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AFFINE SPACE MOTIONS WITH ONLY PLANE TRAJECTORIES

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Abstract: The paper contains the proof of the classification theorem for 3-dimensional affine space motions with only plane trajectories. Such motions are linear submanifolds of dimensions 8,5,3 and 2 in the general affine group and they are explicitly given.

Key words: Affine space kinematics, special motions.

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1. Introduction. The existence of only one nontrivial Euclidean space motion with only plane trajectories was discovered by G. Darboux in 1881 and proved by methods of differential geometry by W. Blaschke in [5]. A generalization of this result to the n-dimensional Euclidean space was published in [6]. On the other hand, the problem of the description of all space motions with only plane trajectories can be naturally generalized to other groups of transformations of the 3-dimensional space.

The solution of this problem for the group of all similarity transformations of the 3-dimensional Euclidean space was published in [7], the solution for isotropic motions in the affine space is obtained in [10], the solution for the 3-dimensional Lobatschewsky space is in [9]. A special class of such motions in the equiaffine space was found in [8].

The present paper solves this problem for the case of the general affine group GA(3) acting in the 3-dimensional affine space A_3. The above mentioned papers show that the number of solutions of this problem for the case of the group GA(3) is too large for an explicit description. Therefore we use a different method for the solution of this problem. We describe all submanifolds of GA(3) with the property that each curve on such a submanifold is a motion with only plane trajectories. This method was already used by the author in [1] for projective motions in the complex projective space. As a result we prove the following **Theorem**. Each maximal F-motion in A_3 is equivalent to one of the following:

a) The 2-dimensional motion $g(\lambda, \mu)=E+\lambda A+\mu B$, where A and B are arbitrary independent matrices.

b) The 3-dimensional motion given by (8).

c) The 5-dimensional motion given by (7).

d) The 8-dimensional motion given by (6).

During the proof of the theorem we shall use some facts which were proved in [1]; for the sake of convenience we shall present them here in an abbreviated form.

Let us choose a fixed affine coordinate system $S=\{0,e_1,e_2,e_3\}$ in a given real affine space A_3. Let GA(3) be the Lie group of all affine transformations of A_3, considered as a subgroup of GL(4) of all matrices of the form $g=\begin{pmatrix}1,0\\ t,\gamma\end{pmatrix}$, where t is a 3-column and $\gamma \in GL(3)$. Each point $X \in A_3$ is identified with the column of its coordinates in $S, X=(1,\xi)^T$, $\xi=(x,y,z)^T$. An affine space motion g(M) is, by definition, an embedding $g:M \longrightarrow GA(3)$ of a manifold M into GA(3). An affine space motion g(M) is called an F-motion, if the trajectory of each point lies in a plane. Obviously, a submanifold of an F-motion is again an F-motion. This means that if we want to find all affine F-motions, it is enough to describe the maximal ones. An F-motion is called maximal, if no open submanifold of it is a submanifold of an F-motion of higher dimension. All maximal F-motions are solutions of the equation

(1) $\qquad |X,AX,BX,g(M)X|=0$ for all $X \in A_3$,

where A and B are matrices from the Lie algebra of GA(3) and vertical bars denote the determinant. This shows that all maximal F-motions are open subsets of a linear space in GA(3), as the general solution of (1) is a linear subspace, from which we remove all singular matrices. Here we consider GA(3) as an open subset of a 12-dimensional linear space determined by the coordinate expression of GA(3) in S.

2. Solution of the equation $|X,AX,BX,CX|=0$. We shall now solve the equation

(2) $$|X,AX,BX,CX|=0,$$

where A,B,C are fixed matrices from the Lie algebra of $GA(3)$. We may suppose that matrices A,B,C are linearly independent, because we always have the obvious solution $C=\lambda A+\mu B$, λ, $\mu\in R$. Let us denote $A=\begin{pmatrix}0 & 0\\ \alpha & a\end{pmatrix}$, where a is a 3×3 matrix and α is a 3-column. For simplicity of denotation let us write a_i for the i-th row of the product $a\xi$, where $X=(1,\xi)^T$, $\xi=(x,y,z)^T$. The same convention will be used for B and C.

We have to consider the following cases:

1) Let $c=\lambda a+\mu b$. Then we may suppose $c=0$ and (2) takes the form $|a\xi+\alpha, b\xi+\beta, \gamma|=0$, where $\gamma\neq 0$. By a change of the base we obtain $\gamma_1=1$, $\gamma_2=\gamma_3=0$ and therefore we obtain

$$\begin{vmatrix} a_2+\alpha_2, & b_2+\beta_2 \\ a_3+\alpha_3, & b_3+\beta_3 \end{vmatrix}=0.$$

In this case we have two possible solutions

a) $\begin{vmatrix} a_1 & b_1, & 1 \\ a_2+\alpha_2, & b_2, & 0 \\ 0 & , 0 , & 0 \end{vmatrix}=0$ b) $\begin{vmatrix} a_1 & , b_1, & 1 \\ a_2+\alpha_2, & 0 , & 0 \\ a_3+\alpha_3, & 0 , & 0 \end{vmatrix}=0.$

From (2) it follows that we must also have

(3) $$|a\xi, b\xi, c\xi|=0$$

for all columns ξ. We may suppose from now on that a,b,c are linearly independent matrices. Let us consider the function $r=$ $=\operatorname{rank}(\lambda a+\mu b+\nu c)$ on the 3-dimensional vector space of real coefficients λ,μ,ν without $(0,0,0)$. Then $r\geq 1$.

2) Let $\max r=1$. Let us suppose at first that (3) is of the form

(4) $$\begin{vmatrix} m_1a_1, n_1a_1, p_1a_1 \\ m_2a_1, n_2a_1, p_2a_1 \\ m_3a_1, n_3a_1, p_3a_1 \end{vmatrix}=0,$$

where a_1 is a nonzero linear form in x,y,z; m_i, n_i, p_i are numbers. If we choose ξ in such a way that $a_1(\xi)\neq 0$, then $|m,n,p|=0$, and this yields that a,b,c are linearly dependent. This is a contradiction and we have no solution in this case.

Let us now suppose that (3) cannot be written in the form (4). Then it is $\begin{vmatrix} \lambda a_1, \lambda b_1, \lambda c_1 \\ \mu a_1, \mu b_1, \mu c_1 \\ \nu a_1, \nu b_1, \nu c_1 \end{vmatrix} = 0$. By a change of the base

we may achieve that $\lambda=1$, $\mu=\nu=0$; a_1, b_1, c_1 are linearly independent. (2) has the form $\begin{vmatrix} a_1+\alpha_1, b_1+\beta_1, c_1+\gamma_1 \\ \alpha_2 \quad , \beta_2 \quad , \gamma_2 \\ \alpha_3 \quad , \beta_3 \quad , \gamma_3 \end{vmatrix} = 0$.

The last two rows are necessarily dependent and by change of the base we may achieve that $\alpha_3=\beta_3=\gamma_3=0$ and this is a solution.

3) Let max $r=2$.

a) Let $a_1=b_1=c_1=0$, a_2, a_3 be linearly independent. If $\alpha_1=\beta_1= =\gamma_1=0$, we have a solution. If $\beta_1^2+\gamma_1^2 \neq 0$, we may suppose that $\gamma_1 \neq 0$, $\alpha_1=\beta_1=0$. Then $b=0$, which is a contradiction. (If c_2 and c_3 are independent, we exchange a and c and get the following case, if c_2 and c_3 are linearly dependent, we subtract a multiple of c from a to have $\alpha_1=0$, a_2 and a_3 remain independent.)

So let finally $\alpha_1 \neq 0$, $\beta_1=\gamma_1=0$. Then we have the solution $b_3=c_3=\beta_3=\gamma_3=0$ in a suitable base.

b) Let $a_1=0$, $b_1 \neq 0$, a_2 and a_3 be linearly independent. Then the determinant of the matrix $a+\lambda b$ must be zero for all λ. It follows that $b_1=ka_2+ma_3$, $c_1=ra_2+sa_3$ for real k,m,r,s. Let further b_1 and c_1 be linearly dependent. Then we add a multiple of b to c to have $c_1=0$, but then $c=0$, which is a contradiction. So b_1 and c_1 are linearly independent and we may suppose $b_1=a_2$, $c_1=a_3$. The solution then is

$$\begin{vmatrix} 0 \;, & a_2, & a_3 \\ a_2, & 0 & ,-a_1 \\ a_3, & a_1, & 0 \end{vmatrix} = 0,$$

where a_i are independent, or $a_1=0$. Together with translations we obtain the following two solutions

i) $\begin{vmatrix} 0 & ,a_2+\alpha_2, & a_3+\alpha_3 \\ a_2+\alpha_2 \;, & 0 & ,-a_1-\alpha_1 \\ a_3+\alpha_3 & ,a_1+\alpha_1, & 0 \end{vmatrix} = 0$, ii) $\begin{vmatrix} \alpha_1 & ,a_2+\beta_1, & a_3+\gamma_1 \\ a_2+\alpha_2, & 0 \;, & 0 \\ a_3+\alpha_3, & 0 \;, & 0 \end{vmatrix} = 0$,

where a_i are independent or $a_1=0$.

4) Let max r=3, min r=1. (3) then can be changed to the equation

(5) $$|\xi, m\xi, n\xi| = 0$$

if we multiply (3) by a^{-1} from the left. We have $\begin{vmatrix} x, m_1, n_1 \\ y, 0, n_2 \\ z, 0, n_3 \end{vmatrix} = 0$, which implies $n_2 = n_3 = 0$. Multiplication by a from left and a change of the base yields now (3) in the form $\begin{vmatrix} a_1, b_1, c_1 \\ a_2, 0, 0 \\ a_3, 0, 0 \end{vmatrix} = 0$. Further we consider also translations:

Let $\beta_2^2 + \beta_3^2 + \gamma_2^2 + \gamma_3^2 \neq 0$. Then we may suppose $\beta_2 \neq 0$, $\gamma_2 = 0$. Let us choose ξ in such a way that $a_2(\xi) = a_3(\xi) = 0$, $a_1(\xi) \neq 0$. Then $\gamma_3 = 0$ and also $\beta_2 = 0$, which is a contradiction. So we have $\beta_2 = \beta_3 = \gamma_2 = \gamma_3 = 0$ and this is a solution.

5) Let max r=3, min r=2. (5) is in the form

$\begin{vmatrix} x, & 0, & n_1 \\ y, & m_2, & n_2 \\ z, & m_3, & n_3 \end{vmatrix} = 0$. We may suppose $n_{11} = 0$, let $n_{12}^2 + n_{13}^2 \neq 0$.

Choose x=0. Then we see that m must have rank 1 and this is a contradiction. So $n_1 = 0$. Then n is a multiple of m, which is impossible and we have no solution in this case.

3. Maximal F-motions. From all solutions of (2) we now construct maximal F-motions. The solution 1a) obviously belongs to the F-motion

(6) $$g(\lambda_i, \mu_i)X = \begin{pmatrix} \lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 \\ \mu_1 x + \mu_2 y + \mu_3 z + \mu_4 \\ z \end{pmatrix}$$

where we omitted the first row consisting of 1. It is easy to see that (6) is also maximal. Really, let $\begin{pmatrix} 0, & 0 \\ \gamma, & c \end{pmatrix}$ belong to this motion. Then we must have $\begin{vmatrix} x, & 0, & c_1 + \gamma_1 \\ 0, & y, & c_2 + \gamma_2 \\ 0, & 0, & c_3 + \gamma_3 \end{vmatrix} = 0$

and so $c_3 = \gamma_3 = 0$.

The solution 1b) belongs to the F-motion

(7) $$g(\lambda, \mu_i)X = X + \lambda \begin{pmatrix} 0 \\ a_2+\alpha_2 \\ a_3+\alpha_3 \end{pmatrix} + \begin{pmatrix} \mu_1 x + \mu_2 y + \mu_3 z + \mu_4 \\ 0 \\ 0 \end{pmatrix}$$

where $a_2+\alpha_2$ and $a_3+\alpha_3$ are linearly independent (otherwise it is a submanifold of (6)). Similarly as above we show that (7) is also maximal.

The solution from 2) belongs to (6), 3a) belongs to (7). The solution 3bi) leads to

(8) $$g(\lambda, \mu, \nu)X = X + \lambda \begin{pmatrix} 0 \\ a_2+\alpha_2 \\ a_3+\alpha_3 \end{pmatrix} + \mu \begin{pmatrix} a_2+\alpha_2 \\ 0 \\ a_1+\alpha_1 \end{pmatrix} + \nu \begin{pmatrix} a_3+\alpha_3 \\ -a_1-\alpha_1 \\ 0 \end{pmatrix},$$

where a_i are linearly independent or $a_1=0$, $a_2+\alpha_2$, $a_3+\alpha_3$ are linearly independent and $\alpha_1 \neq 0$.

To show the maximality of (8), let C belong to (8). Then we may write $c\xi = \begin{pmatrix} \lambda_1 a_1 \\ \lambda_2 a_1 + \nu_2 a_3 \\ \lambda_3 a_1 + \mu_3 a_2 + \nu_3 a_3 \end{pmatrix}$ and $\begin{vmatrix} 0, & a_2, & c_1 \\ a_2, & 0, & c_2 \\ a_3, & a_1, & c_3 \end{vmatrix} = 0$

A suitable choice of ξ now shows that c=0 and a similar way is used for translations.

Finally, 3bii) belongs to (8) as well as 4). This finishes the proof.

Remark. The 8-dimensional motion d) has all trajectories in parallel planes, c) has all trajectories parallel to one direction, projection in this direction gives an affine plane motion with only straight trajectories, b) is centro-affine, it has a fixed point, given by $a_i + \alpha_i = 0$.

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