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A REMARK ON THE WEAK TOPOLOGY OF THE HILBERT SPACE

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Abstract: V.V. Uspenskiĭ [A] asked if every $\aleph_0$-space can be embedded in an $\aleph_0$-space with property k_R. It is shown that the Hilbert space l_2 endowed with the weak topology provides a negative answer to this question.

Key words: Hilbert space, weak topology, $\aleph_0$-space, k_R-space.

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1. Introduction. Let us recall that a regular space X is an $\aleph_0$-space if X has a countable k-network $\mathcal{R}$, i.e. a collection of subsets (not necessarily open) such that whenever $K\subset U$ with K compact and U open in X, then $K\subset P\subset U$ for some $P\in\mathcal{R}$; the class of $\aleph_0$-spaces was introduced by E. Michael [M1], where we refer the reader for the basic properties. A completely regular space X is a k_R-space if arbitrary function $f:X\to R$, whose restriction to every compact $K\subset X$ is continuous on X, see [M2].

V.V. Uspenskii [A] asked if every $\aleph_0$-space can be embedded in an $\aleph_0$-space with property k_R. In this note we shall show that the Hilbert space l_2 endowed with the weak topology (which is an $\aleph_0$-space, see [M1, Cor. 7.10]) provides a negative answer to this question:

Theorem 1. The infinite-dimensional separable Hilbert space equipped with the weak topology cannot be embedded into any $\aleph_0$-space being a k_R-space.

Let us notice that our reasoning shows also that a well-known space V considered by Varadarajan [V, p.98]: the natural numbers extended by the filter of the complements of density 0 sets, provides another example of this kind.[x)]

x) This example was considered also by P. Uryson (see P.S. Aleksandrov, P.S. Uryson: Memuar o kompaktnych topologičeskich prostranstvach, 3rd edition, Moscow 1971 (pp. 119-120)). (Referee's remark)

We shall denote by N the natural numbers and by $|A|$ the cardinality of the set A.

2. Fernique's space F. We shall denote by l_2 the Hilbert space of the square-summable sequences of the real numbers. Let $e_1, e_2, \ldots$ be the standard orthonormal basis in l_2. Following Fernique [HJ, p.268] we shall consider the following subspace of l_2:

$$F = \bigcup_{i,j \in N} \{ie_j\} \cup \{0\}$$

equipped with the topology induced by the weak topology of l_2, i.e. the points ne_i are isolated in F and basic neighbourhoods of the point 0 in F are of the form:

$$(*) \qquad V = \{ne_i : |n\alpha_i| < 1\} \cup \{0\}, \text{ where } \sum_{i=1}^{\infty} \alpha_i^2 < \infty.$$

We shall need the following observation about the space F:

Lemma 2. Let $W_1 \supset W_2 \supset \ldots$ be a sequence of open sets in the space F such that $\bigcap_{i=1}^{\infty} W_i = \{0\}$. Then there exists a set $Y \subset F$ satisfying the conditions: $0 \in \overline{Y}$, $|Y - W_i| < \infty$, for $i = 1, 2, \ldots$ and no sequence of points of the set Y converges to 0.

Proof: Choose inductively for each $n = 1, 2, \ldots$, pairwise disjoint sets $A_n \subset N$ such that $|A_n| = n^2$ and $Y_n = \{ne_i : i \in A_n\} \subset W_n$. We shall show that $Y = \bigcup Y_n$ has the required property. Each set $Y - W_n \subset Y_1 \cup \ldots \cup Y_{n-1}$ is finite and obviously no sequence from Y converges to 0, so it is enough to show that $0 \in \overline{Y}$. Aiming at a contradiction, assume that there exists a neighbourhood V of the form $(*)$ with $Y \cap V = \emptyset$. Then, for each $i \in A_n$, $|n\alpha_i| \geq 1$, but then $\sum_{i \in A_n} \alpha_i^2 \geq |A_n| \frac{1}{n^2} = 1$, which contradicts the fact that the sequence $\alpha_1, \alpha_2 \ldots$ is square summable.

3. Proof of Theorem 1. Let X be any χ_0-space containing the space F defined in sec. 2. We shall show that X is not a k_R-space.

The point 0 is a G_δ-set in X hence there exist sets in X such that

$$W_1 \supset \overline{W}_2 \supset W_2 \supset \ldots \text{ and } \{0\} = \bigcap_{i=1}^{\infty} W_i.$$

By Lemma 2 we can find a set $Y \subset F$ such that $0 \in \overline{Y}$, $|Y - W_i| < \infty$ for $i \in N$ and no sequence of points of Y converges to 0.

Let $y_1, y_2, \ldots$ be an enumeration of the elements of Y. We shall choose an open neighbourhood V_i in X of the points y_i satisfying the following conditions:

(i) $V_i \cap F = \{y_i\}$

(ii) $\overline{\bigcup_{i=1}^{\infty} V_i} \subset \bigcup_{i=1}^{\infty} \overline{V}_i \cup \{0\}$,

(iii) $\overline{V}_i \cap \overline{\bigcup_{j \neq 1} V_j} = \emptyset$.

(iv) no sequence of points of the set $\bigcup_{i=1}^{\infty} V_i$ converges to 0.

To this end we define inductively open sets $U_1, U_2, \ldots$ in X such that $U_i \cap F = \{y_i\}$, $U \notin \overline{U}_i$ for every $i \in N$, $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i \neq j$ and if $y_i \in W_m$ then $U_i \subset W_m$. It is easy to check that $\overline{\bigcup_{i=1}^{\infty} U_i} \subset \bigcup_{i=1}^{\infty} \overline{U}_i \cup \{0\}$. Indeed, if $q \neq 0$ then there exists $m \in N$ such that $q \notin \overline{W}_m$ and the open neighbourhood $X - \overline{W}_m$ of the point q intersects only finitely many sets U_i. In a similar way one can verify that $\overline{U}_i \cap \overline{\bigcup_{j \neq i} U_j} = \emptyset$.

Let us consider a k-network in X consisting of closed sets, let $S_1, S_2, \ldots$ be an enumeration of the elements of the k-network containing 0 and let

$$V_i = U_i - \bigcup \{S_j : j \leq i \text{ and } y_i \notin S_j\}.$$

Obviously, the conditions (i)-(iii) are satisfied. We shall check that (iv) holds as well. Assume on the contrary that there exists a compact set $Z \subset \bigcup_{i=1}^{\infty} V_i$ homeomorphic with a convergent sequence, 0 being the limit point, and let $P = \{y_i \in Y : V_i \cap Z \neq \emptyset\}$; since $0 \notin \overline{V}_i$, the set P is infinite. By the choice of Y, no sequence from Y converges to 0, hence there exists a neighbourhood W of 0 such that P-W is infinite. The set Z-W is finite, so $Z - W \subset \bigcup_{i \leq i_0} V_i$ for some i_0, and the set $Z \cap W$ is compact, so $Z \cap W \subset S_{j_0} \subset W$ for some j_0. Consider $y_{n_0} \in P - W$ with $n_0 > \max(i_0, j_0)$. Then

$$V_{n_0} \cap (Z - W) = \emptyset \text{ and}$$

$$V_{n_0} \cap (Z \cap W) \subset V_{n_0} \cap S_{j_0} = \emptyset \text{ as } y_{n_0} \notin S_{j_0}.$$

Therefore $V_{n_0} \cap Z = \emptyset$, a contradiction with the definition of the set P.

Now, for every $n \in N$ we define a continuous function $f_n : X \to R$ equal to 0 on the set $X - V_n$, and 1 on $\{y_n\}$. Put $f = \max f_n$. In particular, f equals 1 on Y and $f(0)=0$ and since $0 \in \overline{Y}$, f is not continuous at 0. By conditions (i)-(iii) it follows that 0 is the unique point of discontinuity of f.

We shall show that f is continuous on each compact set $K \subset X$, just violating the k_R-property. Let $K \subset X$ be a compact set containing 0. Since compact sets in any χ_0-space are metrizable, condition (iv) implies that $0 \notin \overline{K \cap \bigcup_{i \in N} V}$. It follows that for some neighbourhood W of 0, the function f vanishes on the set $W \cap K$. Hence the restriction $f_{|K}$ is continuous at 0 and f being continuous at any other point in X, $f_{|K}$ is continuous.

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