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DISTRIBUTIVITY IN FINITELY GENERATED ORTHOMODULAR LATTICES

Ladislav BERAN

Abstract: The purpose of this paper is to characterize the distributivity of a finitely generated orthomodular lattice F by the semiprimality of the ideal determined by the lower commutator formed from generators of F.

Key words: Commutativity relation, commutators, distributivity criterion, orthomodular lattice, semiprime ideal.

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1. Preliminaries. In [3] Rav introduced the concept of a semiprime ideal which is an ideal I of a lattice L satisfying

$$x \wedge y \in I \ \&\ x \wedge z \in I \Rightarrow x \wedge (y \vee z) \in I$$

for every $x,y,z \in L$. Here we use this notion as a principal tool for our investigation.

Let L be an orthomodular lattice and let $x_1, x_2, \dots, x_n \in L$. Recall that the upper commutator of $x_1, x_2, \dots, x_n$ is defined by

$$\bar{c} = \overline{\mathrm{com}}(x_1 x_2, \dots, x_n) = \bigwedge (x_1^{e_1} \vee x_2^{e_2} \vee \dots \vee x_n^{e_n})$$

where the superscripts $e_1, e_2, \dots, e_n$ run over $\{-1,1\}$ and $x_i^1 = x_i$, $x_i^{-1} = x_i'$. Dually is defined the lower commutator

$$\underline{c} = \underline{\mathrm{com}}(x_1, x_2, \dots, x_n) = \bigvee (x_1^{e_1} \wedge x_2^{e_2} \wedge \dots \wedge x_n^{e_n})$$

(cf. [2],[1]).

As usual, we write aCb if and only if $a = (a \wedge b) \vee (a \wedge b')$.

Any undefined terminology in this paper will generally conform with [1].

2. Distributivity criterion

Lemma 1. Let $x_1, x_2, \dots, x_n$ be elements of an orthomodular lattice L and let $(\underline{\mathrm{com}}(x_1, x_2, \dots, x_n)]$ be semiprime. Then

$$x_1 \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n)] = x_1 \wedge x_2 \wedge \ldots \wedge x_n.$$

Proof: Let

$$x = x_1 \wedge \bar{c},\ y = x_1',\ z = (x_2 \wedge \ldots \wedge x_n) \vee \underline{c}.$$

Since $\bar{c}C(x_2 \wedge \ldots \wedge x_n)$ and $\bar{c}C\underline{c}$,

$$x \wedge z = x_1 \wedge \bar{c} \wedge [(x_2 \wedge \ldots \wedge x_n) \vee \underline{c}] = x_1 \wedge \bar{c} \wedge (x_2 \wedge \ldots \wedge x_n) \leqq$$
$$\leqq (x_1 \wedge x_2 \wedge \ldots \wedge x_n) \wedge (x_1' \vee x_2' \vee \ldots \vee x_n') = 0.$$

Now, $I = (\underline{c}]$ is semiprime and $x \wedge y = 0 \in I$. Hence $x \wedge (y \vee z) \in I$. Since $\bar{c}Cx_1'$, $\bar{c}C(x_2 \wedge \ldots \wedge x_n)$ and $\bar{c}C\underline{c}$, we have

$$x \wedge (y \vee z) = x_1 \wedge \bar{c} \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n) \vee \underline{c}] =$$
$$= x_1 \wedge \bar{c} \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n)].$$

From $x \wedge (y \vee z) \in I$ we conclude that

$$x_1 \wedge \bar{c} \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n)] \leqq \bar{c} \wedge \underline{c} = 0.$$

Thus

$$x_1 \wedge \bar{c} \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n)] = 0.$$

But

$$x_1 \wedge \bar{c} \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n)] =$$
$$= x_1 \wedge (x_1' \vee x_2' \vee \ldots \vee x_n') \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n)].$$

Let

$$s = x_1 \wedge [x_1' \vee (x_2 \wedge \ldots \wedge x_n)],\ t = (x_1' \vee x_2' \vee \ldots \vee x_n').$$

Then $s \wedge t = 0$ and $s \geqq t'$, so that $s = t'$, by orthomodularity of L.

Corollary 2. If $(\underline{com}(x_1, x_2, \ldots, x_n)]$ is semiprime in an orthomodular lattice, then

$$x_1 C(x_2^{e_2} \wedge \ldots \wedge x_n^{e_n})$$

for any $e_2, \ldots, e_n \in \{-1, 1\}$.

Proof: By symmetry it suffices to prove that $x_1 C(x_2 \wedge \ldots \wedge x_n)$. However, $a\check{C}b$ if and only if $a \wedge (a' \vee b) = a \wedge b$, by [1; Theorem II.3.7]. Consequently, Lemma 1 gives the required result.

Proposition 3. Let $(\underline{com}(x_1, x_2, \ldots, x_n)]$ be a semiprime ideal of an orthomodular lattice. Then

$$\underline{com}(x_1, \ldots, x_n) = \underline{com}(x_2, \ldots, x_n) = \ldots = \underline{com}(x_{n-1}, x_n) = 1.$$

Proof: By Corollary 2 we have $x_1' C(x_2^{e_2} \wedge \ldots \wedge x_n^{e_n})$, so that

$$\underline{\mathrm{com}}(x_1,x_2,\dots,x_n)=\bigvee[x_1\wedge(x_2^{e_2}\wedge\dots\wedge x_n^{e_n})]\vee\bigvee[x_1'\wedge(x_2^{e_2}\wedge\dots\wedge x_n^{e_n})]=$$
$$=[x_1\wedge\bigvee(x_2^{e_2}\wedge\dots\wedge x_n^{e_n})]\vee[x_1'\wedge\bigvee(x_2^{e_2}\wedge\dots\wedge x_n^{e_n})]=$$
$$=(x_1\vee x_1')\wedge\bigvee(x_2^{e_2}\wedge\dots\wedge x_n^{e_n})=\underline{\mathrm{com}}(x_2,\dots,x_n).$$

The remainder follows by induction. Especially,

$$\underline{\mathrm{com}}(x_{n-1},x_n)=\underline{\mathrm{com}}(x_n)=x_n\vee x_n'=1.$$

Corollary 4. Let $x_1,x_2,\dots,x_n$ be elements of an orthomodular lattice such that $(\underline{\mathrm{com}}(x_1,x_2,\dots,x_n)]$ is semiprime. Then $x_i C x_j$ for every $i,j\in\{1,2,\dots\dots,n\}$.

Proof: From symmetry and from Proposition 3 we infer $\underline{\mathrm{com}}(x_i,x_j)=1$ for every $1\leq i\neq j\leq n$. However, $\underline{\mathrm{com}}(x_i,x_j)=1$ is equivalent to $\overline{\mathrm{com}}(x_i,x_j)=[\underline{\mathrm{com}}(x_i,x_j)]'=1'=0$ and this is equivalent to $x_i C x_j$ (cf. [1; Theorem III,2.11]).

Theorem 5. Let F be a finitely generated orthomodular lattice, $F=\langle x_1,\dots,x_n\rangle$. Then F is distributive if and only if $(\underline{\mathrm{com}}(x_1,\dots,x_n)]$ is semiprime.

Proof: 1. If F is distributive, then every ideal of F is semiprime.

2. Suppose, conversely, that $(\underline{\mathrm{com}}(x_1,\dots,x_n)]$ is semiprime. By Corollary 4, $x_i C x_j$ for every $1\leq i,\ j\leq n$, and the proof is completed by applying [1; Theorem II.4.5].

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Department of Algebra, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

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