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ALTERNATING CYCLES AND REALIZATIONS OF A DEGREE SEQUENCE

Z. MAJCHER

Abstract: We find an algorithm for constructing finite sequences of certain graphs (realizations of a degree sequence on a given set) with given initial and final graphs such that each subsequent graph is obtained from the preceding one by a switching.

Key words: Graph, realization of a degree sequence.

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0. Introduction. In this paper, we deal with finite, undirected graphs admitting multiple edges and loops and we also consider some special types of graphs, e.g. graphs without loops, k-graphs, simple graphs.

We are interested in the class $\mathbb{R}_V(d)$ of all graphs being realizations of a degree sequence d on a given set V. The class $\mathbb{R}_V(d)$ is closed under switching operation (see [2]).

One of the most important properties of the class $\mathbb{R}_V(d)$ is contained in the following

Theorem. If $G,H \in \mathbb{R}_V(d)$, then there exists a sequence

(✱) $G^0,G^1,\dots,G^m$ such that $G^0=G$, $G^m=H$ and for every $s \in \{0,1,\dots,m-1\}$ the graph G^{s+1} is obtained from G^s by a switching.

Several proofs of this theorem were presented in the literature. In those proofs different methods have been used for different types of graphs (see [1],[3],[4],[6]), Our aim is to find a method of the proof which is effective, uniform and optimal. In this paper an algorithm for constructing the sequence (✱) is given. This algorithm can be applied to all types of graphs mentioned above. It can generate a shortest sequence (✱), however, in general, solutions are not optimal.

Our method is partially based on ideas contained in [5]. Namely, we make use of the fact that the symmetrical difference $G \dot{-} H$ of two graphs $G,H \in \mathbb{R}_V(d)$ can be decomposed into alternating cycles of some special forms. Therefore, we have to prove several properties of alternating cycles.

1. The set of realizations of a degree sequence and its subsets. Let V be a finite set. We denote by $\mathcal{V}^{(2)}$ the family of all non-empty subsets of V having at most two elements, and by Z^+ - the set of all positive integers.

A graph is an ordered pair (V,E) satisfying the condition:

(1) $$V\neq\emptyset \text{ and } E\subseteq \mathcal{V}^{(2)}\times Z^+.$$

If $e\in E$ and $e=(\{u,v\},n)$ for some $u,v\in V$ and $n\in Z^+$, then the edge e is incident with u and v and has the label n.

We shall write e=unv instead of $e=(\{u,v\},n)$, and e=vnv instead of $e=(\{v\},n)$.

Let G=(V,E) and $u,v\in V$. We denote by $E_G^{(1)}(v)$, $E_G^{(2)}(v)$ and $E_G(u,v)$ the set of all loops incident with v, the set of all edges incident with v and different from loops, and the set of all edges incident both with u and with v - respectively.

The number $deg_G(v)=2|E_G^{(1)}(v)|+|E_G^{(2)}(v)|$ is called the degree of v in G and the number $m_G(u,v)=|E_G(u,v)|$ is called the edge multiplicity of $\{u,v\}$ in G.

A graph G=(V,E) is a multigraph if $E_G^{(1)}(v)=\emptyset$ for every $v\in V$ and G is a k-graph ($k\in Z^+$) if $m_G(u,v)\leq k$ for every $u,v\in V$. A k-multigraph is a multigraph being a k-graph. A 1-multigraph is called a simple graph. A graph without any restrictions will be called sometimes a pseudograph. The class of pseudographs will be denoted by $\mathcal{P}$, the class of multigraphs - by $\mathcal{M}$, k-graphs - by $\mathcal{P}_k$, k-multigraphs - by $\mathcal{M}_k$ and simple graphs - by $\mathcal{S}$. If τ is a class of graphs and $G\in\tau$, then we say that G is of type τ .

Let G=(V,E) be a graph where $V=\{v_1,v_2,\dots,v_n\}$. A sequence d_G of the form

(2) $$d_G=(deg_G(v_1),\ deg_G(v_2),\dots,deg_G(v_n))$$

is called the degree sequence of G.

A sequence $d=(d_1,d_2,\dots,d_n)$ of non-negative integers is graphic if there exists a graph G such that $d=d_G$. Such a graph is called a realization of d.

Let (w_1,w_2,w_3,w_4) be a sequence of vertices of a graph G=(V,E) satisfying the following conditions:

1^0 $w_1\neq w_3$ and $w_2\neq w_4$,

2^0 there exist $n_1,n_2,n_3,n_4\in Z^+$ such that

$e_1=w_1n_1w_2\in E$, $e_3=w_3n_3w_4\in E$ and $e_1\neq e_3$,

$e_2=w_2n_2w_3\notin E$, $e_4=w_4n_4w_1\notin E$ and $e_2\neq e_4$.

Let us denote:

$$G_{(e_1,e_2,e_3,e_4)}=(V,E') \text{ where } E'=(E\setminus\{e_1,e_3\})\cup\{e_2,e_4\}.$$

We say that $G_{(e_1,e_2,e_3,e_4)}$ is obtained from G by a switching operation with respect to the edges e_1,e_3 and e_2,e_4.

We shall write $G_{(w_1,w_2,w_3,w_4)}$ instead of $G_{(e_1,e_2,e_3,e_4)}$ if the switching operation has been done in the following way:

(3)
if $w_1=w_2$ and $w_3=w_4$, then $n_1=m_G(w_1,w_1)$, $n_3=m_G(w_3,w_3)$;
$n_2=m_G(w_1,w_3)+1$, $n_4=m_G(w_3,w_1)+2$;
if $w_1=w_4$ and $w_2=w_3$, then $n_1=m_G(w_1,w_2)$, $n_3=m_G(w_2,w_1)-1$,
$n_2=m_G(w_2,w_2)+1$, $n_4=m_G(w_1,w_1)+1$;
in the remaining cases $n_1=m_G(w_1,w_2)$, $n_3=m_G(w_3,w_4)$,
$n_2=m_G(w_2,w_3)+1$, $n_4=m_G(w_4,w_1)+1$.

If G' is obtained from G by some switching operation, then we also write shortly $G'=sw(G)$.

Let $d=(d_1,d_2,\ldots,d_n)$ be a graphic sequence, $V=\{v_1,v_2,\ldots,v_n\}$ be an arbitrary n-element set and G=(V,E) be a graph. Let $\mathbb{R}_V(d)$ denote the set of all realizations of d on V, that is $G\in\mathbb{R}_V(d)$ if G is a realization of d and the following condition holds:

(4) if $m_G(u,v)=s$ then $E_G(u,v)=\{u1v,u2v,\ldots,usv\}$ for every $u,v\in V$.

It is obvious that if $G\in\mathbb{R}_V(d)$ and $G'=G_{(w_1,w_2,w_3,w_4)}$, then $G'\in\mathbb{R}_V(d)$.

If the realizations of d are required to be graphs of a fixed type τ, then the set of all realizations of d will be denoted by $\mathbb{R}_V(d;\tau)$.

The above definition of a switching operation is suitable for the class of pseudographs. If we consider classes of other types, then this definition must be modified if we want the graph sw(G) to stay in the same class as G. For example we do not like to get loops in the class of graphs without loops. Therefore we have the following definitions:

If $\tau=\mathcal{M}$, then we substitute 1^o by 3^o:

3^o w_1,w_2,w_3,w_4 are pairwise different.

If $\tau=\mathcal{P}_k$, then we add 4^o to the conditions 1^o and 2^o:

4^o $m_G(w_2,w_3)<k$, $m_G(w_4,w_1)<k$.

For $\tau=\mathcal{M}_k$ $(k\geq 2)$ we require conditions 2^o, 3^o and 4^o to be satisfied.

If $\tau=\mathcal{S}$, then we require conditions 2^o, 3^o and 4^o for k=1.

2. Operations on chains and cycles. Let G=(V,E) be a graph. By a chain in G we shall mean a sequence $L=(u_1n_1u_2,u_2n_2u_3,\ldots,u_mn_mu_{m+1})$ of pairwise different edges of G. If $u_1=u_{m+1}$, then we have a cycle. If the edge labels are immaterial, then we shall write $L=u_1u_2\ldots u_mu_{m+1}$ for a chain and $C=u_1u_2\ldots$ $\ldots u_mu_1$ for a cycle.

We shall denote by $V(L)$ and by $E(L)$ the set of all vertices of L and the set of all edges of L respectively. We say that a vertex v is in the k-th position in the chain $L=u_1u_2\ldots u_{m+1}$ if $u_k=v$. Positions k_1 and k_2, where $k_1\neq k_2$, will be called compatible if the number $|k_1-k_2|$ is positive and even.

We define the following operations on chains and on cycles:

For $L=u_1u_2\ldots u_{m-1}u_m$ we define:

(5) $$\overleftarrow{L}=u_mu_{m-1}\ldots u_2u_1.$$

For $C=u_1u_2\ldots u_{i-1}u_iu_{i+1}\ldots u_mu_1$ we define:

(6) $$\overrightarrow{C}^i=u_iu_{i+1}\ldots u_mu_1u_2\ldots u_{i-1}.$$

Let $L_1=u_1u_2\ldots u_m$, $L_2=w_1w_2\ldots w_k$ where $u_m=w_1$. We define:

(7) $$L_1+L_2=u_1u_2\ldots u_mw_2\ldots w_k.$$

For $L=u_1u_2\ldots u_{i-1}u_iu_{i+1}\ldots u_m$ and $C=w_1w_2\ldots w_jw_1$, where $u_i=w_1$, we define:

(8) $$L +_i C=u_1u_2\ldots u_{i-1}w_1w_2\ldots w_jw_1u_{i+1}\ldots u_m.$$

Let $L=u_1u_2\ldots u_{i-1}u_iu_{i+1}\ldots u_m$. We define:

(9) $$L_{/i}=(L_1,L_2),\text{where } L_1=u_1\ldots u_{i-1}u_i, L_2=u_iu_{i+1}\ldots u_m.$$

Let $L=u_1\ldots u_{i-1}u_iu_{i+1}\ldots u_{j-1}u_ju_{j+1}\ldots u_m$, where $u_i=u_j, i<j$. We define:

(10) $$L_{/i,j}=(L_1,C),\text{ where } L_1=u_1\ldots u_{i-1}u_iu_{j+1}\ldots u_m \text{ and } C=u_iu_{i+1}\ldots u_{j-1}u_j.$$

In what follows, the last operation applied to cycles will play an essential role.

A pair $C_{/i,j}=(C_1,C_2)$ will be called a decomposition of C into cycles C_1 and C_2 at positions i and j. A cycle $C=u_1\ldots u_mu_1$ is decomposable if there exist $i,j\in\{1,2,\ldots,m\}$, $i<j$ and C_1, C_2 such that $(C_1,C_2)=C_{/i,j}$.

3. Alternating cycles and their decomposition. For two graphs $G_1=(V,E_1)$, $G_2=(V,E_2)$, the graph $G_1\dot{-}G_2=(V,E_1\dot{-}E_2)$ is the symmetric difference of G_1 and G_2. A cycle $C=(u_1n_1u_2,u_2n_2u_3,\ldots,u_mn_mu_{m+1})$ of $G_1\dot{-}G_2$ is called an alternating cycle or briefly a-cycle if the following condition is satisfied for every $i\in\{1,2,\ldots,m\}$:

(11) $u_in_iu_{i+1}\in E_1$ if i is odd and $u_in_iu_{i+1}\in E_2$ if i is even.

Now we shall study decompositions of an a-cycle into a-cycles.

Lemma 1. If $G_1=(V,E_1)$, $G_2=(V,E_2)$, then an a-cycle C of $G_1\dot{-}G_2$ is decomposable into a-cycles iff there exists a vertex v which occurs in C at two compatible positions. (Obviously, the first and the last vertex in a cycle is counted once.)

Proof. The necessity follows from the definition of an alternating cycle and from (10).

Sufficience. Let $C=u_1u_2\dots u_{i-1}vu_{i+1}\dots u_{j-1}vu_{j+1}\dots u_{2m}u_1$. Then there exists a decomposition $C_{/i,j}=(C_1,C_2)$, where $C_1=u_1u_2\dots u_{i-1}vu_{j+1}\dots u_{2m}u_i$, $C_2=vu_{i+1}\dots u_{j-1}v$. If i and j are both odd, then C_1 and C_2 are a-cycles, if i and j are both even, then C_1 and $\overleftarrow{C}_2$ are a-cycles.

Note that if v occurs in C more than twice, then obviously C is decomposable into a-cycles, since C has always two compatible positions.

If an a-cycle C is decomposable into a-cycles, we shall write briefly C is DAC, otherwise C is NDAC.

Corollary 1. An a-cycle C of a graph $G_1\dot{-}G_2$ is NDAC iff every $v\in V(C)$ occurs in C either exactly once or exactly twice and at non-compatible positions.

Let $C=u_1u_2\dots u_mu_1$ be a cycle in which for some $i,j,k,l\in\{1,2,\dots,m\}$, where $i<j<k<l$, we have $u_i=u_k=u$, $u_j=u_l=v$ and $u\neq v$. Then we say that vertices u and v occur in C alternately.

Lemma 2. Let C be an a-cycle of a graph $G_1\dot{-}G_2$ and C be NDAC. If there exist $u,v\in V(C)$ occurring in C alternately, then there exists an a-cycle C' such that $V(C')=V(C)$, $E(C')=E(C)$ and C' is DAC.

Proof. Let $C=u_1\dots u_i\dots u_j\dots u_k\dots u_l\dots u_{2m}u_1$, where $u_i=u_k=u$ and $u_j=u_l=v$. Let $C_{/i,k}=(C_1,C_2)$. We form an a-cycle $C'=C_1+_i\overleftarrow{C}_2$. Since C is NDAC, neither the positions i,k nor j,l are compatible. Therefore, C_1 and C_2 are not a-cycles, however C' is an a-cycle. Let s be the position of u_j in C'. By the definition of C', we have $s=i+(k-j)$, hence $s+j=i+k$. As $s+j$ is odd, s and j are non-compatible. Hence, s and l are compatible. Thus, by Lemma 1, we can conclude that C' is DAC.

An a-cycle C is essentially non-decomposable into a-cycles, or briefly ENDAC, if C is NDAC and there are no two vertices occurring in C alternately.

On the base of proofs of Lemmas 1 and 2 we can formulate an algorithm for the decomposition of an a-cycle into ENDAC cycles.

Algorithm 1.

INPUT: An a-cycle $C=u_1u_2\dots u_{2m}u_1$ of a graph $G_1\dot{-}G_2$.

OUTPUT: The set $\mathfrak{C}$ of ENDAC cycles such that $E(C)=\bigcup_{D\in\mathfrak{C}} E(D)$.

METHOD:

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ℂ := ∅; x:= 4; k:= 0
F: if there exist i, j such that i<j-2, u_i=u_j and j≥x
   then
     begin
       k:= k+1;
       j_k:= the smallest j such that j≥x and there exists i such that u_i=
       =u_j and i<j-2;
       i_k:= the smallest i such that u_i=u_{j_k} and i<j-2;
       x:= j_k+1
       if j_k-i_k is even
       then
         begin
           (C_k,D_k):= C/i_k,j_k; ℂ := ℂ ∪ {D_k}; C:= C_k;
           go to F
         end
       else
         if there is no y ∈ {1,2,...,k-1} such that i_y< i_k< j_y< j_k
         then go to F
         else
           begin
             s:= the smallest y ∈ {1,2,...,k-1} such that i_y< i_k< j_y< j_k;
             L_1:=u_1...u_{i_s}; L_2:=u_{i_s}...u_{i_k}; L_3:=u_{i_k}...u_{j_s}; L_4:=u_{j_s}...u_{j_k};
             L_5:=u_{j_k}...u_m u_1; D_k:= ←L_2+L_4; ℂ := ℂ ∪ {D_k}; C:= L_1+←L_3+L_5;
             go to F
           end
     end
   else
     begin
       ℂ := ℂ ∪ {C};
       STOP
     end
```

Let us denote by oc(v,C) the number of occurences of a vertex v in a cycle C.

Lemma 3. If $G_1, G_2 \in \mathcal{M}$ and C is an ENDAC cycle of $G_1 \dot{-} G_2$, then there exists $x \in V(C)$ such that oc(x,C)=1.

Proof. Assume that $oc(v,C)>1$ for every $v\in V(C)$. Since C is NDAC, by Corollary 1, we get $oc(v,C)=2$ for every $v\in V(C)$. Let i and j ($i<j$) be the positions of v in C, and let $C'=v\ldots v$ be the subcycle of C taken from the i-th position to the j-th position. We shall show that C' contains a loop. Let $l(C')$ denote the length of C'. We proceed by induction on $l(C')$.

If $l(C')=1$, then $C'=vv$ is a loop.

Assume that the statement holds for each subcycle C' of C with $l(C')<s$, $s>1$.

Let $l(C')=s$. Since $oc(w,C)=2$ for every $w\in V(C)$, there exists $u\in V(C')$ such that $u\neq v$. Since C is ENDAC, the vertices v and u do not occur alternately in C and consequently $l(C')>2$, $oc(u,C')=2$. Then, by inductive assumption, there exists a loop in the cycle $C''=u\ldots u$ being a subcycle of C'.

Thus we get a contradiction with the assumption that $G_1,G_2\in \mathcal{M}$.

Lemma 4. Let $G_1,G_2\in \mathcal{P}$ and C be an ENDAC cycle of $G_1\dot{-}G_2$, $l(C)\geq 4$ and $oc(v,C)=2$ for every $v\in V(C)$. Then there exist $x,y\in V(C)$ such that $L_1=xxyy$ or $L_2=yxxy$ is a subchain of C.

Proof. Let u, v be consecutive vertices of C and $u\neq v$. Since C is ENDAC, so $C'=uv\ldots v\ldots u$ is a subcycle of C. We shall prove, by induction on $k=l(C')$, that C' contains a subchain $L_1=xxyy$ or $L_2=xyyx$.

If $k=3$, then $C'=uvvu$.

Assume that the statement is true for every $k<s$, $s>3$.

Let $l(C')=s$. It must be: 1^o $C'=uv\ldots v\ldots u$, 2^o $C'=uvv\ldots u$.

Case 1^o. Let w be the third vertex of C'. Then C' must be of the form $C'=uvw\ldots w\ldots v\ldots u$. Hence, by the inductive assumption, there exists in $C''=vw\ldots w\ldots v$ a subchain L_1 or L_2.

Case 2^o. If $l(C')=4$, then the proof is completed. Assume that $l(C')>4$ and w is the fourth vertex of C'. Then $C'=uvvw\ldots w\ldots u$. Let z be the fifth vertex in C'. If $z=w$, then we have a subchain $L=vvww$ of C'. If $z\neq w$, then $C'=uvvwz\ldots z\ldots w\ldots u$, and the cycle $C''=wz\ldots z\ldots w$ is contained in C'. Thus the cycle C'' contains the chain of the form xxyy or xyyx, by the inductive assumption.

Theorem 1. If $C=u_1u_2\ldots u_{2m}u_1$ is an ENDAC cycle of $G_1\dot{-}G_2$, then there exists an a-cycle $C'=w_1w_2\ldots w_{2m}w_1$ such that $V(C')=V(C)$, $E(C')=E(C)$, C' is ENDAC and C' is one of the forms I-V:

I $w_i\neq w_1$ for every $i\in\{2,3,\ldots,2m\}$,

II $w_1=w_2$ and $w_3=w_{2m}$, $oc(v,C')=2$ for every $v\in V(C')$,

III $w_1=w_{2m}$ and $w_2=w_{2m-1}$, $oc(v,C')=2$ for every $v\in V(C')$,

IV $w_1=w_{2m}$ and $w_2=w_3$, $oc(v,C')=2$ for every $v\in V(C')$, V $w_1=w_2$ and $w_{2m-1}=w_{2m}$, $oc(v,C')=2$ for every $v\in V(C')$.

Proof. Assume that there exists a vertex v in C such that $oc(v,C)=1$ and i is its position in C. Then $C'=\vec{C}^{i}$ for odd i or $C'=(\overleftrightarrow{C}^{i})$ for even i satisfies condition I.

Assume that $oc(v,C)=2$ for every $v\in V(C)$. Then, by Lemma 4, there exists a subchain $L=u_iu_{i+1}u_{i+2}u_{i+3}$ of the form yxxy or xxyy. In case 1, if i is even, then $C'=\vec{C}^{i+1}$ satisfies II, if i is odd, then $C'=\vec{C}^{i+2}$ satisfies III, in case 2, if i is even, then $C'=\vec{C}^{i+1}$ satisfies IV, if i is odd, then $C'=$ $=\vec{C}^{i+2}$ satisfies V.

Obviously C' is ENDAC in each of the cases.

Remark 1. Theorem 1 provides an easy one-pass method for transforming an ENDAC cycle into an a-cycle which is of type I - V.

4. A-cycles and realizations of a degree sequence

Lemma 5. Let d be a graphic sequence, $G_1,G_2\in \mathbb{R}_V(d)$ and $G_1=(V,E_1)$, $G_2=$ $=(V,E_2)$. Then every non-trivial component of $G_1\dot{-}G_2$ is an Eulerian graph with at least 4 edges and each component has an alternating Euler cycle.

Proof. Since for every $v\in V$ we have

$$|\{e\in E_1\setminus E_2:\ e \text{ inc } v\}|=|\{e\in E_2\setminus E_1:\ e \text{ inc } v\}|,$$

so every non-trivial component of $G_1\dot{-}G_2$ has an alternating Euler cycle.

From (4) it follows:

$$m_{G_1\dot{-}G_2}(u,v)=|m_{G_1}(u,v)-m_{G_2}(u,v)| \text{ for every } u,v\in V(G_1\dot{-}G_2).$$

Thus none of the a-cycles of the graph $G_1\dot{-}G_2$ is of the form C=uvu or C=vvv.

Lemma 6. Let $G_1,G_2\in \mathbb{R}_V(d)$, $G_1=(V,E_1)$, $G_2=(V,E_2)$ and C be an a-cycle of the graph $G_1\dot{-}G_2$. Then the following conditions hold:

1. If $e_1=un_1v$, $e_2=wn_2z$, $e_1\in E_1\setminus E_2$, $e_2\in E_2\setminus E_1$, then $\{u,v\}\neq\{w,z\}$.
2. If u,v,w are consecutive vertices of C, then $u\neq w$.
3. If $G_1,G_2\in \mathbb{R}_V(d;\tau)$, where $\tau\in\{\mathcal{M},\mathcal{M}_k,\mathcal{S}\}$, then every three consecutive vertices of C are different.
4. $|V(C)|\geq 2$.

Proof. The first condition follows from the fact that edges are labelled both in G_1 and in G_2 starting from 1. Conditions 2 - 4 follow from condition 1.

Let $\mathbb{C}$ be a set of a-cycles of the graph $G_1\dot{-}G_2$ such that $\bigcup_{C\in\mathbb{C}}E(C)=$

$=E(G_1 \dot{-} G_2)$. We shall say that $\mathbb{C}$ is an a-cyclic partition of $G_1 \dot{-} G_2$ if each edge of $E(G_1 \dot{-} G_2)$ belongs to exactly one of the a-cycles in $\mathbb{C}$.

If $\{C_1, C_2, \ldots, C_r\}$ is an a-cyclic partition of $G_1 \dot{-} G_2$, then we can form a sequence $(C_1, C_2, \ldots, C_r)$. We say that an a-cycle $C_k = u_1 u_2 \ldots u_{2m} u_1$ $(k=1,2,\ldots,r)$ is closed the most quickly in the sequence $(C_1, \ldots, C_r)$ if for every $s \in \{2,3,\ldots,m-1\}$ and $n \in Z^+$ the following condition holds:

$$u_{2s} n u_1 \in E(G_2) \setminus E(G_1) \Rightarrow u_{2s} n u_1 \in \bigcup_{i \in \{1,\ldots,k-1\}} E(C_i)$$

A sequence $C=(C_1, C_2, \ldots, C_r)$ is called a proper a-cyclic partition of $G_1 \dot{-} G_2$ if for every $k \in \{1,2,\ldots,r\}$, C_k is closed the most quickly.

Example. Let

$E(G_1) \setminus E(G_2) = \{v_1 1 v_2, v_1 1 v_3, v_2 3 v_7, v_3 2 v_4, v_5 1 v_6, v_5 2 v_6, v_5 1 v_8, v_7 1 v_8\}$,
$E(G_2) \setminus E(G_1) = \{v_1 3 v_6, v_1 1 v_8, v_2 1 v_3, v_2 2 v_8, v_3 1 v_5, v_4 1 v_5, v_5 1 v_7, v_6 1 v_7\}$.

Put $C_1 = v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_1$, $C_2 = v_2 v_7 v_5 v_6 v_1 v_3 v_5 v_8 v_2$, $D_1 = v_1 v_2 v_3 v_4 v_5 v_6 v_1$, $D_2 = v_2 v_7 v_5 v_6 v_7 v_8 v_1 v_3 v_5 v_8 v_2$.

Then the partitions $C=(C_1 C_2)$ and $C'=(C_2, C_1)$ are not proper, because $v_6 3 v_1 \in E(G_2) \setminus E(G_1)$ and $v_6 3 v_1 \in E(C_2)$, and similarly $v_3 1 v_2 \in E(G_2) \setminus E(G_1)$ and $v_3 1 v_2 \in E(C_1)$. The partition $C''=(D_1, D_2)$ is a proper a-cyclic partition of $G_1 \dot{-} G_2$.

Remark 2. An a-cyclic partition of $G_1 \dot{-} G_2$ for $G_1, G_2 \in \mathbb{R}_V(d)$ can be constructed using an arbitrary algorithm for finding an Eulerian a-cycle in an Eulerian graph, where the edges should be chosen from G_1 and G_2 in an alternating way. To find a proper a-cyclic partition of $G_1 \dot{-} G_2$ we can use such an algorithm requiring additionally every cycle to be closed the most quickly.

Let $G, H \in \mathbb{R}_V(d;\tau)$ and $G \neq H$. A sequence $G=G^0, G^1, \ldots, G^k = H$ will be called a sequence of intermediate graphs for (G,H) if $G^i \in \mathbb{R}_V(d;\tau)$ and $G^i = sw(G^{i-1})$ for $i \in \{1,2,\ldots,k\}$.

Theorem 2. Let $G, H \in \mathbb{R}_V(d)$ and let $\mathbb{C} = (C_1, C_2, \ldots, C_r)$ be a proper a-cyclic partition of $G \dot{-} H$. If $C_1 = u_1 u_2 \ldots u_{2m} u_1$, then there exists a graph $G^{m-1} \in \mathbb{R}_V(d)$ and a sequence of intermediate graphs $G=G^0, G^1, \ldots, G^{m-1}$ for (G, G^{m-1}) such that $\mathbb{C}' = (C_2, \ldots, C_r)$ is a proper a-cyclic partition of $G^{m-1} \dot{-} H$.

Proof. We shall prove the theorem by induction on m.

For $m=2$ we have $C_1 = u_1 u_2 u_3 u_4 u_1$. From Lemma 6, $u_1 \neq u_3$ and $u_2 \neq u_4$. Let $e_1 = u_1 n_1 u_2$, $e_2 = u_2 n_2 u_3$, $e_3 = u_3 n_3 u_4$, $e_4 = u_4 n_4 u_1$, where n_1, n_2, n_3, n_4 satisfy conditions (3) of Section 1. Then we have:

(12) $\qquad e_1, e_3 \in E(G), \; e_2, e_4 \notin E(G)$

Hence, we can take $G^1 = G_{(e_1,e_2,e_3,e_4)}$.

We can assume that $C_1=(e_1,e_2,e_3,e_4)$, hence $E(G^1 \dot{-} H)=E(\mathfrak{C}')$, where $\mathfrak{C}'=(C_2,\ldots,C_r)$. Therefore $\mathfrak{C}'$ is a proper a-cyclic partition of $G^1 \dot{-} H$.

Assume that the theorem holds for a cycle C_1 of the length $l=2(m-1)$.

Let $C_1=u_1u_2\ldots u_{2m}u_1$ and e_1,e_2,e_3,e_4 satisfy condition (12). From the definition of an a-cycle it follows that $e_1,e_3 \in E(G)\setminus E(H)$, $e_2 \in E(H)\setminus E(G)$. Since $n_4 > m_G(u_4,u_1)$, so $e_4 \notin E(G)$. Put $G^1=G_{(e_1,e_2,e_3,e_4)}$. Since C_1 is closed the most quickly and $l(C_1)>4$, so $e_4 \notin E(H)$. Thus $E(G^1 \dot{-} H)=(E(G \dot{-} H)\setminus\{e_1,e_2,e_3\})\cup\{e_4\}$.

We have $\mathfrak{C}^1=(C',C_2,\ldots,C_r)$, where $C'=u_1u_4\ldots u_{2m}u_1$ and $\mathfrak{C}^1$ is a proper a-cyclic partition of $G^1 \dot{-} H$. Now we can use the inductive assumption.

Remark 3. On the base of the proof of Theorem 2 one can easily formulate an algorithm for the reducing of the first a-cycle in a proper a-cyclic partition of $G \dot{-} H$, where $G,H \in \mathbb{R}_V(d;\mathcal{P})$.

The next theorem concerns the sequences of intermediate graphs in the family $\mathbb{R}_V(d;\tau)$, where $\tau = \mathcal{P}_k$ for $k \geq 2$ or $\tau = \mathcal{M}_k$ for $k \geq 1$. We assume that $\mathcal{M} = \mathcal{M}_k$ for $k=\infty$. Note that the assumption $k \geq 2$ is essential, since for two graphs of type $\mathcal{P}_1$ there need not exist a sequence of intermediate graphs of type $\mathcal{P}_1$ (see Fig. 1).

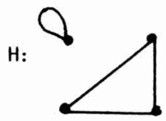

Fig. 1

Theorem 3. Let $G,H \in \mathbb{R}_V(d;\tau)$, where $\tau=\mathcal{P}_k$ for $k \geq 2$ or $\tau=\mathcal{M}_k$ for $k \geq 1$, and $\mathfrak{C}=(C_1,C_2,\ldots,C_n)$ be an a-cyclic partition of the graph $G \dot{-} H$ such that every cycle is of the form I - V (see Th. 1) . Assume that $C_1=u_1u_2\ldots$ $\ldots u_{2m}u_1$ and $(s_0,s_1,\ldots,s_p)$ is a sequence of all positive integers such that:

$$(13)\quad \begin{cases} 1=s_0<s_1<\ldots<s_p=m, \\ m_G(u_1,u_{2i})<k \text{ for } i \in \{s_1,s_2,\ldots,s_p\}, \\ m_G(u_1,u_{2i})=k \text{ for } i \in (\{2,3,\ldots,m\}\setminus\{s_1,s_2,\ldots,s_p\}). \end{cases}$$

Then there exists a graph $G' \in \mathbb{R}_V(d;\tau)$ and there exists a sequence

$$(14)\quad G=G_0^{s_0},G_1^1,\ldots,G_1^{s_1-s_0},\ G_2^1,\ldots,G_2^{s_2-s_1},\ldots,G_p^1,\ldots G_p^{s_p-s_{p-1}}=G'$$

of intermediate graphs for (G,G') such that $\mathfrak{C}'=(C_2,C_3,\ldots,C_n)$ is an a-cyclic partition of the graph $G' \dot{-} H$.

Proof. We shall consider C_1 as a sequence $(e_1,e_2,\dots,e_{2m})$ of edges from $E(G \dot{-} H)$, where e_i is incident with u_i and u_{i+1} for $i=1,2,\dots,2m-1$, and the edge e_{2m} is incident with u_{2m} and u_1.

Denote:

$$(15)\qquad \begin{array}{l} f_1=e_1, \\ f_j=u_1 n_j u_{2j}, \text{ where } n_j=\begin{cases} m_G(u_1,u_{2j})+1 & \text{for } j\in\{s_1,s_2,\dots,s_p\}, \\ k & \text{for } j\in(\{2,3,\dots,m\}\setminus\{s_1,\dots,s_p\}), \end{cases} \end{array}$$

$$(16)\qquad k(r)=\begin{cases} s_0 & \text{for } r=0, \\ s_r-s_{r-1} & \text{for } r=1,2,\dots,p. \end{cases}$$

For $r\in\{1,2,\dots,p\}$ and $i\in\{1,2,\dots,k(r)\}$ we define:

$$(17)\qquad G_r^i=G_a^b(u_1,u_{2q},u_{2q+1},u_{2q+2})=G_a^b(f_q,e_{2q},e_{2q+1},f_{q+1}),$$

where $q=s_r-i$ and $a=r-1$, $b=k(r-1)$ if $i=1$,
$a=r$, $b=i-1$ if $i\neq 1$.

Fig. 2 shows how to construct initial elements of the sequence (14). By means of thick continuous lines we draw these edges of C_1 which belong to $E(G)\setminus E(H)$, by a dashed line we draw edges of C_1 which belong to $E(H)\setminus E(G)$.

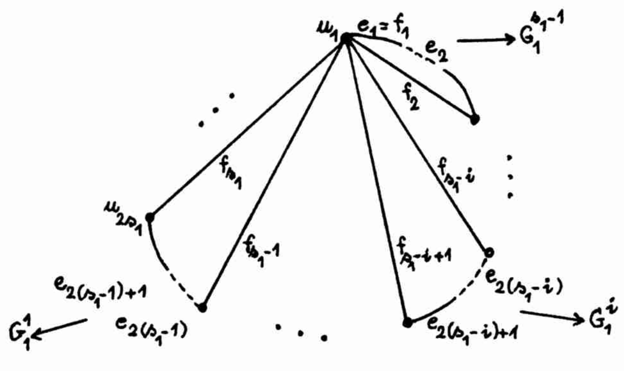

Fig. 2

First let us observe that $f_1=e_1$, $f_m=f_{s_p}=e_{2m}$. We prove that the remaining edges are pairwise distinct. In fact, $e_i\neq e_j^p$ for $i\neq j$ as being edges of C_1; $f_i\neq e_j$ for $i\in\{2,3,\dots,m-1\}$, $j\in\{2,3,\dots,2m-1\}$ since f_i is incident with u_1 and e_j is not (C_1 is of the form I - V); $f_i\neq f_j$ for $i\neq j$ since $u_{2i}\neq u_{2j}$ as being vertices of an NDAC cycle.

We shall show that the switching operations defined hy (17) can be realized, that is, the following conditions are satisfied:

1) $u_1, u_{2q}, u_{2q+1}, u_{2q+2}$ are pairwise different,

2) $f_q \neq e_{2q+1}$, $e_{2q} \neq f_{q+1}$,

3) $f_q, e_{2q+1} \in E(G_a^b)$, $e_{2q}, f_{q+1} \notin E(G_a^b)$,

4) $m_{G_a^b}(u_{2q}, u_{2q+1}) < k$, $m_{G_a^b}(u_{2q+2}, u_1) < k$.

Condition 1) follows from Lemma 6 and from the assumption that C_1 is of the form I - V; condition 2) follows from the above considerations.

Let $r \in \{1,2,\ldots,p\}$, $i \in \{2,3,\ldots,k(r)\}$ and $q = s_r - i$. From (13) and (15) it follows that $f_q, f_{q+1} \in E(G)$, however, from the definition of an a-cycle of $G \dot{-} H$ we have $e_{2q+1} \in E(G) \setminus E(H)$ and $e_{2q} \in E(H) \setminus E(G)$. Let us note that the edges f_q, e_{2q+1}, e_{2q} have not taken part in the earlier switching operations, so $f_q, e_{2q+1} \in E(G_r^{i-1})$ and $e_{2q} \notin E(G^{i-1})$, whereas the edge f_{q+1} has been removed from the graph G_r^{i-1} in the preceding switching operation, hence $f_{q+1} \notin E(G_r^{i-1})$. Thus condition 3) is satisfied.

Since $e_{2q} \in E(H) \setminus E(G)$ and $e_{2q} \notin E(G_r^{i-1})$, so $m_{G_r^{i-1}}(u_{2q}, u_{2q+1}) < k$. Further, since $f_{q+1} \in E(G_r^{i-2}) \setminus E(G_r^{i-1})$, so $m_{G_r^{i-1}}(u_{2q+2}, u_1) < k$. From that it follows that condition 4) is satisfied.

Similarly we prove that conditions 3) and 4) hold if i=1.

From (17) it follows that for $r=1,2,\ldots,p-1$ we have:

$$E(G_r^{k(r)}) = (E(G) \setminus \{e_1, e_3, \ldots, e_{2s_r-1}\}) \cup \{f_{s_r}\} \cup \{e_2, e_4, \ldots, e_{2s_r-2}\},$$

whereas for r=p

$$E(G_p^{k(p)}) = (E(G) \setminus \{e_1, e_3, \ldots, e_{2s_p-1}\}) \cup \{e_2, e_4, \ldots, e_{2s_p-2}, e_{2s_p}\}$$

since, by $s_p = m$, we have $f_{s_p} = e_{2s_p}$.

Thus we can conclude that $E(G' \dot{-} H) = E(G \dot{-} H) \setminus E(C_1)$, and consequently, the sequence $C' = (C_2, \ldots, C_n)$ is an a-cyclic partition of the graph $G' \dot{-} H$.

Remark 4. On the base of the proof of Theorem 3 one can formulate an algorithm for the reducing of the first a-cycle of the form I - V in a-cyclic partition of $G \dot{-} H$, where $G, H \in \mathbb{R}_V(d; \tau)$ for $\tau \in \{\mathcal{P}_k, \mathcal{M}_k, \mathcal{S}\}$, $k \geq 2$.

Now we give a procedure of finding a sequence of intermediate graphs for (G,H), where $G, H \in \mathbb{R}_V(d; \tau)$.

Algorithm 2.

1. Find a proper a-cyclic partition $\mathbb{C} = (C_1, C_2, \ldots, C_n)$ of the graph $G \dot{-} H$,

here $G,H \in \mathbb{R}_V(d;\tau)$. If $\tau = \mathcal{P}$, go to 3.

2. Decompose each cycle C_i of C onto ENDAC cycles and transform each of them to a-cycle of type I - V. Denote also by $\mathbb{C}$ the resulting a-cyclic partition of $G \dot{-} H$.

3. For every cycle of $\mathbb{C}$ use Remark 3 if $\tau \in \{\mathcal{P}, \mathcal{M}\}$ and use Remark 4 if $\tau \in \{\mathcal{P}_k, \mathcal{M}_k, \mathcal{S}\}$ for $k \geq 2$.

Finally we look for the shortest sequence of intermediate graphs for (G,H). Let $G=G^0, G^1, \dots, G^k=H$ be a sequence of intermediate graphs for (G,H). The number k will be called the length of this sequence. The least number k for which there exists a sequence of intermediate graphs for (G,H) will be denoted by $k_0(G,H)$. Therefore $k_0(G,H)$ is the least number of switching operations which must be done to reach H starting from G. In this process we have to take only such switching operations which decrease the number of edges of the graph $G \dot{-} H$. Note that a switching operation applied once to an a-cycle C decreases the number of edges by 2 if $|E(C)| > 4$ and by 4 if $|E(C)|=4$. Hence

(18) $$\frac{s}{2} \leq k_0(G,H) \leq s-1, \text{ where } s=|E(G \dot{-} H)|.$$

The equality $k_0(G,H)=\frac{s}{2}$ holds if each of the edges of $G \dot{-} H$ occurs in a 4-edge a-cycle, and $k_0(G,H)=s-1$ if all edges of $G \dot{-} H$ occur in a given one 2s-edge a-cycle.

Thus we obtain a shortest sequence for (G,H) if the a-cyclic partition of $G \dot{-} H$ which we apply in Step 2 of the last procedure has the greatest number of a-cycles. However, Algorithm 1 does not assure that we deal with an optimal a-cyclic partition of $G \dot{-} H$.

Thus, we pose the following

<u>Problem.</u> Give an algorithm for finding a decomposition of an a-cycle into the greatest number of a-cycles.

Let us notice that (18) can be improved using Lemma 1. Then we get

$\frac{s}{2} \leq k_0(G,H) \leq s-\frac{\Delta}{4}$, where $\Delta = \max\{\deg_{G \dot{-} H}(v)\}_{v \in V(G \dot{-} H)}$.

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Institute of Mathematics of Pedagogical University, Oleska 48, 45-052 Opole, Poland

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