)$  setting  $\eta = \eta_1 + \eta_0$ . The differentiation of (2.2) at  $(\Pi u_0, \lambda_0, \alpha_0)$  by  $\lambda$  and  $v$  respectively gives

$$(4.11) \quad Q[F_u w_{\lambda} + F_{\lambda}] = 0, \quad Lw_{\lambda} = 0$$

$$(4.12) \quad Q[F_u \cdot (I + w_v)] \eta = 0, \quad Lw_v \eta = 0.$$

Since  $F_u \eta = 0$ , then (4.12) implies

$$(4.13) \quad w_v \eta = 0.$$

Taking into account (4.13), further differentiation of (2.2) yields

$$(4.14) \quad Q[F_{uu} + F_u w_{vv}] \eta^2 = 0, \quad Lw_{vv} \eta^2 = 0$$

$$(4.15) \quad Q[F_{uu}w_{\lambda} + F_{u\lambda} + F_{uw_{v\lambda}}]\eta = 0, \quad Lw_{v\lambda}\eta = 0$$

$$(4.16) \quad Q[F_{uu}w_{\lambda}^2 + 2F_{u\lambda}w_{\lambda} + F_{\lambda\lambda} + F_{uw_{\lambda\lambda}}] = 0, \quad Lw_{\lambda\lambda} = 0.$$

Similarly, differentiating (2.4) and using (4.13),

$$(4.17) \quad g_{\lambda} = Q^C[F_{uw_{\lambda}} + F_{\lambda}]$$

$$(4.18) \quad g_{vv}\eta^2 = Q^C[F_{uu} + F_{uw_{vv}}]\eta^2$$

$$(4.19) \quad g_{v\lambda}\eta = Q^C[F_{uw_{\lambda}} + F_{u\lambda} + F_{uw_{v\lambda}}]\eta$$

$$(4.20) \quad g_{\lambda\lambda} = Q^C[F_{uu}w_{\lambda}^2 + 2F_{u\lambda}w_{\lambda} + F_{\lambda\lambda} + F_{uw_{\lambda\lambda}}].$$

By virtue of (4.11) and (4.17), the condition  $g_{\lambda} = 0$  is equivalent to the following one:

$$(4.21) \quad \exists \eta_2 \in W: F_u \eta_2 + F_{\lambda} = 0$$

with the interpretation (4.8).

The last two conditions of (3.1) read (see Remark 2) as

$$(4.22) \quad (bg_{vv}\eta + g_{v\lambda})\eta = 0$$

$$(4.23) \quad bg_{v\lambda}\eta + g_{\lambda\lambda} = 0.$$

It is simple to conclude from (4.14) - (4.20) that (4.22) and (4.23) respectively are equivalent to the following conditions

$$(4.24) \quad \exists \eta_3 \in W: F_u \eta_3 + (bF_{uu}\eta + F_{uu}\eta_2 + F_{u\lambda})\eta = 0$$

and

$$(4.25) \quad \exists \eta_4 \in W: F_u \eta_4 + b(F_{uu}\eta_2 + F_{u\lambda})\eta + F_{uu}\eta_2^2 + 2F_{u\lambda}\eta_2 + F_{\lambda\lambda} = 0.$$

The interpretation of both  $\eta_3$  and  $\eta_4$  is clearly that of (4.9) and (4.10).

Note that (4.1), (4.3), (4.21), (4.24), (4.25) mean that  $(u_0, \lambda_0, \alpha_0, b, \eta_1, \eta_2, \eta_3, \eta_4)$  is a root of  $\mathcal{F}$ , see (4.5). The equivalence of (3.1) and the mentioned conditions were checked throughout.

The proof is completed.

Let us review the inflated mappings for the remaining cases (i) - (v) of Table 2. We shall omit the proofs because they are similar to that of Proposition 2.

**Proposition 3.** A singular point  $x_0 = (u_0, \lambda_0, \alpha_0)$  of codim  $\leq 2$  satisfies the defining conditions (i) of Table 2 if and only if there exists  $\eta_1 \in W$  such that  $(u_0, \lambda_0, \alpha_0, \eta_1)$  is a root of an operator

$$\mathcal{F}: U \times \mathbb{R}_1 \times \mathbb{R}_k \times W \rightarrow [Y]^2$$

which is defined as follows: Let  $\eta_0 \in U$ ,  $L\eta_0 \neq 0$  be given. Then

$$(4.26) \quad \mathcal{F}(u, \lambda, \alpha, \xi_1) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \end{pmatrix}$$

where  $F$  and  $F_u$  are evaluated at  $(u, \lambda, \alpha)$ . The component  $\eta_1$  is interpreted as (4.7).

**Proposition 4.** A singular point  $x_0 = (u_0, \lambda_0, \alpha_0)$  of codim  $\leq 2$  satisfies the defining conditions (ii) and (iii) of Table 2 respectively if and only if there exist  $\eta_1 \in W$ ,  $\eta_2 \in W$  such that  $(u_0, \lambda_0, \alpha_0, \eta_1, \eta_2)$  is a root of an operator

$$\mathcal{F}: U \times \mathbb{R}_1 \times \mathbb{R}_k \times [W]^2 \rightarrow [Y]^3$$

which is defined as follows: Let  $\eta_0 \in U$ ,  $L\eta_0 \neq 0$  be given. Then, for the cases (ii) and (iii) respectively,

$$(4.27) \quad \mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_\lambda \end{pmatrix}$$

and

$$(4.28) \quad \mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_{uu} v^2 \end{pmatrix}$$

where  $v := \xi_1 + \eta_0$ ;  $F$ ,  $F_u$ ,  $F_\lambda$ ,  $F_{uu}$  are evaluated at  $(u, \lambda, \alpha)$ . In both cases (ii) and (iii), the component  $\eta_1$  is interpreted as (4.7). The component  $\eta_2$  means

$$(4.29) \quad \eta_2 = w_\lambda$$

and

$$(4.30) \quad \eta_2 = w_{vv} \eta^2, \quad \eta = \eta_1 + \eta_0$$

for the cases (ii) and (iii) respectively; the derivatives of  $w$  are evaluated at  $(Tu_0, \lambda_0, \alpha_0)$ .

**Proposition 5.** A singular point  $x_0 = (u_0, \lambda_0, \alpha_0)$  of codim  $\leq 2$  satisfies the defining conditions (iv) and (v) of Table 2 respectively if and only if there exist  $\eta_i \in W$ ,  $i=1,2,3$  such that  $(u_0, \lambda_0, \alpha_0, \eta_1, \eta_2, \eta_3)$  is a root of an operator

$$\mathcal{F}: U \times \mathbb{R}_1 \times \mathbb{R}_k \times [W]^3 \rightarrow [V]^4$$

where  $\mathcal{F}$  is defined as follows: Let  $\eta_0 \in U$ ,  $\eta_0 \neq 0$  be given. Then, for the cases (iv) and (v) respectively,

$$(4.31) \quad \mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2, \xi_3) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_{u\lambda} \\ F_u \xi_3 + F_{uu} v^2 \end{pmatrix},$$

and

$$(4.32) \quad \mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2, \xi_3) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_{uu} v^2 \\ F_u \xi_3 + (3F_{uu} \xi_2 + F_{uuu} v^2) v \end{pmatrix},$$

where  $v := \xi_1 + \eta_0$ ;  $F, F_u, F_{u\lambda}, F_{uu}, F_{uuu}$  are evaluated at  $(u, \lambda, \alpha)$ . In both cases (iv) and (v), the component  $\eta_1$  is interpreted via (4.7). The component  $\eta_2$  means (4.29) and (4.30) respectively. The interpretation of  $\eta_3$  is as follows:

$$(4.33) \quad \eta_3 = w_{vv} \eta^2, \quad \eta = \eta_1 + \eta_0$$

and

$$(4.34) \quad \eta_3 = w_{vvv} \eta^3, \quad \eta = \eta_1 + \eta_0$$

respectively.

A numerical application of Propositions 2 - 5 can follow the next recommendation: Guess the kind of a singularity  $x_0$  of  $\text{codim} \leq 2$  which is to be found, then choose the corresponding mapping  $\mathcal{F}$  and find its root. To this end it is vital to know that the root is simple (for Newton method to be applicable), i.e. that the gradient  $D\mathcal{F}$  of  $\mathcal{F}$ , being evaluated in the root, is continuously invertible, i.e. the mapping  $\mathcal{F}$  is regular in the root.

**Theorem 1.** The inflated mappings  $\mathcal{F}$  given by (4.5), (4.26), (4.27), (4.28), (4.31) and (4.32) respectively are regular at a root  $(u_0, \lambda_0, \alpha_0$ , plus the relevant auxiliary variables) if and only if the corresponding mappings  $G$  (see (3.2) and the cases (i) - (v) of Table 2, respectively) are regular at  $(\Pi u_0, \lambda_0, \alpha_0)$ .

**Proof.** We have already proved the above statement for the case (ii), see [2], Proposition 2. The remaining cases can be treated in the same way. Namely, the inverse to  $D\mathcal{F}$  can be constructed explicitly by means of the in-

verse to DG. Since it involves just straightforward calculations, we believe, we could omit it here.

Remark 3. Both Proposition 1 and Theorem 1 provide sufficient conditions for a root of  $\mathcal{F}$  to be simple.

5. Roots of inflated mappings. Suppose  $x_0 = (u_0, \lambda_0, \alpha_0)$  to be a singularity of codim  $\leq 2$ . Let  $(u_0, \lambda_0, \alpha_0$ , plus the auxiliary variables) be the relevant root of the inflated mapping  $\mathcal{F}$ . The aim of this Section is to show that the auxiliary components of the root provide a useful piece of information concerning the bifurcation diagram of the equation  $F(u, \lambda, \alpha_0) = 0$ , i.e. the solution set

$$(5.1) \quad S = \{(u, \lambda) \in U \times \mathbb{R}_1 : F(u, \lambda, \alpha_0) = 0\}.$$

Namely, a parametric description of  $S$  in a neighbourhood of  $(u_0, \lambda_0)$  can be obtained.

We recall the notion of bifurcation equation (2.3). Since  $\dim \text{Ker } F_u(x_0) = 1$ , choose  $\eta \in \text{Ker } F_u(x_0)$ ,  $\eta \neq 0$ , and substitute  $v := v_0 + t\eta$ ,  $v_0 = \Pi^C u_0$ ,  $\lambda := \lambda_0 + \mu$ ,  $\alpha := \alpha_0$  into (2.3) for  $t$  and  $\mu \in \mathbb{R}_1$ , assuming that  $|t|$  and  $|\mu|$  are small enough. It is natural to set  $\eta := \eta_1 + \eta_0$ , see Propositions 2 - 5.

We have to specify  $Q^C$ , see Section 1. It could be defined as follows: Let  $\langle \cdot, \cdot \rangle$  be the pairing of  $Y$  and its dual  $Y^*$ . Denote  $F_u^*(x_0)$  the adjoint operator to  $F_u(x_0)$ . Scale  $\eta^* \in \text{Ker } F_u^*(u_0)$  such that  $\langle \eta, \eta^* \rangle = 1$ . If  $y \in Y$  then define

$$Q^C y = \langle y, \eta^* \rangle \eta.$$

The means of identification of both  $\text{Ker } F_u(x_0)$  and  $Q^C(Y)$  with  $\mathbb{R}_1$  are obvious now. It is natural to define  $h: \mathbb{R}_1 \times \mathbb{R}_1 \rightarrow \mathbb{R}_1$  such that

$$h(t, \mu) = \langle g(v_0 + t\eta, \lambda_0 + \mu, \alpha_0), \eta^* \rangle.$$

Clearly, the solution set  $\mathcal{V}$  to the equation

$$(5.2) \quad h(t, \mu) = 0$$

is locally isomorphic with the solution set  $S$ , (5.1). The isomorphism is represented by the following formulas:

$$(5.3) \quad u = \Pi u_0 + t\eta + w(\Pi u_0 + t\eta, \lambda_0 + \mu, \alpha_0)$$

$$(5.4) \quad \lambda = \lambda_0 + \mu$$

for  $t, \mu \in \mathbb{R}_1$ ,  $|t|$  and  $|\mu|$  small enough.

The point is that some terms of Taylor expansion

$$(5.5) \quad w(\Pi u_0 + t\eta, \lambda_0 + \mu, \alpha_0) = \Pi^C u_0 + t w_V \eta + \mu w_\lambda + \frac{t^2}{2} w_{VV} \eta^2 + \\ + t(\mu w_{V\lambda} \eta + \frac{\mu^2}{2} w_{\lambda\lambda}) + \text{higher order terms (h.o.t.)}$$

are given by components of the root to  $\mathcal{F}$ . Here and in the sequel, the operators  $w$  and  $g$  and their partials are understood to be evaluated at  $(\Pi u_0, \lambda_0, \alpha_0)$ . The operator  $F$  and its partials are evaluated at  $(u_0, \lambda_0, \alpha_0)$ .

We shall demonstrate the idea on an example only: Suppose that  $x_0$  is a point of a pitchfork bifurcation. Thus, due to our definition in Section 3, the conditions (iv) of both Table 2 and Table 3 hold. They are equivalent to the assumption

$$(5.6) \quad h(t, \mu) = at^3 + b\mu t + c\mu^2 + \text{h.o.t.}$$

where  $a \cdot b \neq 0$ ,

$$(5.7) \quad a = \frac{1}{6} \langle g_{VVV} \eta^3, \eta^* \rangle, \quad b = \langle g_{V\lambda} \eta, \eta^* \rangle, \quad c = \frac{1}{2} \langle g_{\lambda\lambda}, \eta^* \rangle.$$

Qualitative analysis (see e.g. [1], Proposition 9.2, p. 95) yields that the solution set  $\mathcal{S}$  to (5.6) consists (locally) of two smooth branches  $\mathcal{S}_1 = \{(t, \mu) : t = t(\mu), |\mu| \text{ small}\}$  and  $\mathcal{S}_2 = \{(t, \mu) : \mu = \mu(t), |t| \text{ small}\}$  where both functions  $t(\mu)$  and  $\mu(t)$  are smooth,  $t(0) = \mu(0) = 0$ . Substituting the functions  $t = t(\mu)$  and  $\mu = \mu(t)$  respectively into (5.2), one obtains the following expansions:

$$(5.8) \quad t(\mu) = -\frac{c}{b} \mu + O(\mu^3)$$

$$(5.9) \quad \mu(t) = -\frac{a}{b} t^2 + O(t^3)$$

The local isomorphism (5.3), (5.4) implies that  $S$  consists (locally) of two smooth branches  $S_1, S_2$ ; the branch  $S_1$  being isomorphic with  $\mathcal{S}_1$ . Applying Proposition 5 (namely the interpretations of  $\eta_1, \eta_2, \eta_3$ ) to (5.5), and taking into account (4.13), one can read (5.3) as follows:

$$(5.10) \quad u = u_0 + t\eta + \mu\eta_2 + \frac{t^2}{2} \eta_3 + O(t\mu) + O(\mu^2) + \text{h.o.t.}$$

Formulas (5.10), (5.4) and the expansions (5.8), (5.9) are ready to supply a local approximation to branches  $S_1$  and  $S_2$ . Of course, the approximation relies upon the constants  $a, b, c$  (see (5.6)) we have not fixed yet.

Let us note at this place that the constants  $a, b, c$  participate in the gradient  $DG$  which is connected with the gradient  $D\mathcal{F}$  of the inflated mapping

$\mathcal{F}$ . Since the gradient  $D\mathcal{F}$  is evaluated in the course of Newton iterations, the constants a, b, c (namely, their approximations) are cheaply available as a by-product. Thus the following calculations could be economised.

By definition, see (2.4), and taking (4.13) into account,

$$(5.11) \quad \begin{cases} g_{vvv} \eta^3 = Q^C(F_{uuu} \eta + 3F_{uu} \eta w_{vv}) \eta^2 \\ g_{v\lambda} \eta = Q^C(F_{u\lambda} + F_{uu} w_{\lambda}) \eta \\ g_{\lambda\lambda} = Q^C(F_{\lambda\lambda} + F_{uu} w_{\lambda}^2 + 2F_{u\lambda} w_{\lambda}) \end{cases}$$

Again, both  $w_{vv} \eta^2$  and  $w_{\lambda}$  are to be replaced by  $\eta_3$  and  $\eta_2$  respectively. Resuming (5.7) and (5.11),

$$(5.12) \quad \begin{cases} a = \frac{1}{6} \langle F_{uuu} \eta^3 + 3F_{uu} \eta \eta_3, \eta^* \rangle \\ b = \langle F_{u\lambda} \eta + F_{uu} \eta \eta_2, \eta^* \rangle \\ c = \frac{1}{2} \langle F_{\lambda\lambda} + F_{uu} \eta_2^2 + 2F_{u\lambda} \eta_2, \eta^* \rangle. \end{cases}$$

We just proved

**Proposition 6.** Suppose  $x_0 = (u_0, \lambda_0, \alpha_0)$  to be a point of a pitchfork bifurcation; let  $(u_0, \lambda_0, \alpha_0, \eta_1, \eta_2, \eta_3)$  be the relevant root of the mapping  $\mathcal{F}$ , see (4.31). Then the solution set S, see (5.1), consists (locally) of two smooth branches  $S_1$  and  $S_2$ :

$$\begin{aligned} S_1 &= \{(u, \lambda) : u = u_0 - (\frac{c}{b} \eta - \eta_2) \mu + O(\mu^2), \lambda = \lambda_0 + \mu, \text{ for } \mu \in \mathbb{R}_1, |\mu| \text{ small}\} \\ \text{and} \\ S_2 &= \{(u, \lambda) : u = u_0 + t \eta - (\frac{a}{b} \eta_2 - \frac{1}{2} \eta_3) t^2 + O(t^3), \lambda = \lambda_0 - \frac{a}{b} t^2 + O(t^3), \text{ for } \\ &\quad t \in \mathbb{R}_1, |t| \text{ small}\} \end{aligned}$$

where  $\eta = \eta_1 + \eta_0$  and the constants a, b, c are given by (5.12).

Solution sets S for the remaining singularities of codim  $\neq 2$  can be approximated similarly following the above example, see [7].

**Remark 4.** As we have already hinted, the gradient  $D\mathcal{F}$  provides the leading terms of the equation (5.2) which contributes to a description of the solution set S. Let us note that  $D\mathcal{F}$  is a source of much more complex information. For example, one can approximate transition sets (see [1], Chapter III) of the bifurcation problem  $F(u, \lambda, \alpha) = 0$  or one can (qualitatively) describe all the solution sets  $S_\alpha = \{(u, \lambda) : F(u, \lambda, \alpha) = 0\}$  for  $\alpha$  being close to

$\alpha_0$ , etc. We shall discuss it elsewhere (we refer to [7] for a preliminary version of the mentioned results).

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