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MINIMAL CONVEX-VALUED WEAK* USCO CORRESPONDENCES AND THE RADON-NIKODÝM PROPERTY

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Abstract: We show that the minimal convex-valued weak* usco correspondences form a suitable generalization of maximal monotone operators. Using these correspondences, we develop Fitzpatrick's result about generic continuity of monotone operators and characterize closed convex sets with the Radon--Nikodým property.

Key words: Asplund space, Baire space, Banach space, convex analysis, convex function, maximal monotone operator, minimal convex-valued weak* usco correspondence, strongly weak* exposed point, subdifferential map, support function, sublinear functional, weak* dentable set.

Classification: 46B20, 46B22

0. Introduction

The minimal convex valued weak* usco correspondences have been introduced in [16] to prove that a Banach space is in the Stegall class $\mathcal{S}$ [27] whenever there is a weak* lower semicontinuous rotund function on its dual. In the present paper we use these correspondences to develop the following theorem due to S. Fitzpatrick.

0.1. Theorem [9]. Let X be a real Banach space and let K be a closed linear subspace of the dual Banach space X^* such that every bounded subset of K is weak* dentable. Let T be a monotone operator on X and D be an open subset of X. If $Tx \neq \emptyset$ for x in D and $K \cap Tx \neq \emptyset$ for x in a dense subset of D, then T is single-valued and norm to norm upper semicontinuous at each point of a dense G_δ subset of D.

Basic properties of minimal convex-valued weak* usco correspondences are given in Section 1. Their connection with convex analysis is described in Section 2. Main results are contained in Section 3. Closed convex sets with the Radon-Nikodým property are characterized in Section 4 (Corollary 4.4).

Here a closed convex subset K of a Banach space is said to have the Radon-Nikodým property (abbreviated RNP) if every closed convex bounded subset of K is the closed convex hull of its strongly exposed points [4].

Theorems 2.11, 3.5 and 3.15 form a skeleton of the present paper.

Theorem 2.11 is suggested by the works of P. S. Kenderov [20] and J. P. R. Christensen and P. S. Kenderov [7].

Theorem 3.5 generalizes Theorem 0.1 on account of Theorem 2.1 and the "three convex sets lemma" [25, Lemma 2.2], [4, Thm. 4.3.1 (w^*)].

Theorem 3.15 is suggested by the works [3], [23], [24], [8], [25] due to E. Bishop, I. Namioka, R. R. Phelps and J. B. Collier. Many results of these works are analysed in Giles' book [12].

Theorem 2.1 and Corollary 4.4 have been preliminarily communicated in [17].

1. Weak* convex-valued usco correspondences

Throughout the paper it will be assumed that D and Y are topological spaces. In applications D will be a Baire space (i. e. every open nonempty subset of D is of the second Baire category) and Y will be of the form (X^*, w^*), where X^* is a dual Banach space and w^* is its weak* topology.

We define the set $m(D, Y)$ writing $F \in m(D, Y)$ if and only if F is a set-valued correspondence assigning a nonempty subset $F(d)$ of Y to each point $d \in D$. The set $m(D, Y)$ will be considered as a partially ordered set with order $\leqq$, defining, for $E, F \in m(D, Y)$, $E \leqq F$ if and only if $E(d) \subset F(d)$ holds whenever $d \in D$. For $F \in m(D, Y)$, $G \subset D$ and $M \subset Y$ we put

$$F(G) := \bigcup \{ F(d) : d \in G \},$$

$$(1) \qquad F^{-1}(M) := \{ d \in D : M \cap F(d) \neq \emptyset \}.$$

According to [7] we denote by USCO(D, Y) the set of all usco correspondences [7] from D into Y, therefore, $F \in$ USCO(D, Y) if and only if $F \in m(D, Y)$ and F is an upper semicontinuous compact-valued correspondence.

We define usco(D, Y) to be the set of all minimal elements (relative to order $\leqq$) of the set USCO(D, Y). Minimal usco correspondences have been used, for instance, in [6], [7], [21], [26], [27] and [16].

<u>1.1. Theorem</u> [7]. Let Y be a Hausdorff space and F be in USCO(D, Y). Then there exists a correspondence $E \in$ usco(D, Y) having the property $E \leqq F$.

Minimal usco correspondences can be characterized by the following way.

<u>1.2. Theorem</u> [16]. Let Y be a Hausdorff space and F be in USCO(D, Y). Then the following conditions are equivalent.

(i) $F \in$ usco(D, Y).

(ii) The implication $G \subset F^{-1}(M) \Rightarrow F(G) \subset M$ is satisfied whenever G is an open subset of D and M is a closed subset of Y.

(iii) For every pair $[G, V]$, where G is open in D, V is open in Y and $V \cap F(G) \neq \emptyset$, there exists an open set U with the properties

$$\emptyset \neq U \subset G, \quad F(U) \subset V .$$

In what follows it will be considered a real Banach space $X \neq \{0\}$. We denote by X^* the corresponding dual Banach space and by w^* the weak* topology for the set X^*.

For any set $M \subset X^*$ we write $\bar{M}$, $\bar{M}^*$ and $\overline{co}^* M$ for the norm closure, weak* closure and weak* closed convex hull of the set M, respectively.

<u>1.3. Definition</u> [16]. The weak* convexification of a correspondence $F \in m(D, X^*)$ is the correspondence co $F \in m(D, X^*)$ defined by the formula

$$(\text{co } F)(d) := \overline{co}^* F(d) \text{ whenever } d \in D .$$

<u>1.4. Proposition</u> [16]. $F \in USCO(D,(X^*, w^*)) \Rightarrow \text{co } F \in USCO(D,(X^*,w^*))$.
Accordingly to [16] we define

$$USCOC(D,(X^*,w^*)) := \{ F \in USCO(D,(X^*,w^*)) : \text{co } F = F \} .$$

Thus, $F \in USCOC(D,(X^*,w^*))$ holds if and only if, using the weak* topology, F is a convex-valued usco correspondence from D into X^*.

We denote by $uscoc(D,(X^*,w^*))$ the set of all minimal elements (relative to order $\leq$) of the set $USCOC(D,(X^*,w^*))$.

<u>1.5. Theorem</u> [16]. Let F be in $USCOC(D,(X^*,w^*))$. Then there exists a correspondence $E \in uscoc(D,(X^*,w^*))$ with the property $E \leq F$.

There is a characterization of the set $uscoc(D,(X^*,w^*))$ similar to Theorem 1.2.

<u>1.6. Theorem</u> [16]. Let F be in $USCOC(D,(X^*,w^*))$. Then the following conditions are equivalent.

(i) $F \in \mathrm{uscoc}(D,(X^*,w^*))$.

(ii) The implication $G \subset F^{-1}(M) \Rightarrow F(G) \subset M$ is satisfied whenever G is an open subset of D and M is a weak* closed convex subset of X^*.

(iii) For every pair [G, M], where G is an open subset of D, M is a weak* closed convex subset of X^* and $F(G) \cap (X^* \setminus M) \neq \emptyset$, there exists an open set U with the properties

$$\emptyset \neq U \subset G, \quad F(U) \subset X^* \setminus M .$$

(iv) For every pair [G, H], where G is an open subset of D, H is a weak* open halfspace in X^* and $F(G) \cap H \neq \emptyset$, there exists an open set U with the properties

$$\emptyset \neq U \subset G, \quad F(U) \subset H .$$

<u>1.7. Corollary</u> [16]. $F \in \mathrm{usco}(D,(X^*,w^*)) \Rightarrow \mathrm{co}\, F \in \mathrm{uscoc}(D,(X^*,w^*))$.

<u>1.7'. Corollary</u> [16]. If $E \in \mathrm{usco}(D,(X^*,w^*))$, $F \in \mathrm{uscoc}(D,(X^*,w^*))$ and $E \leqq F$, then $\mathrm{co}\, E = F$.

Theorem 1.1 and Corollaries 1.7 and 1.7' tell us that the weak* convexification maps the set $\mathrm{usco}(D,(X^*,w^*))$ onto the set $\mathrm{uscoc}(D,(X^*,w^*))$.

<u>1.8. Corollary</u> [7]. Let D be a Baire space and F be in $\mathrm{usco}(D,(X^*,w^*))$. Then the correspondence F is openly locally bounded on D, that is, for every open nonempty subset G of D there is an open nonempty subset U of G such that the set F(U) is bounded.

<u>1.8'. Corollary</u>. Let D be a Baire space and F be in $\mathrm{uscoc}(D,(X^*,w^*))$. Then the correspondence F is openly locally bounded on D.

<u>Proof.</u> Following the idea due to J. P. R. Christensen

and P. S. Kenderov [7], we take in consideration an open nonempty set $G \subset D$ and the corresponding dual unit ball B of X^* (being a weak* compact barrel in X^*). As $X^* = \bigcup\{n B : n = 1, 2, \dots\}$, we have

$$G = G \cap D = G \cap F^{-1}(X^*) = G \cap F^{-1}\left(\bigcup_{n=1}^{\infty} n B\right) = \bigcup_{n=1}^{\infty} (G \cap F^{-1}(n B)).$$

The set G endowed with the relativized topology is a Baire space and each set $G \cap F^{-1}(n B)$ is closed in G. Therefore there are an open set U and a natural number n with $\emptyset \neq U \subset G \cap F^{-1}(n B)$. We have $\emptyset \neq U \subset G$ and $U \subset F^{-1}(n B)$. It follows $F(U) \subset n B$ by virtue of Condition (ii) of Theorem 1.6.

We note that Corollary 1.8, too, is a consequence of Corollary 1.8′ on account of Corollary 1.7.

<u>1.9. Definition.</u> Let F be in $m(D, X^*)$. Then the set $C(F, D, X^*)$ is defined as follows : $d \in C(F, D, X^*)$ if and only if $d \in D$ and, using the norm topology of X^*, F is upper semicontinuous and single-valued at d.

<u>1.10. Proposition</u>. Suppose $F \in m(D, X^*)$ and $d \in D$. Then $d \in C(F, D, X^*)$ if and only if there exists an $x^* \in X^*$ such that for every norm neighbourhood V of the point $0 \in X^*$ there exists an open set $G \subset D$ with the properties $d \in G$ and $F(G) \subset x^* + V$.

In what follows we fix a countable local basis $\mathcal{V}$ for the norm topology of X^* formed by weak* closed absolutely convex sets. For instance, it can be supposed

$$\mathcal{V} = \{n^{-1} B : n = 1, 2, \dots\},$$

where B is the dual unit ball in X^*.

The complete proof of the following technical lemma is given in [16].

1.11. Lemma. Let $F \in m(D, X^*)$ and $G(F, V) := \bigcup \{G \subset D : G$ is open and $F(G) - F(G) \subset V\}$ for each $V \in \mathcal{V}$. Then $C(F, D, X^*) = \bigcap \{G(F, V) : V \in \mathcal{V}\}$.

1.12. Remark. As every set $G(F, V)$ is open and $\mathcal{V}$ is a countable family, the set $C(F, D, X^*)$ always is a G_δ subset of D.

The following corollary can be regarded as a method to prove that $C(F, D, X^*)$ is a dense G_δ subset of D.

1.13. Corollary. Let D be a Baire space and let F be in $m(D, X^*)$. If for every pair $[G, V]$, where G is an open nonempty subset of D and $V \in \mathcal{V}$, there exists an open set U with the properties $\emptyset \neq U \subset G$ and $F(U) - F(U) \subset V$, then $C(F, D, X^*)$ is a dense G_δ subset of D.

Proof. If G is an arbitrary open nonempty subset of D and $V \in \mathcal{V}$, then, by hypothesis, the open set $G(F, V)$ meets G and hence $G(F, V)$ is dense in D. Applying Baire Category Theorem and Lemma 1.11., we obtain the required result.

1.14. Proposition [16]. Let F be in $usco(D, (X^*, w^*))$ (or in $uscoc(D, (X^*, w^*))$), E be in $m(D, X^*)$ and $E \leqq F$. Then $C(E, D, X^*) = C(F, D, X^*)$.

Proof. Since the inclusion $C(F, D, X^*) \subset C(E, D, X^*)$ is obvious, it suffices to prove the converse. Let $d \in C(E, D, X^*)$, $V \in \mathcal{V}$ and $x^* \in E(d)$. Then there is an open set $G \subset D$ with $d \in G$ and $E(G) \subset x^* + V$. As $E \leqq F$, it follows

$$G \subset F^{-1}(E(G)) \subset F^{-1}(x^* + V).$$

Now Condition (ii) of Theorem 1.2 (or Theorem 1.6) tells us that $F(G) \subset x^* + V$. Hence $d \in C(F, D, X^*)$, by Proposition 1.10.

2. Connection with convex analysis

Let $f : X \to \overline{R}$ be a convex function. The subdifferential map $\partial f : X \to X^*$ is defined by setting $\partial f(x) := \emptyset$ if $f(x) \notin R$ and

$$\partial f(x) := \bigcap \left\{ \{ x^* \in X^* : \langle h, x^* \rangle \leqq f(x+h) - f(x) \} : h \in X \right\}$$

if $f(x) \in R$. Here $\langle .,. \rangle$ denotes the pairing between X and X^*. If $f(x) \in R$ and $\varepsilon > 0$ then the ε - subdifferential $\partial_\varepsilon f(x)$ of f at the point $x \in X$ is defined by

$$\partial_\varepsilon f(x) := \bigcap \left\{ \{ x^* \in X^* : \langle h, x^* \rangle \leqq f(x+h) - f(x) + \varepsilon \} : h \in X \right\}.$$

If the function f is finite and continuous on an open set $D \subset X$ then, according to Moreau's result [22], the restriction $\partial f \mid D$ of the subdifferential map ∂f to the set D belongs to USCOC$(D,(X^*,w^*))$. Now, using monotonicity of subdifferential maps and applying Kenderov's result [20] (for it, see the proof of Theorem 1.28 of [25], too), we see that the correspondence $\partial f \mid D$ satisfies Condition (iv) of Theorem 1.6.

Similarly, let us consider a maximal monotone operator $T : X \to X^*$ having the property that $Tx \neq \emptyset$ for any x in an open set $D \subset X$. Then, accordingly to Kenderov's results [18], [20], the restriction $T \mid D$ of T to D belongs to USCOC$(D,(X^*,w^*))$ and satisfies Condition (iv) of Theorem 1.6 as well. Therefore it holds the following

2.1. Theorem. Let D be an open subset of X, $f : X \to \overline{R}$ be a convex function finite and continuous on D and let $T : X \to X^*$ be a maximal monotone operator such that $Tx \neq \emptyset$ whenever $x \in D$. Then both correspondences $\partial f \mid D$ and $T \mid D$ belong to uscoc$(D,(X^*,w^*))$.

Let $\{M_\gamma : \gamma \in (\Gamma, \leqq)\}$ be a net of nonempty subsets of the dual Banach space X^* and $x^* \in X^*$. Then we write

$$\lim_{\gamma \in \Gamma} M_\gamma = x^*$$

if and only if for every $V \in \mathcal{V}$ (see Section 1) there is a γ_0 in Γ with $\bigcup \{ M_\gamma : \gamma_0 \leqq \gamma \in \Gamma \} \subset x^* + V$.

2.2. Theorem [2], [11]. Let $f : X \to \bar{R}$ be a convex function finite and continuous on an open set $D \subset X$, $x_0 \in X$ and $x_0^* \in X^*$. Then the following conditions are equivalent.

(i) The Fréchet derivative $f'(x_0)$ of f at x_0 is x_0^*.

(ii) $\lim_{\varepsilon \downarrow 0} \partial_\varepsilon f(x_0) = x_0^*$.

(iii) $x_0 \in C(\partial f \mid D, D, X^*)$ and $x_0^* \in \partial f(x_0)$.

(iv) There exists a correspondence $F \in m(D, X^*)$ such that

$$F \leqq \partial f \mid D, \quad x_0 \in C(F, D, X^*) \text{ and } x_0^* \in F(x_0).$$

2.3. Remark. The equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) are due to E. Asplund and R. T. Rockafellar [2] and the implication (iv) $\Longrightarrow$ (i) is due to J. R. Giles [11]. The implication (iv) $\Longrightarrow$ (iii) follows from Theorem 2.1 and Proposition 1.14.

Let $p : X \to R$ be a sublinear functional. It is a well-known fact [14] that, at any point $x \in X$, it holds

(2) $$\partial p(x) = \{ x^* \in \partial p(0) : \langle x, x^* \rangle = p(x) \}.$$

This relation can be modified as follows.

2.4. Proposition [16]. Let $p : X \to R$ be a sublinear functional. Then for every pair $[\varepsilon, x]$, where $\varepsilon > 0$ and $x \in X$, it holds

$$\partial_\varepsilon p(x) = \{ x^* \in \partial p(0) : \langle x, x^* \rangle \geqq p(x) - \varepsilon \}.$$

If $x \in X$ and $M \subset X^*$, then, following [14], we set

$$s(x \mid M) := \sup \{ \langle x, x^* \rangle \ : x^* \in M \} .$$

The function p defined on X by the formula

$$p(x) := s(x \mid M) \text{ whenever } x \in X$$

is called the support function of the set M.

The next theorem catalogizes some well-known facts about continuous sublinear functionals and support functions [13].

2.5. Theorem. Let $p : X \longrightarrow R$ be a continuous sublinear functional and M be a bounded nonempty subset of X^*. Then
(i) $s(. \mid M)$ is a continuous sublinear functional on X,
(ii) $p = s(. \mid \partial p(0))$ and
(iii) $p = s(. \mid M) \Longrightarrow \overline{co}^* M = \partial p(0)$.

2.6. Definition [23]. Let M be a bounded nonempty subset of X^*, $0 \neq x \in X$, $\alpha > 0$ and let $p : X \longrightarrow R$ be the support function of the set M. Then the weak* slice of the set M determined by x and α is the set

$$S(M, x, \alpha) := \{ x^* \in M : \langle x, x^* \rangle > p(x) - \alpha \} .$$

In the proof of the following lemma we shall apply the well-known inclusion

$$\overline{M} \cap G \subset \overline{M \cap G}$$

satisfied for any $M \subset Y$ and any open $G \subset Y$.

2.7. Lemma. Let M be a convex bounded nonempty subset of X^*, $0 \neq x \in X$, $0 < \varepsilon < \alpha$ and let $p : X \longrightarrow R$ be the support function of M. Then

$$\partial_{\varepsilon} p(x) \subset \overline{S(M, x, \alpha)}^* \subset \partial_{\alpha} p(x).$$

Proof. Define

$$H_\alpha \ := \{ x^* \in X^* : \langle x, x^* \rangle > p(x) - \alpha \} ,$$

$$H^\varepsilon \ := \{ x^* \in X^* : \langle x, x^* \rangle \geqq p(x) - \varepsilon \} .$$

According to Proposition 2.4 and Theorem 2.5 we have

$$\partial_\varepsilon p(x) = \partial p(0) \cap H^\varepsilon = \bar{M}^* \cap H^\varepsilon \subset \bar{M}^* \cap H_\alpha \subset$$

$$\subset \overline{M \cap H_\alpha}^* = \overline{S(M, x, \alpha)}^* \subset \overline{\bar{M}^* \cap \bar{H}^*_\alpha}^* =$$

$$= \overline{\partial p(0) \cap H^\alpha}^* = \partial p(0) \cap H^\alpha = \partial_\alpha p(x).$$

In [23] I. Namioka and R. R. Phelps gave the definition of strongly weak* exposed points for weak* compact convex subsets of dual Banach spaces. This definition can be slightly extended as follows.

2.8. Definition. Let M be a convex bounded nonempty subset of X^*, $0 \neq x \in X$ and $x^* \in X^*$. Then the element x strongly exposes the set M at x^* if and only if it holds

$$\lim_{\alpha \downarrow 0} S(M, x, \alpha) = x^* .$$

A point $x^* \in X^*$ is said to be a strongly weak* exposed point of the set M provided that there is an element $0 \neq x \in X$ strongly exposing the set M at x^*.

2.9. Proposition. Let M be a convex bounded nonempty subset of X^*, $0 \neq x \in X$, $x^* \in X^*$ and let $p : X \longrightarrow R$ be the support function of the set M. Then the element x strongly exposes the set M at the point x^* if and only if $p'(x) = x^*$. Further, every strongly weak* exposed point of M belongs to $\bar{M}$.

Proof. Consider the following relations:

(i) $\lim_{\alpha \downarrow 0} S(M, x, \alpha) = x^*$,

(ii) $\lim_{\alpha \downarrow 0} \overline{S(M, x, \alpha)}^* = x^*$,

(iii) $\lim_{\varepsilon \downarrow 0} \partial_\varepsilon p(x) = x^*$ and

(iv) $p'(x) = x^*$.

As the family $\mathcal{V}$ consists of weak* closed subsets, (i) is equivalent to (ii). The equivalences (ii)$\Longleftrightarrow$(iii) and (iii)$\Longleftrightarrow$(iv) follow from Lemma 2.7 and Theorem 2.2, respectively. Further it follows from (i) that $x^* \in \mathbf{M}$.

<u>2.10. Lemma</u>. Let M be a convex bounded nonempty subset of X^*, E be the set of all strongly weak* exposed points of M and let $p : X \longrightarrow R$ be the support function of M. Then

$$\{ x \in X : x \neq 0 \text{ and } p'(x) \text{ exists} \} \subset \{ x \in X : p(x) = s(x \mid E) \} .$$

<u>Proof</u>. Suppose that $0 \neq x \in X$ and $p'(x)$ exists. Then $p'(x) \in E$ and $p'(x) \in \partial p(x)$ by Theorem 2.2. As $E \subset \mathbf{M} \subset \mathbf{M}^*$ and $s(. \mid \mathbf{M}^*) = p$, it follows from (2) that

$$p(x) = \langle x, p'(x) \rangle \leqq s(x \mid E) \leqq s(x \mid \mathbf{M}^*) = p(x) .$$

We close this section by the theorem proved firstly in [15] for subdifferential maps of continuous convex functions.

<u>2.11. Theorem</u>. Let F be in usco$(D,(X^*,w^*))$ (or in uscoc$(D,(X^*,w^*))$), G be an open nonempty subset of D such that the set $F(G)$ is bounded and let $p : X \longrightarrow R$ be the support function of the set $F(G)$. Then for every pair $[\varepsilon, h]$, where $\varepsilon > 0$ and $0 \neq h \in X$, there exists an open set U such that

$$\emptyset \neq U \subset G, \quad F(U) \subset \partial_\varepsilon p(h) .$$

<u>Proof</u>. Consider $\varepsilon > 0$, $0 \neq h \in X$ and define

$$H_\varepsilon := \{ x^* \in X^* : \langle h, x^* \rangle > p(h) - \varepsilon \} .$$

Since $F(G) \cap H_{\varepsilon} \neq \emptyset$, there is an open set U satisfying

$$\emptyset \neq U \subset G, \quad F(U) \subset F(G) \cap H_{\varepsilon}$$

on account of Condition (iii) of Theorem 1.2 (or Condition (iv) of Theorem 1.6). It follows from Theorem 2.5 and Proposition 2.4 that

$$F(G) \cap H_{\varepsilon} \subset \partial p(0) \cap H_{\varepsilon} \subset \partial_{\varepsilon} p(h) .$$

3. Main result

In the present section we assume that K is a subset of the dual Banach space X^*.

3.1. Remark. According to (1) the following conditions are equivalent for any correspondence $F \in m(D, X^*)$.

(i) The set $F^{-1}(K)$ is dense in D.

(ii) $F(U) \cap K \neq \emptyset$ whenever U is an open nonempty subset of D.

(iii) There is a dense subset A of D satisfying $F(d) \cap K \neq \emptyset$ whenever $d \in A$.

We recall that the family $\mathcal{V}$ is a local basis for the norm topology of the dual Banach space X^* and it consists of weak* closed absolutely convex sets.

3.2. Definition. We say that a continuous sublinear functional $p : X \to R$ has arbitrarily small approximative subdifferentials provided that for each $V \in \mathcal{V}$ there is a pair $[\varepsilon, h]$ such that $\varepsilon > 0$, $0 \neq h \in X$ and $\partial_{\varepsilon} p(h) - \partial_{\varepsilon} p(h) \subset V$.

3.3. Definition. We say that a continuous sublinear functional $p : X \to R$ is K - lower semicontinuous (K - l. s. c.) on X if there exists a subset M of the set K such that $p = s(. \mid M)$.

3.4. Lemma. Suppose

(i) $F \in usco(D,(X^*,w^*))$ or $F \in uscoc(D,(X^*,w^*))$,

(ii) $F^{-1}(K)$ is dense in D,

(iii) G is an open nonempty subset of D and

(iv) the set $F(G)$ is bounded.

Then the support function $p : X \longrightarrow R$ of the set $F(G)$ is K - lower semicontinuous on X.

Proof. Fix $0 \neq h \in X$. It suffices to prove

$$p(h) - \varepsilon \leqq s(h \mid K \cap F(G)) \text{ whenever } \varepsilon > 0.$$

Fix $\varepsilon > 0$. According to Theorem 2.11 there is an open set U such that $\emptyset \neq U \subset G$ and

(3) $\quad F(U) \subset \partial_{\varepsilon} p(h)$.

According to Remark 3.1 there exists an x^* in $K \cap F(U)$. Using (3) and Proposition 2.4, we obtain

$$p(h) - \varepsilon \leqq \langle h, x^* \rangle \leqq s(h \mid K \cap F(U)) \leqq s(h \mid K \cap F(G)) .$$

3.5. Theorem. Let D be a Baire space, F be in $usco(D,(X^*,w^*))$ or in $uscoc(D,(X^*,w^*))$ and let us suppose

(i) the set $F^{-1}(K)$ is dense in D and

(ii) every continuous sublinear functional $p : X \longrightarrow R$ being K - lower semicontinuous on X has arbitrarily small approximative subdifferentials.

Then $C(F, D, X^*)$ is a dense G_{δ} subset of D.

Proof. Let G be an open nonempty subset of D and $V \in \mathcal{V}$. According to Corollary 1.13 it suffices to find an open set U with the properties

(4) $\quad \emptyset \neq U \subset G, \quad F(U) - F(U) \subset V$.

According to Corollary 1.8 or 1.8′ there is an open set Q such that $\emptyset \neq Q \subset G$ and the set F(Q) is bounded. Now let us set $p := s(. \mid F(Q))$. It follows from (i), Lemma 3.4 and Theorem 2.5 that p is a continuous sublinear functional being K - l. s. c. on X. It follows from (ii) that there is a pair $[\varepsilon, h]$ such that $\varepsilon > 0$, $0 \neq h \in X$ and

$$(5) \qquad \partial_\varepsilon p(h) - \partial_\varepsilon p(h) \subset V .$$

Theorem 2.11 tells us that there is an open set U such that $\emptyset \neq U \subset Q$ and $F(U) \subset \partial_\varepsilon p(h)$. It follows from (5) that the set U satisfies (4).

3.6. Lemma. Let $p : X \to R$ be a continuous sublinear functional. Then p is K - lower semicontinuous on X if and only if

$$(6) \qquad p = s(. \mid K \cap \partial p(0)) .$$

Proof. (6) implies that p is K - l. s. c. on X. Conversely, if $p = s(. \mid M)$ and $M \subset K$, then, according to Theorem 2.5, $M \subset K \cap \partial p(0)$ and

$$p = s(. \mid M) \leqq s(. \mid K \cap \partial p(0)) \leqq s(. \mid \partial p(0)) = p .$$

3.7. Lemma. Let $p : X \to R$ be a continuous sublinear functional such that the set $(\partial p)^{-1}(K)$ is dense in X. Then p is K - lower semicontinuous on X.

Proof. According to Remark 3.1 there is a dense subset A of X such that for each $x \in A$ there is an $x^* \in K \cap \partial p(x)$. According to (2) we have $x^* \in K \cap \partial p(0)$ and $\langle x, x^* \rangle = p(x)$. This means that the continuous functionals p and $s(. \mid K \cap \partial p(0))$ coincide on the dense set A; therefore they coincide on X everywhere. According to Lemma 3.6 p is K - l. s. c. on X.

3.8. Lemma. Let $F \in m(D, X^*)$, $d \in C(F, D, X^*)$ and $x^* \in F(d)$.

If the set $F^{-1}(K)$ is dense in D, then $x^* \in \overline{K}$.

Proof. Every norm neighbourhood of x^* contains a point of K.

3.9. Definition [23]. A bounded nonempty subset M of X^* is said to be weak* dentable provided that for each $V \in \mathcal{V}$ there exists a pair $[\alpha, x]$ such that $\alpha > 0$, $0 \neq x \in X$ and $S(M, x, \alpha) - S(M, x, \alpha) \subset V$.

3.10. Lemma. Let K be a convex subset of X^*. If every bounded nonempty subset of K is weak* dentable then every continuous sublinear functional $p : X \to R$ being K - lower semicontinuous on X has arbitrarily small approximative subdifferentials.

Proof. Let $V \in \mathcal{V}$. Suppose $p : X \to R$ is a continuous sublinear functional having the property

$$p = s(. \mid K \cap \partial p(0))$$

and take in consideration Lemma 3.6. The set $M := K \cap \partial p(0)$ is a convex bounded nonempty subset of K and $p = s(. \mid M)$. If every bounded nonempty subset of K is weak* dentable, then there is a pair $[\alpha, x]$ such that $\alpha > 0$, $0 \neq x \in X$ and

$$S(M, x, \alpha) - S(M, x, \alpha) \subset V .$$

As V is a weak* closed absolutely convex set, it holds

$$\overline{S(M, x, \alpha)}^* - \overline{S(M, x, \alpha)}^* \subset V .$$

Choose an ε such that $0 < \varepsilon < \alpha$. Lemma 2.7 tells us that $\partial_\varepsilon p(x) \subset \overline{S(M, x, \alpha)}^*$ and hence $\partial_\varepsilon p(x) - \partial_\varepsilon p(x) \subset V$.

To convert Lemma 3.7, we firstly recall one result due to E. Bishop and R. R. Phelps.

3.11. Theorem [3]. Let M be a closed convex bounded nonempty subset of X. Then there exists a dense subset A of X^* such that for each $x^* \in A$ there is an $x \in M$ with the property $\langle x, x^* \rangle = \sup \{ \langle z, x^* \rangle : z \in M \}$.

In what follows we shall assume that K is a closed convex subset of X^*. We denote by $w^* | K$ the relativized weak* topology for the set K.

The following definition is suggested by Theorem 3.11.

3.12. Definition. We shall say that the set K has the weak* Bishop-Phelps property (w^*BPP) if for every $w^* | K$ - closed convex bounded nonempty subset of K there exists a dense subset A of X such that for each $x \in A$ there is an $x^* \in X^*$ with the properties

$$(7) \qquad x^* \in M, \quad \langle x, x^* \rangle = \sup \{ \langle x, z^* \rangle : z^* \in M \} .$$

3.13. Remark. Every weak* closed convex subset of X^* has the w^*BPP. If K is a closed convex subset of a Banach space Z and $Z^* = X$, then the set K regarded as a closed convex subset of X^* has the w^*BPP by virtue of Theorem 3.11. It follows from Asplund's work [1] that, if X is an Asplund space, every closed convex subset of X^* has the w^*BPP.

3.14. Lemma. Let K have the w^*BPP and let $p : X \to R$ be a continuous sublinear functional. If p is K - l. s. c. on X then the set $(\partial p)^{-1}(K)$ is dense in X.

Proof. Suppose $p : X \to R$ is a continuous sublinear functional having the property $p = s(. | K \cap \partial p(0))$. Then the set $M := K \cap \partial p(0)$ is a $w^* | K$ - closed convex bounded nonempty subset of K and $p = s(. | M)$. Using (2) we see that the condition (7) can be expressed by $x^* \in K \cap \partial p(x)$. According to Definition 3.10 the set

$$\{ x \in X : K \cap \partial p(x) \neq \emptyset \} = (\partial p)^{-1}(K)$$

contains a dense subset of X.

3.15. Theorem. Let a closed convex subset K of the dual Banach space X^* have the weak* Bishop-Phelps property. Then the following conditions are equivalent.

(i) Every bounded nonempty subset of K is weak* dentable.

(ii) Every continuous sublinear functional $p : X \to R$ being K - lower semicontinuous on X has arbitrarily small approximative subdifferentials.

(iii) The set $C(F, D, X^*)$ is a dense G_δ subset of D whenever D is a Baire space, $F \in \mathrm{uscoc}(D,(X^*,w^*))$ and $F^{-1}(K)$ is dense in D.

(iv) The set $\{ x \in D : f'(x) \text{ exists} \}$ is a dense G_δ subset of D whenever D is an open subset of X, $f : X \to \bar{R}$ is a convex function finite and continuous on D and $(\partial f)^{-1}(K)$ is dense in D.

(v) Every continuous sublinear functional $p : X \to R$ being K - lower semicontinuous on X is Fréchet differentiable on a dense subset of X.

(vi) Every $w^* | K$ - closed convex bounded nonempty subset of K is the $w^* | K$ - closed convex hull of its strongly weak* exposed points.

(vii) Every $w^* | K$ - closed convex bounded nonempty subset of K has strongly weak* exposed points.

Proof. The implication (i) $\Longrightarrow$ (ii) follows from Lemma 3.10, (ii) $\Longrightarrow$ (iii) follows from Theorem 3.5, (iii) $\Longrightarrow$ (iv) follows from Theorems 2.1 and 2.2, (iv) $\Longrightarrow$ (v) follows from Lemma 3.14 and (vi) $\Longrightarrow$ (vii) is obvious. Thus it remains to prove the implications (v) $\Longrightarrow$ (vi) and (vii) $\Longrightarrow$ (i).

(v) $\Longrightarrow$ (vi): Let M be a $w^* | K$ - closed convex bounded nonempty subset of K, E be the set of all strongly weak* exposed points of the set M and $p := s(\cdot \mid M)$. Clearly

(8) $\quad M = K \cap \overline{M}^*$.

It follows from (v) that the set $\{x \in X : x \neq 0,\ p'(x) \text{ exists}\}$ is dense in X and, according to Lemma 2.10, this set is contained in the closed set $\{x \in X : p(x) = s(x \mid E)\}$. Hence $p = s(\cdot \mid E)$ and $\overline{co}^* E = \overline{M}^*$. Now (8) implies

$$M = K \cap \overline{co}^* E .$$

As $E \subset \overline{M} = M \subset K$, the set $K \cap \overline{co}^* E$ is the $w^* \mid K$ - closed convex hull of E.

<u>(vii) $\Longrightarrow$ (i):</u> Suppose B is a bounded nonempty subset of K and $V \in \mathcal{V}$. Let $M := K \cap \overline{co}^* B$. Then, according to Theorem 2.5,

(9) $\quad s(\cdot \mid M) = s(\cdot \mid B)$

and M is a $w^* \mid K$ - closed convex bounded nonempty subset of K. It follows from (vii) that there exist elements α, x and x^* such that $\alpha > 0$, $0 \neq x \in X$, $x^* \in X$ and

(10) $\quad S(M, x, \alpha) \subset x^* + 2^{-1} V$.

It follows from Definition 2.6 and from (9) that $S(B, x, \alpha) \subset S(M, x, \alpha)$. According to (10) we have

$$S(B, x, \alpha) - S(B, x, \alpha) \subset V ,$$

hence the set B is weak* dentable.

4. Some applications

In [7] J. P. R. Christensen and P. S. Kenderov proved that X is an Asplund space if and only if the set

$C(F, D, X^*)$ is a dense G_δ subset of D whenever D is a Baire space and $F \in usco(D,(X^*,w^*))$. Setting

$$K = X^*$$

in Theorem 3.15 and taking in consideration the equivalence (iii)$\Longleftrightarrow$(iv), we obtain the following

<u>4.1. Corollary</u>. X is an Asplund space if and only if the set $C(F, D, X^*)$ is a dense G_δ subset of D whenever D is a Baire space and $F \in uscoc(D,(X^*,w^*))$.

From the corollary the above Christensen-Kenderov result can be derived by applying of Theorem 1.1, Corollary 1.7 and Proposition 1.14. Further, Theorem 3.15 contains some characterizations of Asplund spaces which can be found in [23] and [25].

Now let us suppose that the Banach space X is of the form

$$X = Z^* ,$$

where Z is a Banach space. Setting K = Z, regarding K as a closed convex subset of X^* and taking in consideration Remark 3.13 and the equivalences (i)$\Longleftrightarrow$(vi)$\Longleftrightarrow$(vii), we have the following result due to R. R. Phelps:

<u>4.2. Theorem</u> [24]. The following conditions for a Banach space Z are equivalent.

(i) Every bounded nonempty subset of Z is dentable.

(ii) Every closed convex bounded nonempty subset of Z is the closed convex hull of its strongly exposed points.

(iii) Every closed convex bounded nonempty subset of Z has strongly exposed points.

As the properties of Theorem 4.2 characterize Banach spaces with the Radon-Nikodým property, it follows from the

Brøndsted-Rockafellar theorem [5] that the equivalence (iv) $\Longleftrightarrow$ (vi) of Theorem 3.15 gives Collier's characterization for Banach spaces with the RNP:

4.3. Theorem [8]. A Banach space Z has the Radon-Nikodým property if and only if the dual Banach space Z^* is a weak* Asplund space.

Finally, taking in consideration the equivalence (iii) $\Longleftrightarrow$ (vi) of Theorem 3.15, we obtain the following characterization for closed convex sets with the RNP.

4.4. Corollary. A closed convex subset K of a Banach space Z has the Radon-Nikodým property if and only if, regarding K as a closed subset of the second dual Banach space Z^{**}, the set $C(F, D, Z^{**})$ is a dense G_δ subset of D whenever D is a Baire space, $F \in \mathrm{uscoc}(D,(Z^{**},w^*))$ and the set $F^{-1}(K)$ is dense in D.

We know by [4, Theorem 5.8.1 (i)] that the Cartesian product $X := \prod\{X_i : 1 \leq i \leq n\}$ of Banach spaces X_i with the RNP has the same property. To see how the corollary works, we reprove this result. Thus, let D be a Baire space, $F \in \mathrm{uscoc}(D,(X^{**},w^*))$ and let $F^{-1}(X)$ be dense in D. Identifying X^{**} with $\prod\{X_i^{**} : 1 \leq i \leq n\}$ and taking in consideration that the natural projection $p_i : X^{**} \to X_i^{**}$ is continuous relative to the weak* topologies, we see that the correspondence $F_i := p_i \circ F \in \mathrm{USCOC}(D,(X_i^{**},w^*))$ satisfies Condition (ii) from Theorem 1.6. Hence $F_i \in \mathrm{uscoc}(D,(X_i^{**},w^*))$. As

$$F_i^{-1}(X_i) = F^{-1}(p_i^{-1}(X_i)) \supset F^{-1}(X) ,$$

the set $F_i^{-1}(X_i)$ is dense in D. Hence $C(F_i, D, X_i^{**})$ is a dense G_δ subset of D and therefore the same holds for the set

$$C(F, D, X^{**}) = \bigcap \left\{ C(F_i, D, X_i^{**}) : 1 \leqq i \leqq n \right\} .$$

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