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# A GENERALIZATION OF THE INTERIOR MAPPING THEOREM OF CLARKE AND POURCIAU

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**Abstract:** We prove the following generalization of a result of Clarke and Pourciau. A mapping acting between two (super) reflexive Banach spaces which is locally approximable by convex subsets of linear surjections is locally surjective. The main tool of the proof is a modification of the Caristi´s fixed point principle. We also show that this tool can be used for deriving theorems of Cramer and Ray, Džumabaev, and Graves.

**Key words:** Reflexive Banach space, Clarke´s generalized Jacobian, interior mapping theorem.

**Classification:** 58C15, 47H15

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**1. Introduction.** A special case of the well known theorem due to Graves [9], see Corollary 3, asserts that the image of a neighbourhood of $x_0 \in X$ under a mapping $F$ acting between Banach spaces $X$ and $Y$ is a neighbourhood of $Fx_0$ provided that $F$ is continuously Fréchet differentiable at $x_0$ and the derivative of $F$ at $x_0$ is surjective. Clarke [2] for $X=Y=R^n$ and Pourciau [11] for $X=R^n$ and $Y=R^k$, $k \leq n$, have generalized this result for Lipschitz, not necessarily differentiable, mappings by showing

**Theorem 1.** Let $F:D(F) \subset R^n \longrightarrow R^k$, $k \leq n$, be a Lipschitz mapping and let $x_0$ be an interior point of the domain $D(F)$ of $F$. Let $\partial F(x_0)$ denote the set of $n \times k$-matrices obtained as the closed convex hull of all possible limits

$$\lim_{m \to \infty} DF(x_m),$$

where $x_m \longrightarrow x_0$ and the derivatives $DF(x_m)$ exist.

If $\partial F(x_0)$ consists of matrices of maximal rank only, then $F(D(F))$ is a neighbourhood of $Fx_0$.

Let us suppose that $\partial F(x_0)$ contains matrices of maximal

rank only. Then, owing to the finite dimensionality, a compactness argument ensures that there is an $\alpha > 0$ such that. for every $L \in \partial F(x_0)$ and every $y \in R^k$ there exists $x \in R^n$ satisfying

$$Lx=y \text{ and } \|y\| \geq \alpha \|x\|.$$

Moreover, if $\beta \in (0, \alpha)$ is given, then by using the mean value theorem [11, Theorem 3.1, Proposition 3.2] and the compactness once more, we can find an $r > 0$ such that for any $x_1$, $x_2$ in the closed ball $B(x_0.r)$ centred at $x_0$ and of radius r there exists an $L \in \partial F(x_0)$ such that

$$\|Fx_1-Fx_2-L(x_1-x_2)\| \leq \beta \|x_1-x_2\|.$$

These observations have led the first named author in [7] to generalize the above theorem to Hilbert spaces. The result obtained there asserts that if the above relations hold when replacing $\partial F(x_0)$ by a convex bounded subset of the space $\mathcal{L}(X,Y)$ of continuous linear mappings from X to Y, then the closure of F(D(F)) is a neighbourhood of $Fx_0$. Recently Ursescu [13] has shown by a more direct and simpler method that $Fx_0$ is in fact in the interior of F(D(F)). It should be noted that this can also be derived from the quoted result of [7] by using the Pták´s closed graph theorem [10].

In this paper we go on in generalizing this result:

**Theorem 2.** Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be two reflexive Banach spaces, $r > 0$, $\varrho > 0$, $\beta \geq 0$, let $F:D(F) \subset X \longrightarrow Y$ be a mapping and let $x_0 \in D(F)$. Let us suppose that either F is continuous and its domain D(F) is closed or that F has a closed graph and Y is superreflexive. Moreover, let there exist a convex bounded subset $\mathcal{M}$ of $\mathcal{L}(X,Y)$ such that whenever $x \in B(x_0,r) \cap D(F)$ and $h \in X$, there are $\varepsilon \in (0,1]$ and $L \in \mathcal{M}$ fulfilling

$$\|F(x-\varepsilon h)-Fx+\varepsilon Lh\| \leq \varepsilon \beta \|h\|. \tag{1}$$

Finally, let us assume that the mappings from $\mathcal{M}$ are uniformly open in the sense that, for each $L \in \mathcal{M}$ and each $y \in Y$, there exists $x \in X$ such that

$$Lx=y \text{ and } \|y\| \geq (\beta + \varrho)\|x\|. \tag{2}$$

Then the open ball $\mathring{B}(Fx_0, \varrho r)$ of centre $Fx_0$ and radius $\varrho r$ is included in $F(B(x_0,r) \cap D(F))$.

Recall that in [7] it is required that whenever x and $x-\varepsilon h$ belong to $B(x_0,r)$, then (1) holds with some $L \in \mathcal{M}$ . It should

be noted that the case $\beta > 0$ can be reduced to that of $\beta = 0$; see Remark 5.

The proof consists of three steps. First, from a variant of the Caristi´s fixed point principle (Lemma 1), we derive an interior mapping theorem (Corollary 2). Then we prove Lemmas 2 and 3 which show that the hypotheses of Theorem 2 lead to the situation occurred in Corollary 2.

We also show how Lemma 1 may be used to derive the interior mapping theorems of Cramer and Ray [3, Theorem 2.1], Džumabaev [6], and Graves [9, Theorem 1].

**2. Caristi´s principle and its consequences.** We shall use the Caristi´s fixed point principle [1],[5] in the following slight reformulation and generalization.

**Lemma 1.** Let $Z$ be a set and let $d_0, d_1, \ldots, d_k$ be pseudometrics on $Z$. Suppose further that

$$d(z,\bar{z}) = \max \{d_0(z,\bar{z}), \ldots, d_k(z,\bar{z})\}, \quad z, \bar{z} \in Z,$$

is a metric in which $Z$ is a complete metric space. Let us consider functions $f_0, f_1, \ldots, f_k : Z \longrightarrow [0, +\infty)$ which are lower semicontinuous with respect to $d$. Finally fix $z_0 \in Z$ and let us assume that, for any $z \in Z$ fulfilling $f_0(z) > 0$ and

$$(3) \qquad d_i(z_0,z) \leqq f_i(z_0) - f_i(z), \quad i = 0, \ldots, k,$$

there exists $\bar{z} \in Z$, $\bar{z} \neq z$, such that

$$(4) \qquad d_i(z,\bar{z}) \leqq f_i(z) - f_i(\bar{z}), \quad i = 0, \ldots, k.$$

Then there exists $z \in Z$ such that $f_0(z) = 0$ and $d_i(z_0,z) \leqq f_i(z_0)$, $i = 0, \ldots, k$.

**Proof.** A simple induction argument ensures that there exists a sequence $\{z_0, z_1, \ldots\} \subset Z$ such that for all $n = 0, 1, \ldots$

$$d_i(z_n, z_{n+1}) \leqq f_i(z_n) - f_i(z_{n+1}), \quad i = 0, \ldots, k,$$

and

$$d_0(z_n, z_{n+1}) \geqq \tfrac{1}{2} s_n,$$

where

$$s_n = \sup \{d_0(z_n,z) : z \in Z, d_i(z_n,z) \leqq f_i(z_n) - f_i(z), \ i = 0, \ldots, k\}.$$

Clearly

$$d_i(z_n, z_{n+m}) \leqq d_i(z_n, z_{n+1}) + \ldots + d_i(z_{n+m-1}, z_{n+m}) \leqq$$

$$\leqq f_i(z_n)-f_i(z_{n+1})+\ldots+f_i(z_{n+m-1})-f_i(z_{n+m})=f_i(z_n)-f_i(z_{n+m})$$

for all n, m and

$$f_i(z_0)\geqq f_i(z_1)\geqq \ldots \geqq 0.$$

It follows that $\{z_n\}$ is a Cauchy sequence in each pseudometric $d_i$, hence in the metric d. As d is complete, $\{z_n\}$ converges in the metric d to some $z\in Z$. Then from the lower semicontinuity of $f_i$ we have for all n

$$d_i(z_n,z)=\lim_{m\to\infty} d_i(z_n,z_{n+m})\leqq f_i(z_n)-\lim_{m\to\infty} f_i(z_{n+m})\leqq f_i(z_n)-f_i(z)$$

and, especially,

$$d_i(z_0,z)\leqq f_i(z_0)-f_i(z)\leqq f_i(z_0),\ i=0,\ldots,k.$$

If $f_0(z)=0$, we are done. Further let us assume that $f_0(z)>0$. Then, by the assumptions, there exists $\bar{z}\in Z$, $\bar{z}\neq z$, such that

$$d_i(z,\bar{z})\leqq f_i(z)-f_i(\bar{z}).$$

For each i, we add this inequality and

$$d_i(z_n,z)\leqq f_i(z_n)-f_i(z),$$

and we obtain

$$d_i(z_n,\bar{z})\leqq d_i(z_n,z)+d_i(z,\bar{z})\leqq$$

$$\leqq f_i(z_n)-f_i(z)+f_i(z)-f_i(\bar{z})=f_i(z_n)-f_i(\bar{z}).$$

The definition of $s_n$ then yields that $s_n\geqq d_0(z_n,\bar{z})$. But $s_n\leqq 2d_0(z_n,z_{n+1})\longrightarrow 0$. Hence

$$d_0(z,\bar{z})=\lim_{n\to\infty} d_0(z_n,\bar{z})\leqq \lim_{n\to\infty} s_n=0,$$

a contradiction with $\bar{z}\neq z$. We have thus shown that the possibility $f_0(z)>0$ cannot occur and so the proof is completed.

**Remark 1.** a) In applications the existence of $\bar{z}$ is often required for any $z\in Z$ with $f_0(z)>0$, which strengthens a little the assumptions of Lemma 1.

b) It is obvious but useful to realize that the functions $f_i$ can be replaced by $\Phi_i\circ f_i$, where $f_i$ are as in Lemma 1 and $\Phi_i\colon [0,+\infty)\longrightarrow [0,+\infty)$ are nondecreasing lower semicontinuous with $\Phi_i(s)=0$ if and only if $s=0$.

c) Another useful variant of b) is to replace (3) and (4) by

(3´) $d_i(z_0,z)\leqq A_i(f_i(z_0))(f_i(z_0)-f_i(z))$ and

(4′) $\quad d_i(z,\bar{z}) \leqq A_i(f_i(z))(f_i(z)-f_i(\bar{z}))$

respectively. If $A_i \geqq 0$, if $\Phi_i$ fulfil the same hypotheses as in b), and if

(5) $\quad A_i(u)(u-v) \leqq \Phi_i(u) - \Phi_i(v)$ whenever $0 \leqq v < u$,

then, according to b), there is $z \in Z$ such that $f_0(z)=0$ and $d_i(z_0,z) \leqq \Phi_i(f_i(z_0))$. Often used situations in which (5) holds are, for example:

A: $(0,+\infty) \longrightarrow (0,+\infty)$, $\int_0^c A(s)ds < +\infty$ for all $c > 0$, and

$\alpha$) A is nonincreasing. Then, if we take $\Phi(u) = \int_0^u A(s)ds$, we get for $0 \leqq v < u$

$$\Phi(u) - \Phi(v) = \int_v^u A(s)ds \geqq A(u)(u-v).$$

Hence (5) is satisfied.

$\beta$) A is nondecreasing. Then the choice $\Phi(u) = \int_0^{2u} A(s)ds$ yields for $0 \leqq v < u$

$$\Phi(u) - \Phi(v) = \int_{2v}^{2u} A(s)ds \geqq \int_u^{2u} A(s)ds \geqq A(u)u > A(u)(u-v)$$

if $2v \leqq u$, and

$$\Phi(u) - \Phi(v) \geqq A(2v)(2u-2v) > A(u)(u-v)$$

if $2v > u$. Thus (5) holds again.

$\gamma$) A(s)s is nondecreasing. Then for $\Phi(u) = \int_0^{eu} A(s)ds$ and $0 \leqq v < u$ we have

$$\Phi(u) - \Phi(v) = \int_{ev}^{eu} A(s)ds \geqq \int_u^{eu} A(s)s\,\frac{1}{s}\,ds \geqq A(u)u \int_u^{eu} \frac{1}{s}\,ds =$$
$$= A(u)u > A(u)(u-v)$$

if $ev \leqq u$, and

$$\Phi(u) - \Phi(v) \geqq A(ev)ev \int_{ev}^{eu} \frac{1}{s}\,ds > A(u)u \cdot \ln\left(\frac{u}{v}\right) > A(u)(u-v)$$

if $ev > u$.

d) Requiring stronger versions of (3′), (4′) we can get better choices for $\Phi$. For example, if $A_i(s)$s are nondecreasing and if we replace (3′),(4′) by the inequalities

$$d_i(z_0,z) \leqq A_i(f_i(z_0))(f_i(z_0) - \max(q_i f_i(z_0), f_i(z))),$$

$$d_i(z,\bar{z}) \leqq A_i(f_i(z))(f_i(z) - \max(q_i f_i(z), f_i(\bar{z})))$$

respectively, where $q_i \in [0,1)$ are fixed, then we can take $\Phi_i(u) = \int_0^{ue^{1-q_i}} A_i(s)ds$.

**Corollary 1.** Let (X,d), (Y,d) be complete metric spaces, $r>0$, $\varphi>0$, $c>0$, let $F:D(F)\subset X\to Y$ be a mapping and let $x_0\in D(F)$. Let us assume that F is continuous and D(F) is closed (or that F has a closed graph only). Finally, suppose that, for any $x\in B(x_0,r)\cap D(F)$ and any $y\in B(Fx_0,\varphi r)$, $y\neq Fx$, there exists $\bar{x}\in D(F)$, $\bar{x}\neq x$, such that

$$\varphi\, d(x,\bar{x}) \leqq d(Fx,y)-d(F\bar{x},y) \tag{6}$$

(and moreover, if F has a closed graph only, that

$$cd(Fx,F\bar{x}) \leqq d(Fx,y)-d(F\bar{x},y)). \tag{7}$$

Then $B(Fx_0,\varphi r)\subset F(B(x_0,r)\cap D(F))$.

**Proof.** Fix $y\in B(Fx_0,\varphi r)$. We are to find an $x\in B(x_0,r)\cap D(F)$ such that $Fx=y$. Denote $Z=D(F)$

$$d_0(x,\bar{x})=\varphi d(x,\bar{x}),\quad d_1(x,\bar{x})=cd(Fx,F\bar{x}),\quad x,\bar{x}\in Z,$$

$$f_0(x)=f_1(x)=d(Fx,y),\quad x\in Z.$$

If F is continuous and D(F) is closed, take k=0, while in the parenthetic case consider k=1. Clearly Z is complete and $f_0$, $f_k$ are continuous in the metric $\max(d_0,d_k)$. Also, the inequalities (6) and (7) pass exactly to (4). The assumptions of Lemma 1 are thus verified and so there exists an $x\in D(F)$ such that $f_0(x)=0$, i.e., $Fx=y$, and that $d_0(x_0,x)\leqq f_0(x_0)$, which implies that $x\in B(x_0,r)$.

**Remark 2.** a) For slightly weaker assumptions of the above corollary see the exact formulation of Lemma 1.

b) In the same way as in Remark 1 - b),c),d) one can replace (6),(7) by using the functions A and $\Phi$. In fact, the version of Corollary 1 obtained by the use of d) implies [3, Theorem 2.1].

c) Corollary 1 can be extended to multivalued mappings. Thus, if $F:D(F)\subset X\longrightarrow 2^Y$ is upper semicontinuous closed valued and D(F) is closed, then (6) should be replaced by

$$\varphi\, d(x,\bar{x}) \leqq \operatorname{dist}(Fx,y)-\operatorname{dist}(F\bar{x},y),$$

while if F has a closed graph only, then (6) and (7) should read as

$$\max(\varphi\, d(x,\bar{x}),cd(v,\bar{v})) \leqq d(v,y)-d(\bar{v},y),$$

where $(x,v)$ and $(\bar{x},\bar{v})$ lie in the graph of $F$.

d) We also notice that, if $F$ is continuous, the completeness of $Y$ is not necessary. A similar remark applies also to the consequences of Corollary 1.

**Corollary 2.** Let $X$, $Y$ be Banach spaces, $r > 0$, $\varrho > 0$, $q \in [0,1)$, let $F: D(F) \subset X \to Y$ be a mapping and let $x_0$ be in $D(F)$. Let us assume that $F$ is continuous and $D(F)$ is closed (or that $F$ has a closed graph only). Finally, suppose that, for any $x \in B(x_0,r) \cap D(F)$ and any $y \in B(Fx_0, \varrho r)$, $y \neq Fx$, there exist $0 \neq h \in X$ and $\varepsilon \in (0,1]$ such that

$$\varrho \|h\| + \frac{1}{\varepsilon}\|F(x-\varepsilon h) - Fx + \varepsilon(Fx-y)\| \leqq \|Fx-y\| \tag{8}$$

(and moreover, if $F$ has a closed graph only, that

$$\frac{1}{\varepsilon}\|F(x-\varepsilon h) - Fx + \varepsilon(Fx-y)\| \leqq q\|Fx-y\|). \tag{9}$$

Then $B(Fx_0, \varrho r) \subset F(B(x_0,r) \cap D(F))$.

**Proof.** Take $x \in B(x_0,r) \cap D(F)$, $y \in B(Fx_0, \varrho r) \setminus \{Fx\}$ arbitrarily. By the hypotheses find $h$ and $\varepsilon$ corresponding to $x$ and $y$. The triangle inequality then yields

$$\|F(x-\varepsilon h) - y\| \leqq \|F(x-\varepsilon h) - Fx + \varepsilon(Fx-y)\| + (1-\varepsilon)\|Fx-y\|.$$

Thus, by (8),

$$\|F(x-\varepsilon h) - y\| \leqq \|Fx-y\| - \varrho\varepsilon\|h\|,$$

and after denoting $\bar{x} = x - \varepsilon h$, we get

$$\varrho\|x-\bar{x}\| \leqq \|Fx-y\| - \|F\bar{x}-y\|,$$

which is the inequality (6). If (9) holds, then

$$\|F\bar{x}-y\| \leqq \varepsilon q\|Fx-y\| + (1-\varepsilon)\|Fx-y\| = (1-\varepsilon(1-q))\|Fx-y\|,$$

and

$$\|Fx-F\bar{x}\| \leqq \|F\bar{x}-Fx+\varepsilon(Fx-y)\| + \varepsilon\|Fx-y\| \leqq \varepsilon(1+q)\|Fx-y\| =$$

$$= \frac{1+q}{1-q}(\|Fx-y\| - (1-\varepsilon(1-q))\|Fx-y\|) \leqq \frac{1+q}{1-q}(\|Fx-y\| - \|F\bar{x}-y\|)$$

and so (7) is verified. It means that Corollary 1 can be applied and consequently $B(Fx_0, \varrho r) \subset F(B(x_0,r) \cap D(F))$.

In the proof of Theorem 2 we shall need only Corollary 2. But we feel that further consequences of this corollary should also be mentioned.

**Corollary 3.** Let X, Y be Banach spaces, $r > 0$, $\varphi > 0$, $\beta \geq 0$, let $F:D(F) \subset X \longrightarrow Y$ be a mapping with a closed graph and let $x_0 \in D(F)$. Let us assume that there exists $L \in \mathscr{L}(X,Y)$ such that, for every $x \in B(x_0,r) \cap D(F)$ and every $h \in X$, there is $\varepsilon \in (0,1]$ fulfilling

$$\|F(x-\varepsilon h)-Fx+\varepsilon Lh\| \leq \varepsilon\beta\|h\|. \tag{10}$$

Finally, suppose that the L is such that to each $y \in Y$ there is $x \in X$ satisfying $Lx=y$ and $\|y\| \geq (\beta+\varphi)\|x\|$.

Then $B(Fx_0, \varphi r) \subset F(B(x_0,r) \cap D(F))$.

**Proof.** Fix $x \in B(x_0,r) \cap D(F)$ and $y \in B(Fx_0, \varphi r)$, $y \neq Fx$. Find $h \in X$ such that $Lh=Fx-y$ and $(\beta+\varphi)\|h\| \leq \|Fx-y\|$. Let $\varepsilon$ correspond to x and h. Then

$$\frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y)\| = \frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon Lh\| \leq$$

$$\leq \beta\|h\| \leq \frac{\beta}{\beta+\varphi}\|Fx-y\|$$

and so both (8) and (9) hold. Now apply Corollary 2.

**Remark 3.** The above corollary is a slight improvement of the result of Graves [9, Theorem 1], where (10) is required to hold whenever x and $x-\varepsilon h$ belong to $\mathring{B}(x_0,r)$. Another proof of the theorem of Graves, by using Nadler's contraction principle for multivalued mappings is due to Szilágyi [12].

**Corollary 4.** Let X, Y be Banach spaces, $r > 0$, $\alpha > 0$, $\Theta \in (0,1)$, let $F:D(F) \subset X \longrightarrow Y$ be a mapping with a closed graph and let $x_0 \in D(F)$. Let us assume that, for every x from $B(x_0,r) \cap D(F)$, there are $\alpha_x > \alpha$, $\beta_x \in [0, \Theta\alpha_x)$, $\sigma_x > 0$ and a mapping $C_x: \mathring{B}(0, \sigma_x) \subset X \longrightarrow Y$ such that $\|Fu-Fx-C_x(u-x)\| \leq \beta_x\|u-x\|$ whenever $u \in \mathring{B}(x_0,r) \cap \mathring{B}(x, \sigma_x)$, and that for every $y \in Y$ there is $h \in X$ satisfying $\|y\| \geq \alpha_x\|h\|$ and $C_x(\varepsilon h) = \varepsilon y$ for all $\varepsilon > 0$ sufficiently small.

Then $\mathring{B}(Fx_0,(1-\Theta)\alpha r) \subset F(\mathring{B}(x_0,r) \cap D(F))$.

**Proof.** Choose a fixed $\hat{r} \in (0,r)$. Take arbitrary x in $B(x_0\hat{r}) \cap$ $\cap D(F)$ and y in $B(Fx_0,(1-\Theta)\alpha\hat{r}) \setminus \{Fx\}$. Find $h \in X$ and $\varepsilon_0 \in (0,1)$ such that $\|Fx-y\| \geq \alpha_x\|h\|$ and $C_x(-\varepsilon h) = \varepsilon(Fx-y)$ whenever $\varepsilon \in$ $\in (0, \varepsilon_0)$. Fix an $\varepsilon \in (0, \varepsilon_0)$ so small that

$$\|x-\varepsilon h-x\| < \sigma_x \text{ and } \|x-\varepsilon h-x_0\| < r.$$

Then we can estimate

$$\frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y)\| = \frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx-C_x(-\varepsilon h)\| \leq \beta_x \|h\| <$$

$$\leq \Theta\|Fx-y\|$$

and

$$(1-\Theta)\alpha\|h\| + \frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y\| \leq \frac{(1-\Theta)\alpha}{\alpha_x}\|Fx-y\|+$$

$$+\Theta\|Fx-y\| < \|Fx-y\|.$$

It means that Corollary 2 applies. Hence $B(Fx_0,(1-\Theta)\alpha\hat{r})$ $F(B(x_0,\hat{r})\cap D(F))$ and by letting $\hat{r}$ go to r the result follows.

**Remark 4.** This corollary is a slight improvement of the result of Džumabaev [6], where the $C_x$ are assumed to have inverses and an additional condition $\alpha_x > tr(1-\Theta)\alpha/\alpha_x$ with a fixed $t\in(0,1)$ is required.

**Corollary 5.** Let X, Y be Banach spaces, $r>0$, $\varrho>0$, and $q\in[0,1)$. Let $F:D(F)\subset X\longrightarrow Y$ be a continuous mapping (or a mapping with a closed graph), Gâteaux differentiable on $B(x_0,r)\subset$ $\subset D(F)$. Let for every $x\in B(x_0,r)$ and every $y\in B(Fx_0,\varrho r)$, $y\neq Fx$, there exist $0\neq h\in X$ such that

$$\varrho\|h\|+\|Fx-y-DF(x)h\|\leq\|Fx-y\|$$

(and moreover, if F has a closed graph only, that

$$\|Fx-y-DF(x)h\|\leq q\|Fx-y\|).$$

Then $\mathring{B}(Fx_0,\varrho r)\subset F(\mathring{B}(x_0,r))$.

**Proof.** Take $\hat{\varrho}\in(0,\varrho)$, $\hat{q}\in(q,1)$, and $0<\tilde{r}<\hat{r}<r$. Let $\hat{F}$ be the restriction of F to $B(x_0,\hat{r})$. Then $D(\hat{F})$ is closed and the above inequalities ensure that there exists $\varepsilon\in(0,1)$ such that the assumptions of Corollary 2 hold with r, $\varrho$, q, and F replaced by $\tilde{r}$, $\hat{\varrho}$, $\hat{q}$, and $\hat{F}$ respectively. Hence $B(Fx_0,\hat{\varrho}\tilde{r})\subset F(B(x_0,\tilde{r}))$ and we conclude the proof by letting $\hat{\varrho}$ converge to $\varrho$ and $\tilde{r}$ converge to r.

**Corollary 6** ([9, Theorem 3]). Let $L_0\in\mathcal{L}(X,Y)$ and let there exist $\alpha>0$ such that to every $y\in Y$ there is $x\in X$ satisfying $L_0x=y$ and $\|y\|\geq\alpha\|x\|$.

If $L\in\mathcal{L}(X,Y)$ is such that $\|L-L_0\|<\alpha$, then $(\alpha-\|L-L_0\|)B_Y\subset L(B_X)$, where $B_X$ and $B_Y$ denote the closed unit

balls in X and Y respectively.

**Proof.** Let $L$ be as above and take any $x \in X$ and any $y \in Y \setminus \{Lx\}$. Find $h \in X$ such that $L_0 h = Lx - y$ and $\alpha \|h\| \leq \|Lx - y\|$. Then $h \neq 0$ and

$$(\alpha - \|L - L_0\|)\|h\| + \|L(x-h) - Lx + (Lx - y)\| = (\alpha - \|L - L_0\|)\|h\| + \|Lh - L_0 h\| \leq$$
$$\leq \alpha \|h\| \leq \|Fx - y\|.$$

Thus (8) holds with $\varepsilon = 1$ and so, by Corollary 2

$$(\alpha - \|L - L_0\|)B_Y = B(L(0), (\alpha - \|L - L_0\|)) \subset L(B(0,1)) = L(B_X).$$

**Remark 5.** The above corollary enables us to reduce in Theorem 2 the case $\beta > 0$ to that of $\beta = 0$. Let us show it. Define

$$\mathcal{M}' = \{L' \in \mathcal{L}(X,Y) : \|L' - L\| \leq \beta \text{ for some } L \in \mathcal{M}\}.$$

Let $x \in B(x_0, r) \cap D(F)$ and $h \in X$ be given. Clearly, we may suppose that $h \neq 0$. Let $G \subset X$ be a hyperplane such that $\|h+g\| \geq \|h\|$ for all $g \in G$. Define

$$L'(th+g) = -\frac{t}{\varepsilon}(F(x - \varepsilon h) - Fx) + Lg, \quad t \in R, \ g \in G,$$

where $\varepsilon > 0$ and $L \in \mathcal{M}$ correspond to $x$ and $h$. Then $F(x - \varepsilon h) - Fx + \varepsilon L'h = 0$. Further, $L'$ is linear and by (1)

$$\|L'(th+g) - L(th+g)\| = \|\frac{t}{\varepsilon}(F(x - \varepsilon h) - Fx) + L(th)\| \leq \frac{|t|}{\varepsilon}\varepsilon\beta\|h\| =$$
$$\leq \frac{|t|}{\varepsilon}\varepsilon\beta\|h\| = |t|\beta\|h\| \leq \beta\|th+g\|$$

for all $t \in R$ and all $g \in G$. Hence $\|L' - L\| \leq \beta$, and so $L' \in \mathcal{M}'$. Now applying Corollary 6 we get that

$$\varrho B_Y \subset (\beta + \varrho - \|L' - L\|)B_Y \subset L'(B_X).$$

From this inclusion it easily follows that for every $y \in Y$ there is $x \in X$ such that $L'x = y$ and $\|y\| \geq \varrho\|x\|$. Thus we have shown that the assumptions of Theorem 2 are fulfilled with $\beta$ and $\mathcal{M}$ replaced by 0 and $\mathcal{M}'$ respectively.

**3. Geometrical lemmas and the proof of Theorem 2.** If $X$ is a Banach space, let $X^*$ denote its dual, $X^{**}$ its second dual, $\varkappa : X \longrightarrow X^{**}$ the canonical embedding and $\langle x^*, x \rangle$ the value of $x^* \in X^*$ at $x \in X$. If $L \in \mathcal{L}(X,Y)$, then $L^*$ means the adjoint to $L$.

**Lemma 2.** Let $X$ and $Y$ be Banach spaces, $\mathcal{M} \subset \mathcal{L}(X,Y)$ be a

convex set, and $\alpha > 0$. Let us consider the following assertions:

(i) for every $y^* \in Y^*$ there is $0 \neq x \in X$ such that $\langle y^*, Lx\rangle \geq \alpha \|y^*\| \|x\|$ whenever $L \in \mathcal{M}$;

(ii) for every $y^* \in Y^*$ and every $L \in \mathcal{M}$ $\|L^* y^*\| \geq \alpha \|y^*\|$;

(iii) whenever $y \in Y$ and $L \in \mathcal{M}$ then there is $x \in X$ such that $Lx=y$ and $\|y\| \geq \alpha \|x\|$.

Then (i) $\Longrightarrow$ (ii), (iii) $\Longrightarrow$ (ii) and, if X is reflexive, then all the assertions are equivalent.

**Proof.** (i) $\Longrightarrow$ (ii). Let $y^* \in Y^*$ and $L \in \mathcal{M}$. By (i) there is $0 \neq x \in X$ such that $\langle y^*, Lx\rangle \geq \alpha \|y^*\| \|x\|$. Hence

$$\|L^* y^*\| \|x\| \geq \langle L^* y^*, x\rangle = \langle y^*, Lx\rangle \geq \alpha \|y^*\| \|x\|, \quad \|L^* y^*\| \geq \alpha \|y^*\|.$$

(iii) $\Longrightarrow$ (ii). For $y^* \in Y^*$ and an arbitrary $\sigma > 0$ find $y \in Y$, $\|y\| = 1$, such that $\langle y^*, y\rangle \geq (1-\sigma)\|y^*\|$. Then by (iii), for any $L \in \mathcal{M}$ there exists $x \in X$ such that $Lx=y$ and $1 = \|y\| \geq \alpha \|x\|$. Hence

$$\|L^* y^*\| \|x\| \geq \langle L^* y^*, x\rangle = \langle y^*, Lx\rangle = \langle y^*, y\rangle \geq (1-\sigma)\|y^*\| \geq$$
$$\geq (1-\sigma)\alpha \|y^*\| \|x\|, \|L^* y^*\| \geq (1-\sigma)\|y^*\|.$$

And since $\sigma > 0$ was arbitrary, we get (ii).

Let X be reflexive by the end of the proof. Let us prove (ii) $\Longrightarrow$ (i). We shall proceed as Clarke in the proof of [2, Lemma 3]. Fix $0 \neq y^* \in Y^*$. Let us remark that the set $\{L^* y^* : L \in \mathcal{M}\}$ is convex and disjoint from $\{x^* \in X^* : \|x^*\| < \alpha \|y^*\|\}$. Hence by the separation theorem and reflexivity there is $x \in X$, $x \neq 0$, such that for any $L \in \mathcal{M}$

$$\langle y^*, Lx\rangle = \langle L^* y^*, x\rangle \geq \sup \{\langle x^*, x\rangle : \|x^*\| < \alpha \|y^*\|\} = \alpha \|y^*\| \|x\|.$$

It remains to prove (ii) $\Longrightarrow$ (iii). By (ii) $L^*$ maps $Y^*$ onto the closed subspace $Z = L^*(Y^*)$ of $X^*$ and there exists $S \in \mathcal{L}(Z, Y^*)$ such that $\|S\| \leq 1/\alpha$ and $S(L^* y^*) = y^*$ for all $y^* \in Y^*$. Then $S^*$ maps $Y^{**}$ into $Z^*$. Fix now $y \in Y$, $y \neq 0$. Then $S^*(\mathcal{J}(y))$ is in $Z^*$ and hence, by the Hahn Banach theorem, there exists $x^{**} \in X^{**}$ such that $\|x^{**}\| = \|S^* \mathcal{J}(y)\|$ and that $\langle x^{**}, z\rangle = \langle S^* \mathcal{J}(y), z\rangle$ for all $z \in Z$. As X is reflexive, we can write $x^{**} = \mathcal{J}(x)$ with some $x \in X$. Then we have

$$\langle y^*, Lx\rangle = \langle L^* y^*, x\rangle = \langle \mathcal{J}(x), L^* y^*\rangle = \langle S^* \mathcal{J}(y), L^* y^*\rangle =$$
$$= \langle \mathcal{J}(y), SL^*(y^*)\rangle = \langle \mathcal{J}(y), y^*\rangle = \langle y^*, y\rangle$$

for all $y^* \in Y^*$. Hence $Lx=y$. Moreover $\|S^*\| = \|S\| \leq 1/\alpha$ and so

$$\|x\| = \|x^{**}\| = \|S^* \varkappa(y)\| \geq \frac{1}{\alpha} \|\varkappa(y)\| = \frac{1}{\alpha}\|y\| .$$

Thus (iii) holds.

**Lemma 3.** Let X, Y be Banach spaces, let $\alpha > 0$, $\gamma \in (0,\alpha)$ be given and let $\mathcal{M} \subset \mathcal{L}(X,Y)$ be a nonempty convex bounded set such that for every $y^* \in Y^*$ there exists $0 \neq x \in X$ satisfying

$$\langle y^*, Lx\rangle \geq \alpha \|y^*\| \|x\| \text{ whenever } L \in \mathcal{M} .$$

If the norm of Y is Fréchet differentiable off the origin, then for every $0 \neq y \in Y$ there are $t \in (0,1/\gamma)$ and $g \in X$ such that

(11) $\|g\| = \|y\|$ and $\|y - tLg\| < (1 - \gamma t)\|y\|$ whenever $L \in \mathcal{M}$.

If the norm of Y is uniformly Fréchet differentiable on the unit sphere, then there exists $t \in (0,1/\gamma)$ such that for every $0 \neq y \in Y$ there is $g \in X$ fulfilling (11).

**Proof.** Let $0 \neq y \in Y$ be given. Let $\|y\|'$ denote the Fréchet derivative of $\|\cdot\|$ at y. By assumptions, to $y^* = \|y\|'$, there exists $g \in X$, $\|g\| = \|y\|$, such that

$$\langle \|y\|', Lg\rangle \geq \alpha \|y\|' \| \|g\| = \alpha \|y\|$$

for all $L \in \mathcal{M}$. Denote $c = \sup \{\|L\| : L \in \mathcal{M}\}$. As $\mathcal{M}$ is bounded, c is finite. Since the norm is Fréchet differentiable at y, there is $t > 0$ such that

$$\|y - z\| \leq \|y\| - \langle \|y\|', z\rangle + \frac{\alpha - \gamma}{2c}\|z\|$$

whenever $z \in Y$ and $\|z\| \leq tc\|y\|$. We note that if the norm on Y is uniformly Fréchet differentiable on the unit sphere, then t can be chosen independently of the concrete y. As $\|tLg\| \leq tc\|y\|$ for all $L \in \mathcal{M}$, we have

$$\|y - tLg\| \leq \|y\| - \langle \|y\|', tLg\rangle + \frac{\alpha - \gamma}{2c}\|tLg\| \leq$$

$$\leq \|y\| - \alpha t\|y\| + \frac{\alpha - \gamma}{2c} tc\|y\| < (1 - \gamma t)\|y\|,$$

which was to prove.

**Proof of Theorem 2.** According to Troyanski [4, p. 164] $Y^*$ admits an equivalent locally uniformly rotund norm. If Y is superreflexive, so is $Y^*$ [4, p. 87] and by Enflo [4, p. 87] there exists an equivalent uniformly rotund norm on $Y^*$. Further, it is known and easy to check [8] that such norms can be taken arbit-

rarily close, in the sense of Banach-Mazur distance, to the original norm on $Y^*$. Hence, by an easy duality argument [4] we get that $Y$ admits an equivalent norm which is Fréchet (or uniformly Fréchet) differentiable on the unit sphere and is arbitrarily close to the original norm on $Y$. Thus we may assume that the original norm on $Y$ is Fréchet (uniformly Fréchet) differentiable on the unit sphere and that the assumptions of Theorem 2 hold with $\beta$ and $\varrho$ replaced by $\beta+\sigma$ and $\varrho-2\sigma$ respectively, where $\sigma$ is some fixed number from $(0,\varrho/3)$.

Take $x\in B(x_0,r)\cap D(F)$ and $y\in B(Fx_0,(\varrho-3\sigma)r)$, $y\neq Fx$. From (2), by applying subsequently Lemmas 2 and 3 we can find $t\in (0,1/(\beta+\varrho-2\sigma))$ and $g\in X$ such that

$$\|g\|=\|Fx-y\| \text{ and } \|Fx-y-tLg\|<(1-(\beta+\varrho-2\sigma)t)\|g\|$$

for all $L\in\mathcal{M}$. Let us note that in the case of uniform Fréchet differentiability the $t$ does not depend on the choice of $x$ and $y$. From the hypotheses choose $\varepsilon\in(0,1]$ and $L\in\mathcal{M}$ such that

$$\|F(x-\varepsilon tg)-Fx+\varepsilon L(tg)\|\leq\varepsilon(\beta+\sigma)\|tg\|.$$

Then the last two inequalities yield

$$\frac{1}{\varepsilon}\|F(x-\varepsilon tg)-Fx+\varepsilon(Fx-y)\|\leq\frac{1}{\varepsilon}\|F(x-\varepsilon tg)-Fx+\varepsilon L(tg)\|+$$

$$+\|Fx-y-tLg\|<(\beta+\sigma)\|tg\|+(1-(\beta+\sigma-2\sigma)t)\|g\|=$$

$$=\|Fx-y\|-(\varrho-3\sigma)\|tg\|=(1-(\varrho-3\sigma)t)\|Fx-y\|,$$

$$(\varrho-3\sigma)\|tg\|+\frac{1}{\varepsilon}\|F(x-\varepsilon tg)-Fx+\varepsilon(Fx-y)\|<\|Fx-y\|.$$

It means that (8) and (9) hold with $h=tg$, $q=1-(\varrho-3\sigma)t$, and with $\varrho$ replaced by $\varrho-3\sigma$. Thus by Corollary 2 $B(Fx_0,(\varrho-3\sigma)r)\subset F(B(x_0,r)\cap D(F))$. And since $\sigma>0$ could be arbitrarily small, the conclusion of Theorem 2 follows.

**Remark 6.** From the above proof one can see that the version of Theorem 2 with $F$ continuous and $D(F)$ closed holds under weaker assumptions. Namely, the reflexivity of $Y$ can be replaced by the requirement that the set of equivalent Fréchet differentiable norms on $Y$ is dense in the sense of Banach-Mazur distance. We do not know whether this case occurs if one such norm exists.

**Final note.** After this paper had been prepared for publication, we learned about the paper of P.H. Dien, Some results

on locally Lipschitzian mappings, Acta Math. Vietnamica 6(1981), 97-105. Here a theorem similar to our Theorem 2 is presented under a little stronger assumptions. Its proof is based on the Ekeland's variational principle.

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