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EXISTENCE OF SOLUTIONS OF THE DARBOUX PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract: We consider the existence of solutions of the classical Darboux problem for the partial differential equation $u'''_{x_1x_2x_3}=$ $=f(x_1,x_2,x_3,u,u'_{x_1},u'_{x_2},u'_{x_3},u''_{x_1x_2},u''_{x_1x_3},u''_{x_2x_3})$ via a fixed point theorem of Sadovskii. Here f is a continuous function with values in a Banach space satisfying some regularity condition expressed in terms of the measure of noncompactness α.

Key words: Hyperbolic partial differential equations, Darboux conditions, existence solutions in a Banach space, measure of noncompactness.

Classification: 35A05, 35L15, 34G20.

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1. Introduction. In the present note we consider the following hyperbolic partial differential equation:

$$(+)\quad u'''_{x_1x_2x_3}=f(x_1,x_2,x_3,u,u'_{x_1},u'_{x_2},u'_{x_3},u''_{x_1x_2},u''_{x_1x_3},u''_{x_2x_3})$$

with suitable initial boundary conditions of the Darboux type.

Equations of the type (+) (in Euclidean spaces) are considered in papers by Kwapisz, Palczewski and Pawelski [9], Conlan [5], Castellano [3], Palczewski [11], Frasca [8], Chu and Diaz [4], and others. Below, we prove the existence theorem for the case where f is a continuous function with values in a Banach space satisfying some regularity condition expressed in terms of the measure of noncompactness α . The proof is based on the fixed point theorem of Sadovskii ([12], Theorem 3.4.4).

2. Notations and preliminaries. Let a_i (i=1,2,3) be positive real numbers. We put $I_i=[0,a_i]$ and $V=I_1\times I_2\times I_3$. Throughout this paper E is a Banach space with norm $\|\cdot\|$, and f is an E-valued continuous function defined on the product $\Omega=V\times E\times E^3\times E^3$. By C(V,E) we represent the standard Banach space of all E-valued continuous functions on V. Moreover, let $C^*(V,E)$ denote the class of E-valued functions $(x_1,x_2,x_3)\mapsto u(x_1,x_2,x_3)$ continuous on V to-

gether with their partial derivatives $u'_{x_1}, u'_{x_2}, u'_{x_3}, u''_{x_1x_2}, u''_{x_1x_3}, u''_{x_2x_3}$ and $u'''_{x_1x_2x_3}$.

The measure of noncompactness $\alpha(A)$ of a nonempty bounded subset A of E is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter $\leq \varepsilon$. For the properties of α the reader is referred to [2],[6],[7],[12].

We shall use in the sequel the following immediate adaptation of Lemma 2.2 of [1] (cf. [10]): If P is a compact subset of V, then $\alpha(\cup\{W(\xi): \xi \in P\}) = \sup\{\alpha(W(\xi)): \xi \in P\}$ for a bounded equicontinuous subset W of C(V,E) (here $W(\xi)$ stands for the set of all $w(\xi)$ with $w \in W$).

We state the Sadovskii fixed point theorem as follows.

Let $\mathfrak{X}$ be a closed convex subset of C(V,E). Let Φ be a function which maps each nonempty subset W of $\mathfrak{X}$ to a real nonnegative $\Phi(W)$ with (1) $\Phi(\{w\} \cup W) = \Phi(W)$ for $w \in \mathfrak{X}$, (2) $\Phi(\overline{\text{conv}}\, W) = \Phi(W)$ ($\overline{\text{conv}}\, W$ is the closed convex hull of W), and (3) if $\Phi(W)=0$ then $\overline{W}$ (the closure of W) is compact in C(V,E). Assume that F is a continuous mapping of $\mathfrak{X}$ into itself such that $\Phi(F[W]) < \Phi(W)$ whenever $\Phi(W) > 0$. Then F has a fixed point in $\mathfrak{X}$.

3. Formulation of the problem and result. We write $J_{jk} = I_j \times I_k$ for $j,k = 1,2,3$ with $j<k$. Let us determine E-valued functions σ_1, σ_2 and σ_3 continuous respectively on J_{23}, J_{13} and J_{12}, including the second mixed derivatives, and fulfilling the conditions

$$\sigma_1(0,x_3) = \sigma_2(0,x_3), \quad \sigma_1(x_2,0) = \sigma_3(0,x_2), \quad \sigma_2(x_1,0) = \sigma_3(x_1,0)$$

for $x_i \in I_i$ $(i=1,2,3)$.

By (PD) we shall denote the problem of finding a function $u \in C^*(V,E)$ satisfying (+) and the initial conditions

$$u(0,x_2,x_3) = \sigma_1(x_2,x_3), \quad u(x_1,0,x_3) = \sigma_2(x_1,x_3), \quad u(x_1,x_2,0) = \sigma_3(x_1,x_2)$$

for all (x_j, x_k) in J_{jk}.

We shall write the right side of (+) shortly as $f(\xi,u,R,Q)$, where $\xi = (x_1,x_2,x_3)$ and $R=(r_1,r_2,r_3)$, $Q=(q_{12},q_{13},q_{23})$ with $r_i(\xi) = u'_{x_i}(\xi)$, $q_{jk}(\xi) = u''_{x_jx_k}(\xi)$. Moreover, let $\Theta=(0,0,0)$ (here 0 is the zero of E).

Our result reads as follows.

Theorem. Let f be uniformly continuous on bounded subsets of Ω. Assume that the following conditions hold:

1^0 $\|f(\xi,u,\Theta,\Theta)\| \leq c_1 + c_2\|u\|$ for $\xi \in V$ and $u \in E$.

2^0 $\|f(\xi,u,R,Q)-f(\xi,u,\bar{R},\bar{Q})\| \le \omega(\sum_i \|r_i-\bar{r}_i\| + \sum_{j<k} \|q_{jk}-\bar{q}_{jk}\|)$

for $(\xi,u,R,Q) \in \Omega$ and $(\xi,u,\bar{R},\bar{Q}) \in \Omega$, where $t \mapsto \omega(t)$ is a nonnegative continuous nondecreasing and subadditive function with $\omega(0)=0$ only for $t=0$ and

$$\int_0^{\eta} \frac{dt}{\omega(t)} = +\infty$$

for $\eta > 0$.

3^0 $\alpha(f[\xi,A]) \le L \cdot \max\{\alpha(A_i): 1 \le i \le 7\}$ for $\xi \in V$ and any set A which is the product of nonempty bounded subsets A_i of E.

Under these assumptions, the problem (PD) admits at least one solution on V.

Proof. Put $u'''_{x_1x_2x_3}=s$. For the convenience we assume that $\sigma_i \equiv 0$ for i=1,2,3. Then, (PD) is equivalent to solving the functional-integral equation

$$(*)\quad s(x,y,z)=f(x,y,z, \int_0^x\int_0^y\int_0^z s(t_1,t_2,t_3)dt_1dt_2dt_3,$$
$$\int_0^y\int_0^z s(x,t_2,t_3)dt_2dt_3,\ \int_0^x\int_0^z s(t_1,y,t_3)dt_1dt_3,$$
$$\int_0^x\int_0^y s(t_1,t_2,z)dt_1dt_2,$$
$$\int_0^z s(x,y,t_3)dt_3,\ \int_0^y s(x,t_2,z)dt_2,\ \int_0^x s(t_1,y,z)dt_1)$$

in C(V.E).

Let $\lambda=1+c_1+c_2+7\,\omega(1)$. Let Γ be the set of all $(\xi,u,R,Q) \in \Omega$ such that $\|u\| \le \lambda^{-2}\exp(3\lambda)$, $\|r_i\| \le \lambda^{-1}\exp(3\lambda)$ and $\|q_{jk}\| \le \exp(3\lambda)$ for i,j,k= =1,2,3 with j< k. We set:

$$\tilde{\omega}(\eta)=\sup\{\|f(\xi,u,R,Q)-f(\bar{\xi},\bar{u},R,Q)\| : (\xi,u,R,Q),\ (\bar{\xi},\bar{u},R,Q) \in \Gamma \text{ with } \|u-\bar{u}\| + \sum_i |x_i-\bar{x}_i| \le \eta\}$$

and

$$\rho(\eta)=\tilde{\omega}((1+\exp(3\lambda))\eta)+\omega((2+\lambda)\exp(3\lambda)\eta)$$

for $\eta \ge 0$.

According to the lemma of [11] the equation

$$h(x,y;\eta)=\rho(\eta)+\omega(\int_0^x\int_0^y h(t_1,t_2;\eta)dt_1dt_2 + \int_0^x h(t_1,y;\eta)dt_1 + \int_0^y h(x,t_2;\eta)dt_2)$$

has a continuous solution h such that $h(x,y,0) \equiv 0$. Denote by $\mathfrak{X}$ the set of all $w \in C(V,E)$ with

$$\|w(\xi)\| \le \lambda \cdot \exp(\lambda \sum_i x_i)$$

and

$$\|w(\xi)-w(\bar\xi)\| \le h(\bar x_1,\bar x_2;|\bar x_3-x_3|)+h(\bar x_1,x_3;|\bar x_2-x_2|)+h(x_2,x_3;|\bar x_1-x_1|)$$

for $\xi=(x_1,x_2,x_3)\in V$ and $\bar\xi=(\bar x_1,\bar x_2,\bar x_3)\in V$.

Let F be determined by the right side of $(*)$. It is easy to verify that $\mathfrak{X}$ is a closed con vex equicontinuous and bounded subset of $C(V,E)$, and F is a continuous mapping of $\mathfrak{X}$ into itself.

Let $r>\max(1,L)$. Define

$$\Phi(W)=\sup\{\exp(-r\xi)\,\alpha(W(\xi)):\ \xi\in V\}$$

for a nonempty subset W of $\mathfrak{X}$. By properties of α and Ascoli theorem, our function Φ satisfies the conditions (1) - (3) listed in Section 2.

Let W be a subset of $\mathfrak{X}$ with $\Phi(W)>0$. To prove the theorem it remains to be shown that $\Phi(F[W])<\Phi(W)$.

Fix (x,y,z) in V. Consider the continuous function $\psi(\xi)=\alpha(W(\xi))$. Let $\varepsilon>0$ be arbitiary and $\sigma=\sigma(\varepsilon)$ a positive number such that $\xi'=(t_1',t_2',t_3')\in V$ and $\xi''=(t_1'',t_2'',t_3'')\in V$ with $|t_i'-t_i''|<\sigma$ (i=1,2,3) implies $|\psi(\xi')-\psi(\xi'')|<\varepsilon$. We divide the intervals $[0,x]$, $[0,y]$ and $[0,z]$ into m parts

$$x_0=0<x_1<\ldots<x_m=x,\quad y_0=0<y_1<\ldots<y_m=y,\quad z_0=0<z_1<\ldots<z_m=z$$

in such a way that

$$\max\{|x_i-x_{i-1}|,\ |y_i-y_{i-1}|,\ |z_i-z_{i-1}|: i=1,2,\ldots,m\}<\sigma.$$

Define

$$P_{ijk}=[x_{i-1},x_i]\times[y_{j-1},y_j]\times[z_{k-1},z_k],\quad W_{ijk}=\bigcup\{W(\xi):\ \xi\in P_{ijk}\}$$

for $i,j,k=1,2,\ldots,m$. Moreover, let ξ_0 be a point in P_{ijk} such that $\psi(\xi_0)=\sup\{\psi(\xi):\ \xi\in P_{ijk}\}$.

Denote by $A_0=\int_0^x\int_0^y\int_0^z W(t_1,t_2,t_3)dt_1dt_2dt_3$ the set of all $\int_0^x\int_0^y\int_0^z w(t_1,t_2,t_3)dt_1dt_2dt_3$ with $w\in W$. Applying the integral mean value theorem we obtain

$$\alpha(A_0)\le\alpha\Big(\sum_{i,j,k=1}^{m}\operatorname{mes}(P_{ijk})\overline{\operatorname{conv}}(W_{ijk})\Big)=$$

$$=\sum_{i,j,k=1}^{m}\operatorname{mes}(P_{ijk})\sup\{\psi(\xi):\ \xi\in P_{ijk}\}\le\sum_{i,j,k=1}^{m}\iiint_{P_{ijk}}(|\psi(t_1,t_2,t_3)-\psi(\xi_0)|+$$

$$+\psi(t_1,t_2,t_3))dt_1dt_2dt_3<\varepsilon xyz+\int_0^x\int_0^y\int_0^z\psi(t_1,t_2,t_3)dt_1dt_2dt_3\le$$

$$\le\varepsilon xyz+\Phi(W)\int_0^x\int_0^y\int_0^z\exp(r(t_1+t_2+t_3))dt_1dt_2dt_3;$$

therefore

$$\alpha(A_0)<r^{-3}\exp(r(x+y+z))\cdot\Phi(W).$$

Further, by A_i (i=1,2,3) and A_{ijk} (j,k=1,2,3) with $j<k$) we represent

the sets

$$\int_0^y\int_0^z W(x,t_2,t_3)dt_2dt_3,\quad \int_0^x\int_0^z W(t_1,y,t_3)dt_1dt_3,\quad \int_0^x\int_0^y W(t_1,t_2,z)dt_1dt_2$$

and

$$\int_0^z W(x,y,t_3)dt_3,\quad \int_0^y W(x,t_2,z)dt_2,\quad \int_0^x W(t_1,y,z)dt_1,$$

respectively. Arguments analogous to the above imply that

$$\alpha(A_i) < r^{-2}\cdot\exp(r(x+y+z))\cdot\Phi(W)$$

and

$$\alpha(A_{jk}) < r^{-1}\exp(r(x+y+z))\cdot\Phi(W).$$

Consequently,

$$\alpha(F[W](x,y,z)) \le$$

$$\le L\cdot\max\{\alpha(A_0),\ \alpha(A_i),\ \alpha(A_{jk}): i,j,k=1,2,3 \text{ with } j<k\} <$$

$$< r^{-1}L\cdot\exp(r(x+y+z))\cdot\Phi(W)$$

for all $(x,y,z)\in V$. This shows that $\Phi(F[W])\le r^{-1}L\cdot\Phi(W)$. Now, applying Sadovskii´s theorem, we infer that F has a fixed point in $\mathfrak{X}$ and the proof is complete.

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$$\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = f(x_1, x_2, x_3, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^2 u}{\partial x_1, \partial x_3}, \frac{\partial^2 u}{\partial x_2 \partial x_3}),$$

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