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ON ANALYTICAL DIMENSION OF RINGS OF BOUNDED UNIFORMLY CONTINUOUS FUNCTIONS

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Abstract: Analytical dimension of the ring of bounded uniformly continuous real-valued functions on an arbitrary uniform space is characterized by properties of the space. For pseudometrizable spaces some more satisfactory characterizations are obtained.

Key words: Uniform space, uniform dimension of spaces and mappings, analytical dimension.

Classification: 54E15, 54F45

M. Katětov was the first who examined (see [6] and [7]) the relations between the dimension of a topological space X and properties of $C^*(X)$ - the ring of all bounded continuous real-valued functions on X endowed with the usual sup-norm ($C^*(X)=C(X)$ for compact X, $C(\emptyset) = \{\emptyset\}$). For this purpose, he introduced a concept of the analytical dimension.

Let us recall basic definitions. A subring C_1 of a real commutative topological algebra C with unit is said to be analytically closed provided C_1 is a subalgebra containing the unit, C_1 is a closed subset and $y \in C$, $y^2 \in C_1$ imply $y \in C_1$. A subset B of C is called an analytical base of C if there is no analytically closed subring C_1 with $B \subset C_1 \neq C$. The least cardinal number of an analytical base of C is called the analytical dimension of C and will be denoted by Ad C.

If X is a non-void compact metric space, then dim X=Ad C(X) (see [6]). Since the values of dim are non-negative integers or ∞ and the values of Ad are cardinals, such equalities should be understood in the sense that either both sides are finite and equal or they are both infinite.

Katětov in [7] generalized this result for any compact Hausdorff space and, using the equality dim βX=dim X, for any Tihonov space, he used, however, a modified concept of the analytical dimension of the algebra C; let us denote it by ad C. This is defined as the least cardinal number such that for any countable $M \subset C$ there is an analytically closed subalgebra A with $M \subset A$, Ad $A \leqq$ ad C. Clearly ad $C \leqq$ Ad C and the values of ad are countable cardinals only.

If X is a uniform space (specially a metric space) we denote by $U^*(X)$ the subalgebra of $C^*(X)$ consisting of all bounded uniformly continuous functions. The analytical dimension of $U^*(X)$ was examined in [4] and [1], the basic result asserts that Ad $U^*(X) = \Delta d\ X$ for any non-void metric space X. The symbols Δd and δd denote the great and the small uniform (covering) dimensions (for definitions see [5]). The aim of this paper is to search for properties of an arbitrary uniform space X which correspond to the values of ad $U^*(X)$ and Ad $U^*(X)$. The matter of ad will be simple (Theorem 1 below). Then Ad will be characterized by the existence of mappings with certain dimensonal properties.

Given a uniform space $(X, \mathcal{U})$ (where $\mathcal{U}$ is the filter of uniform entourage, see e.g. [8], no separation axiom is assumed) and U in $\mathcal{U}$, we say that a collection $\mathcal{K}$ of subsets of X is a U-cover of a subset $Z \subset X$ if for each $z \in Z$, $U[z] \cap Z \subset K$ for some K in $\mathcal{K}$. All mappings for uniform spaces are supposed to be uniformly continuous. Let us repeat the definition of the uniform dimension Δd of mappings from [3] and at the same time define a new concept of a D-mapping, which will be, however, used in Theorems 6 and 7 only.

Definition 1. Let $(X, \mathcal{U})$, $(Y, \mathcal{U})$ be uniform spaces, $f: X \to Y$. Assume that for each U in $\mathcal{U}$ there exist V in $\mathcal{V}$, W in $\mathcal{U}$ and a natural number m such that, if $M \subset Y$, $M \times M \subset V$, then there exists a collection $\mathcal{K}$ such that $K \times K \subset U$ for each $K \in \mathcal{K}$ and $\mathcal{K}$ is a W-cover of $f^{-1}[M]$ with order at most m. Then we will say that f is a D-mapping. If the number m can be chosen fixed, then the least possible non-negative value of m-1 is defined to be $\Delta d\ f$. If such a fixed number does not exist or f is not a D-mapping, we set $\Delta d\ f = \infty$.

If f is the mapping of a non-void uniform space X onto a one-point space then $\Delta d\ f = \Delta d\ X$ and f is a D-mapping if and only if X is distal (see [2]). In [3], distality was called property (f).

However, for the purposes of this paper we also need another concept of uniform dimension of mappings which would relate to the dimension Δd similarly as, for dimensions of spaces, δd to Δd. We will use the following definition.

Definition 2. Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be uniform spaces, Y precompact, $f: X \longrightarrow Y$. Then $\delta d\ f$ is defined as the smallest non-negative integer n with the following property: for each finite uniform cover $\mathcal{G}$ of X there exist V in $\mathcal{V}$ and W in $\mathcal{U}$ such that if $M \subset Y$, $M \times M \subset V$ then there exists a collection $\mathcal{K}$ such that $\mathcal{K}$ refines $\mathcal{G}$ and $\mathcal{K}$ is a W-cover of $f^{-1}[M]$ with order at most n+1. If such a number does not exist we set $\delta d\ f = \infty$.

The following properties of $\delta d\ f$ are almost evident.

Proposition 1. Let X, Y be uniform spaces, Y precompact, $f: X \longrightarrow Y$. Then $\delta d\ f \leqq \Delta d\ f$. If X is precompact then $\delta d\ f = \Delta d\ f$. If $X \neq \emptyset$ and Y is one-point then $\delta d\ f = \delta d\ X$.

It might be surprising that in Definition 2, mappings with precompact range are considered only. However, first, it will be quite sufficient for our purposes, moreover, the definition of $\delta d\ f = 0$ will suffice. Secondly, this paper is not devoted to a detailed study of $\delta d\ f$ and I do not know now what definition would be the most suitable in general case.

We will also use Hausdorff modification of a uniform space. This concept has appeared in the literature under various names (see e.g. [9],[10],[11]); let us recall some facts and agree on the terminology. Let $(X, \mathcal{U})$ be a uniform space. Then there exists a finest uniformity $h\mathcal{U}$ on $X/\cap\mathcal{U}$ such that the mapping $q: (X, \mathcal{U}) \longrightarrow (X/\cap\mathcal{U}, h\mathcal{U})$ defined by $x \in q(x)$ is uniformly continuous. The uniform space $(X/\cap\mathcal{U}, h\mathcal{U})$ is Hausdorff and will be termed the Hausdorff modification of $(X, \mathcal{U})$ and shortly denoted by hX. The mapping q will be called the canonical projection. If Y is any Hausdorff uniform space, $f: X \longrightarrow Y$, then there exists a unique mapping $hf: hX \longrightarrow Y$ such that $f = hf \circ q$.

The projection q is uniformly open and if $G\subset X$ is open then $q^{-1}[q[G]]=G$. Using these properties and the fact that, for any uniform cover $\mathcal{G}$ of a uniform space, the collection of the interiors of all sets from $\mathcal{G}$ is again a uniform cover, one easily proves

Proposition 2. For any uniform space X, $\delta d\ X=\delta d\ hX$, $\Delta d\ X=\Delta d\ hX$.

Let us turn to the analytical dimensions.

Lemma 1. Let X be a uniform space, hX its Hausdorff modification, $q:X\longrightarrow hX$ the canonical projection. Let shX be the Samuel compactification of hX, $e:hX\longrightarrow shX$ the (proximal) embedding. Then for each $f\in U^*(X)$ there is a unique $\sigma(f)\in C(shX)$ such that $f=\sigma(f)\circ e\circ q$. The mapping $\sigma:U^*(X)\longrightarrow C(shX)$ is an isometry onto and also a linear, ring and lattice isomorphism. If $L\subset U^*(X)$, L separates far subsets of X, then $\sigma[L]$ separates distinct points of shX.

Proof. Besides using properties of q, the proof is quite similar to the proof of Lemma 1 in [4].

Theorem 1. If X is a non-void uniform space, then $ad\ U^*(X)=$ $=\delta d\ X$.

Proof. By Lemma 1, $U^*(X)$ and $C(shX)$ are isometrically isomorphic algebras, hence $ad\ U^*(X)=ad\ C(shX)$. Now, by Proposition 2 and by V.2 in [5] we have $\delta d\ X=\delta d\ hX=\delta d\ shX=\dim shX$. By Proposition 4 in [7], $ad\ C(shX)=\dim shX$. Therefore $ad\ U^*(X)=$ $=\delta d\ X$.

The characterization of Ad*U (X) will be more difficult (Theorem 5 below). The letter I always stands for the unit interval $[\![0,1]\!]$ endowed with the usual metric. If A is a non-void set then I^A denotes the usual product uniform space, each element $x\in I^A$ should be understood as $x=(x_\alpha;\alpha\in A)$ and for $\alpha\in A$, pr_α denotes the usual projection $x\longmapsto x_\alpha$ of I^A onto I. For sets and collections in pseudometric spaces, the symbols dist(x,Z), diam Z, mesh $\mathcal{C}$ and σ-discreteness have the usual meaning.

The following lemma is only a reformulation of Lemma 2 in [4].

Lemma 2. Let X be a topological space, F a finite non-void set, $f:X \longrightarrow I^F$ continuous. Let L be a sublattice and a submodule of $C^*(X)$ that contains all $pr_\alpha \circ f$ for $\alpha \in F$ and all constant functions. Suppose that J, K are intervals, $K \subset J \subset I^F$, K is closed and J is open in I^F. Then there exists a non-negative $h \in L$ such that $h(x)=1$ for $x \in f^{-1}[K]$, $h(x)=0$ for $x \in f^{-1}[I^F \setminus J]$.

Theorem 2. Let X be a uniform space, $A \neq \emptyset$, $f:X \longrightarrow I^A$, $\sigma d\ f=0$. Then $\{pr_\alpha \circ f : \alpha \in A\}$ is an analytical base of $U^*(X)$.

Proof. Let L be an analytically closed subring of $U^*(X)$ that contains all $pr_\alpha \circ f$, notice that L is a sublattice, too. We will prove that L separates far subsets of X. Let C, D be far subsets of X. Then $\{X \setminus C, X \setminus D\}$ is a uniform cover of X. Since $\sigma d\ f=0$, choose for this cover a finite non-void $F \subset A$ and $\delta > 0$ such that if

$$V=\{(x,y) \in I^A \times I^A ; |x_\alpha - y_\alpha| < \delta \text{ for } \alpha \in F\}$$

and $M \times M \subset V$ then there are far subsets M', M'' of X such that $f^{-1}[M]=M' \cup M''$, $M' \subset X \setminus C$, $M'' \subset X \setminus D$. Let p_F denote the projection $(x_\alpha ; \alpha \in A) \longmapsto (x_\alpha ; \alpha \in F)$, p_α^F the projections $(x_\alpha ; \alpha \in F) \longmapsto x_\alpha$. Let $J_1, \dots, J_r$ be intervals open in I^F such that diam $p_\alpha^F[J_i] < \delta$ for $i=1,\dots,r$ and $\bigcup(J_i ; i=1,\dots,r)=I^F$. Choose, for each i, a closed interval $K_i \subset J_i$ such that $\bigcup(K_i ; i=1,\dots,r)=I^F$. For the mapping $p_F \circ f$ and each J_i, K_i choose in L a function $h_i : X \longrightarrow I$ by Lemma 2. Now let, for $i=1,\dots,r$, M_i' and M_i'' be far sets in X such that $f^{-1}[p_F^{-1}[J_i]]=M_i' \cup M_i''$, $M_i' \subset X \setminus C$, $M_i'' \subset X \setminus D$. Put $k_i(x)=h_i(x)$ for $x \in X \setminus M_i'$, $k_i(x)=-h_i(x)$ for $x \in M_i'$, $i=1,\dots,r$. Then $k_i \in U^*(X)$, $k_i^2=h_i^2 \in L$ (or $|k_i|=|h_i| \in L$), L is analytically closed, hence $k_i \in L$. Finally, put $g(x)=\Sigma(k_i(x)+|k_i|(x) ; i=1,\dots,r)$, clearly $g \in L$. Let $x \in C$. Then there exists i such that $p_F(f(x)) \in K_i$. Hence $h_i(x)=1$, $x \in M_i''$, $k_i(x)=h_i(x)$, thus $g(x) \geqq 2$. Let $x \in D$. Then for any i, either $p_F(f(x)) \in J_i$ and $x \in M_i'$, $k_i(x)=-h_i(x)$, or $p_F(f(x)) \notin J_i$ and $h_i(x)=0$, $k_i(x)=0$; consequently, $g(x)=0$.

Now apply Lemma 1, use the mapping σ. We know $\sigma[L]$ separates points of shX, hence by Stone-Weierstrass Theorem, $\sigma[L]=C(\mathrm{sh}X)$. Thus $L=U^*(X)$ and the proof is complete.

Notice that Theorem 2 generalizes Theorem 1 from [4].

Lemma 3. Let X, Y be uniform spaces, Y compact, $f:X \longrightarrow Y$. Then $\sigma d\, f=0$ if and only if for each finite uniform cover $\mathcal{G}$ and $q \in Y$ there exists a neighbourhood V of q and a uniformly discrete collection $\mathcal{K}_V$ that refines $\mathcal{G}$ and such that $f^{-1}[V]=\cup\mathcal{K}_V$.

Proof. To prove the sufficiency, using the compactness of Y, we take a finite uniform cover by the neighbourhoods V.

Lemma 4. Let $\mathcal{G}=(G_j;j \in B)$ be a uniform cover of a uniform space X. Then there exists a family $(g_j;j \in B)$ where $g_j:X \longrightarrow I$ such that if $Z \subset X$, $Z \subset G_j$ for no $j \in B$ then there exists $j \in B$ with diam $g_j[Z] \geqq 1$.

Proof. Let d be a uniformly continuous pseudometric on X such that for each $x \in X$, $\{y \in X;d(x,y)<1\} \subset G_j$ for some $j \in B$. Put for $x \in X$, $j \in B$, $g_j(x)=\min\{1,d\text{-dist}(x,X \setminus G_j)\}$. Let $Z \subset X$, $Z \subset G_j$ for no j. Then $Z \neq \emptyset$, choose $z \in Z$. There is j with $g_j(z)=1$. Further, there exists $y \in Z \setminus G_j$, hence $g_j(y)=0$. Therefore diam $g_j[Z] \geqq g_j(z)-g_j(y)=1$.

Lemma 5. Let $\mathcal{W}$ be a filter of subsets of a set Y, let X be a uniform space, $f:X \longrightarrow Y$ a mapping. Let S be the set of all $g \in U^*(X)$ with the following property: for each $\delta>0$ there exist $V \in \mathcal{W}$ and a uniformly discrete collection $\mathcal{H}$ such that $\cup\mathcal{H}=f^{-1}[V]$ and mesh $\{g[H];H \in \mathcal{H}\} \leqq \delta$. Then S is an analytically closed subring of $U^*(X)$.

Proof: is identical with the first part of the proof of Theorem 2 in [4] (namely a),b) and c) on page 384). We omit it here.

Theorem 3. Let $(X,\mathcal{U})$ be a uniform space, $A \neq \emptyset$, $f:X \longrightarrow I^A$. Let $\{pr_\alpha \circ f; \alpha \in A\}$ be an analytical base of $U^*(X)$. Then $\sigma d\, f=0$.

Proof. Suppose that $\sigma d\, f>0$. By Lemma 3, there is a finite uniform cover $\mathcal{G}$ of X and $q \in Y$ such that, for no neighbourhood V of q, $f^{-1}[V]$ can be expressed as in Lemma 3. Keep these $\mathcal{G}$ and q fixed. Let $\mathcal{W}$ be the filter of all neighbourhoods of q. Let $S \subset U^*(X)$ be defined by X, Y, f and $\mathcal{W}$ as in Lemma 5. By Lemma 5, S is an analytically closed subring of $U^*(X)$. Let

$\alpha \in A$. If $\sigma > 0$ put

$$V = \{z \in I^A; |z_\alpha - q_\alpha| < \sigma/2\}.$$

Clearly, diam $(pr_\alpha \circ f)\ f^{-1}[V] \leq$ diam $pr_\alpha[V] \leq \sigma$, thus $pr_\alpha \circ f \in S$. Since $\{pr_\alpha \circ f; \alpha \in A\}$ is an analytical base of $U^*(X)$, we have $S = U^*(X)$.

Now, let us use the properties of $\mathcal{G}$ and q. Given $V \in \mathcal{W}$ and $U \in \mathcal{U}$ define a relation $\sim$ on the set $f^{-1}[V]$: $x \sim y$ means there exists $x = x_0, x_1, \dots, x_k = y$ such that $x_i \in f^{-1}[V]$ and $(x_{i-1}, x_i) \in U \cap U^{-1}$ for each i. Clearly, $\sim$ is an equivalence on $f^{-1}[V]$, let $\mathcal{M}_{U,V}$ be the collection of all classes defined by this equivalence. Each $\mathcal{M}_{U,V}$ is uniformly discrete and, by the property of $\mathcal{G}$, no $\mathcal{M}_{U,V}$ refines $\mathcal{G}$. Suppose $\mathcal{G} = (G_j; j \in B)$ where B is finite. For $j \in B$, take the functions g_j from Lemma 4 and put

$$T_j = \{(U,V) \in \mathcal{U} \times \mathcal{W};\ \text{mesh}\ \{g_j[Z]; Z \in \mathcal{M}_{U,V}\} \geq 1\}.$$

By Lemma 4, $\bigcup(T_j; j \in B) = \mathcal{U} \times \mathcal{W}$. The set $\mathcal{U} \times \mathcal{W}$ is directed by the relation $\prec$ defined by $(U_1,V_1) \prec (U_2,V_2) \equiv U_1 \supset U_2$ and $V_1 \supset V_2$. As B is finite we can choose j such that T_j is cofinal in $(\mathcal{U} \times \mathcal{W}, \prec)$. Let $\sigma < 1$, $V \in \mathcal{W}$ and let $\mathcal{H}$ be the collection for $g = g_j$ from Lemma 5. Then $\mathcal{H}$ is refined by some $\mathcal{M}_{U,V}$. Take $(U_1,V_1) \in T_j$ with $(U,V) \prec (U_1,V_1)$. Now $(U,V) \in T_j$ and hence $g_j \notin S$ which is a contradiction with $S = U^*(X)$.

Let us recall Theorem 2 from [4] in a slightly stronger form.

Theorem 4. Let X be a pseudometric space, $f: X \longrightarrow I^A$ where A is countable. Let $\{pr_\alpha \circ f; \alpha \in A\}$ be an analytical base of $U^*(X)$. Then $\Delta d\ f = 0$.

Theorem 2 in [4] concerned metric spaces and finite A only. But the proof is the same. We use only the fact that each point of I^A has a countable neighbourhood base.

Every precompact metric space Y can be uniformly embedded into I^A with a countable A. Thus Theorems 2 and 4 imply the following assertion.

Corollary 1. Let X be a pseudometric space, Y a precompact metric space, $f: X \longrightarrow Y$. Then $\sigma d\ f = 0$ if and only if $\Delta d\ f = 0$.

Using a constant function f, the following well-known result follows.

Corollary 2. If X is a pseudometric space, then $\sigma d\ X=0$ if and only if $\Delta d\ X=0$.

Of course, for pseudometric spaces X and countable Ad $U^*(X)$, Theorem 3 is weaker than Theorem 4. On the other hand, Corollary 1 might be proved independently and Theorem 4 would be a consequence of Corollary 1 and Theorem 3. However, all the techniques used for the proof are the same in both ways.

Theorem 4 does not hold for non-pseudometrizable uniform spaces X, even for finite A. In fact, let X be any uniform space with $\sigma d\ X=0$ and $\Delta d\ X=\infty$ (see e.g. [5], V.5). Then, we have by Theorem 2, Ad $U^*(X) \leqq 1$ (moreover, equal to zero - see Theorem 5 below), but the existence of $f:X \longrightarrow I^A$ with arbitrary, finite or infinite, A and $\Delta d\ f=0$ would imply (by Theorem 8 in [3]) that X is distal and consequently $\sigma d\ X=\Delta d\ X$ ([5], V.5).

In the following summarizing theorem, A may be empty as well; $I^{\emptyset}$ is a one-point space.

Theorem 5. Let X be a uniform space. Then A $U^*(X)$ is the least cardinality of a set A such that there exists $F:X \longrightarrow I^A$ with $\sigma d\ f=0$.

Proof. Suppose Ad $U^*(X)=0$, thus $\emptyset$ is an analytical base of $U^*(X)$. Let $f_0:X \longrightarrow I$ be any constant function. Now, $\{f_0\}$ is an analytical base of $U^*(X)$, too, and by Theorem 3, $\sigma d\ f_0=0$. Thus $\sigma d\ f=0$ for $f:X \longrightarrow I^{\emptyset}$, too. On the contrary, if $\sigma d\ f=0$, then necessarily $\sigma d\ f_0=0$ for any constant $f_0:X \longrightarrow I$ and, by Theorem 2, $\{f_0\}$ is an analytical base of $U^*(X)$. But each constant function can be excluded from any analytical base, thus $\emptyset$ is an analytical base. The rest of the proof follows from Theorems 2 and 3, we need only the fact that in any analytical base of $U^*(X)$ each function can be replaced by a function mapping X into I.

Katětov proved ([6], Theorem 3) the following similar theorem: If X is a compact Hausdorff space then Ad C(X) is the least cardinality of a set A such that there exists $f:X \longrightarrow I^A$ with ind $f^{-1}[y] \leqq 0$ for each $y \in I^A$.

This theorem directly follows from Theorem 5, because dim $f^{-1}[y] \leqq 0$ for any $y \in I^A$ is equivalent, by Theorem 3 in [3], with $\Delta d\ f=0$ and this is, by Proposition 1, equivalent with $\sigma d\ f=0$. On the other hand, Theorem 5 can be derived from the Katětov's theorem. It is more complicated, it needs still Lemma 1 with the equality $\sigma d\ f=\Delta d\ \sigma(f)$. This follows from a modification of Theorem 2 in [3] and other assertions.

Theorem 5 characterizes the value of Ad $U^*(X)$ by means of existence of certain mappings, thus by no intrinsic properties of X. We can prove only the following connections with dimension.

Proposition 3. Let X be a uniform space. Then $\sigma d\ X \leqq$ $\leqq$ Ad $U^*(X)$. If $\sigma d\ X \leqq 0$ then Ad $U^*(X)=0$.

Proof. If Ad $U^*(X)$ is finite, take $f:X \longrightarrow I^A$ from Theorem 5 for a suitable A. Applying Theorem 5 from [3] for f as the map of the precompact modification of X, we get the first inequality. It also follows from Theorem 1. The second assertion directly follows from Theorem 5.

Let us show that finite $\sigma d\ X$ admits Ad $U^*(X)$ being uncountable. Let X be the space "long line", i.e. the lexicographical product of countable ordinals with $I \setminus \{1\}$. Then $\Delta d\ X = \sigma d\ X =$ $=$dim $X=1$. But if A is countable, $f:X \longrightarrow I^A$ is continuous then, for some $y \in I^A$, $f^{-1}[y]$ must contain a segment, thus $\sigma d\ f > 0$.

Nevertheless, for a pseudometric space X, Theorem 5 implies an intrinsic condition for finiteness of Ad $U^*(X)$. For a countable A and $f:X \longrightarrow I^A$, $\sigma d\ f=0$ is equivalent with $\Delta d\ f=0$, by Corollary 1. Let $A=\{1,\dots,n\}$. By Theorem 7 in [3], there is $f:X \longrightarrow I^A$ with $\Delta d\ f=0$ if and only if $\Delta d\ X \leqq n$. See also Theorem 3 in [4].

Now, we are going to present a similar characterization for infinite countable A.

Theorem 6. Let X be a pseudometric space, let A be the set of all positive integers. Then the following statements are equivalent:

(1) X is distal.

(2) There exists $f:X \longrightarrow I^A$ with $\Delta d\ f=0$.

(3) There exist a distal space Y and a D-mapping $f:X \longrightarrow Y$.

Proof. The implication (2) $\Longrightarrow$ (3) is obvious. The proof of

(3) $\Longrightarrow$ (1) is easy, in fact the same as the proof of Theorem 8 in [3]. Suppose (1) holds, $X \neq \emptyset$ and let us prove (2).

Since X is distal, there exists, for each j=1,2,... a uniform cover $\mathcal{K}_j$ of X with finite order m_j such that mesh $\mathcal{K}_j \leqq 2^{-j}$. Clearly, $m_j \geqq 1$. Put $s_0=0$, $s_j=m_1+\ldots+m_j$, $N_j=\{i \in A; s_{j-1} < i \leqq s_j\}$ for j=1,2,... . We may suppose that (see e.g. [5], IV.25) $\mathcal{K}_j = \bigcup\{\mathcal{G}_i; i \in N_j\}$ where each $\mathcal{G}_i$ is an η_j-discrete collection for some $\eta_j > 0$ with mesh $\mathcal{G}_i \leqq 2^{-j}$. Choose, for j=1,2, .., $0 < \delta_j \leqq 1$ such that for each $x \in X$ there exist $i \in N_j$ and $G \in \mathcal{G}_i$ such that $\{y \in X; \mathrm{dist}(x,y) < \delta_j\} \subset G$. For any $i \in N_j$, $x \in X$, put $f_i(x)=\min\{1, \mathrm{dist}(x, Y \setminus \bigcup \mathcal{G}_i)\}$. Clearly, $f_i : X \longrightarrow I$ is uniformly continuous. Now, define $f: X \longrightarrow I^A$ by $pr_i \circ f = f_i$ for all $i \in A$. Let us show that $\Delta d\, f=0$. Given $\varepsilon > 0$, choose j with $2^{-j} \leqq \varepsilon$. Put

$$V=\{(u,v) \in I^A \times I^A; |pr_i(u)-pr_i(v)| < \delta_j \text{ for } i \in N_j\}.$$

Let $M \subset I^A$, $M \times M \subset V$. Suppose $f^{-1}[M] \neq \emptyset$. Choose $x \in X$ with $f(x) \in M$. By the properties of $\mathcal{K}_j$, there exists $i \in N_j$ such that $f_i(x) \geqq \delta_j$. Now if $y \in X$, $f(y) \in M$ then $f_i(y) > 0$, hence $y \in \bigcup \mathcal{G}_i$. Thus $\mathcal{G}_i$ is a desired $\eta_i$cover of $f^{-1}[M]$ with mesh at most ε.

Notice that similarly as in Theorem 7 in [3] one can prove that the set of all mappings f with $\Delta d\, f=0$ contains a dense G_δ-subset in a certain function space. But in the case of infinite A a direct simple construction of the desired mapping f is possible and here preferred.

Corollary 3. Let X be a pseudometric space. Then Ad $U^*(X)$ is countable if and only if X is distal.

Observe that in Theorem 7, the proof of (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) needed neither the countability of A nor the pseudometrizability of X. Given a uniform space X and a set A with sufficiently large cardinality then the proof of (1) $\Longrightarrow$ (2) is also possible and is very similar. Thus the following theorem holds.

Theorem 7. Let X be a uniform space. Then the following statements are equivalent:

(1) X is distal.

(2) There exist a set A and $f: X \longrightarrow I^A$ with $\Delta d\, f=0$.

(3) There exist a distal space Y and a D-mapping $f:X \longrightarrow Y$.

Compare this assertion with the fact that for any uniform space X there exist a set A and $f:X \longrightarrow I^A$ with $\sigma d\ f=0$. It follows directly from Theorem 3.

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