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ON THE MULTIPLICITY POINTS OF MONOTONE OPERATORS ON SEPARABLE
BANACH SPACES II

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Abstract: The results from [1] are sharpened, e.g. it is proved that the set of multiplicity points of a monotone operator on a separable real Banach space can be written as a union of countably many subsets of Lipschitz hypersurfaces, having "finite convexity on curves with finite convexity".

Key words: Multiplicity points of monotone operators, finite convexity, Lipschitz surfaces in Banach spaces.

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Let T be a monotone operator on a real Banach space X , i.e. $T: X \rightarrow \exp X^*$ and $\langle x-y, x^*-y^* \rangle \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$. Denote by $\text{co}Tx$ a convex hull of the set Tx and put

$$A_n = \{x \in X: \dim(\text{co}Tx) \geq n\},$$

$$A^n = \{x \in X: \text{co}Tx \text{ contains a ball of codimension } n\}.$$

In [1] there was proved that if X (or X^* , respectively) is separable then A_n (or A^n , resp.) is representable as a countable union of Lipschitz fragments of codimension n (of dimension n , resp.), where F (see Definition 2) has "linearly finite convexity" (i.e. uniformly bounded convexity on lines).

By finer calculations with the Lipschitz fragments constructed in [1], the stronger result is obtained: they are in fact CFC-fragments (see Definition 3).

X will always be a real Banach space; by $\Omega(x, r)$ we shall denote an open ball in X with centre x and radius $r > 0$.

Definition 1. Let $S \subset \mathbb{R}$ and $c: S \rightarrow X$. If $\text{card } S \geq 3$ we define

$$\mathcal{K}(c, S) = \sup_{j=1}^{\infty} \left\| \frac{c(s_{j+1}) - c(s_j)}{s_{j+1} - s_j} - \frac{c(s_j) - c(s_{j-1})}{s_j - s_{j-1}} \right\|,$$

where "sup" is taken over all finite sequences $s_0 < s_1 < \dots < s_{k+1}$ in M . We put $\mathcal{K}(c, S) = 0$ if $\text{card } S \leq 2$.

$\mathcal{K}(c, S)$ is called convexity of c on S .

Basic properties of mappings with finite convexity can be found in [1], part 2.

Definition 2. Let $B \subset X$, $n \in \mathbb{N}$, and $n < \dim X$. We shall say that B is a Lipschitz fragment of dimension n (of codimension n , resp.) iff the following is satisfied:

There exist subspaces W and Z of X and a set $M \subset W$ such that

(i) $X = W \oplus Z$

(ii) $\dim W = n$ (codim $W = n$, resp.)

(iii) $B = \{w + F(w) : w \in M\}$ where $F : M \rightarrow Z$ is a Lipschitz mapping.

(\oplus denotes a topological sum.)

Fragments with $M=W$ are called surfaces.

Definition 3. Let $B \subset X$ be a Lipschitz fragment. We shall say that B is CFC-fragment (of the same dimension or codimension) iff W, Z, M, F from Definition 2 can be chosen in such way that for any mapping $c : S \rightarrow M$ with $S \subset \mathbb{R}$ the following inequality holds:

$$\mathcal{K}(F \circ c, S) \leq a \cdot \mathcal{K}(c, S) + b \cdot \text{Lip}(c),$$

where a and b are nonnegative constants independent on c and

$$\text{Lip}(c) = \sup \left\{ \left\| \frac{c(s) - c(s')}{s - s'} \right\| : s, s' \in S, s \neq s' \right\}.$$

Theorem. Let T be a monotone operator on a separable Banach space X and $n < \dim X$ be a positive integer. Then A_n is representable as a union of countably many CFC-fragments of codimension n . If the dual space X^* is separable then A^n is representable as a countable union of CFC-fragments of dimension n .

Proof. We shall prove both the propositions of the theorem simultaneously. Without any loss of generality we can assume that T is maximal monotone, hence Tx is always convex.

There was proved in [1] that if X (or X^* , resp.) is separable then A_n (or A^n , resp.) can be written as a countable union of Lipschitz fragments B of codimension n (of dimension n , resp.), each of them having the following properties:

- (I) $B = \{w + F(w) : w \in M\}$, $M \subset W$, $F : M \rightarrow Z$
 where W, Z, M, F are as in Definition 2.
- (II) There exist subspaces V, Y of X^* such that
 $X^* = V \oplus Y$, $V = Z^\perp$, $Y = W^\perp$.
- (III) For any $x \in B$ there exist $t_x \in Tx$ and a topological
 complement P_x of V in X^* such that
 $\|t_x\| < m$, $\|\sigma_x\| < q$, $(t_x + P_x) \cap \Omega(t_x, r) \subset Tx$
 where $\sigma_x : X^* \rightarrow P_x$ is a projection in the direction
 of V and m, q, r are positive constants independent
 on $x \in B$.
 (Our constants m, r correspond to constants $m + \frac{r}{2}$,
 $\frac{r}{2}$ from [1], 3.9.)
- (IV) $t_x - t_y \in V$ for any $x, y \in B$.

Now it is sufficient to prove that B is in fact CFC-fragment.

Let $B \subset \mathbb{R}$ and $c : S \rightarrow M$ be arbitrary. If $\text{card } S \neq 2$ then
 $\mathcal{K}(F \circ c, S) = 0$ by Definition 1. So let $\text{card } S \geq 3$ and $s_0 < s_1 < \dots$
 $\dots < s_{k+1}$, $s_j \in S$ ($j = 0, 1, \dots, k+1$). Denote

$$\begin{aligned} w_j &= c(s_j), \quad x_j = w_j + F(w_j) \\ t_j &= t_{x_j}, \quad \sigma_j = \sigma_{x_j}. \end{aligned}$$

Let y^* be an arbitrary functional from a unit sphere in Y .

Put

$$t_j^+ = t_j + \frac{r}{q} \sigma_j(y^*).$$

The fact $t_j^+ \in Tx_j$ follows from (III).

Now for any $i, j \in \{0, 1, \dots, k+1\}$, the monotonicity of T and
 properties (II), (III) imply:

$$\begin{aligned} 0 &\leq \langle x_i - x_j, t_i - t_j^+ \rangle = \\ &= \langle w_i - w_j + F(w_i) - F(w_j), t_i - t_j + \frac{r}{q}(y^* - \sigma_j(y^*)) - \frac{r}{q}y^* \rangle = \\ &= \langle w_i - w_j, t_i - t_j + \frac{r}{q}(y^* - \sigma_j(y^*)) \rangle - \frac{r}{q} \langle F(w_i) - F(w_j), y^* \rangle. \end{aligned}$$

Hence

$$(1) \quad \langle F(w_i) - F(w_j), y^* \rangle \leq \langle w_i - w_j, \frac{q}{r}(t_i - t_j) + y^* - \sigma_j(y^*) \rangle.$$

By the same way it is possible to obtain

$$(2) \quad - \langle F(w_i) - F(w_j), y^* \rangle \leq \langle w_i - w_j, \frac{q}{r}(t_i - t_j) - y^* + \sigma_i(y^*) \rangle$$

using $0 \leq \langle x_i - x_j, t_i^+ - t_j \rangle$.

For simplicity let us denote $Q(j,i) = \frac{w_j - w_i}{s_j - s_i}$ if $i \neq j$. The inequalities (1),(2) give for any $j \in \{1, 2, \dots, k\}$:

$$\begin{aligned} & \left\langle \frac{F(w_{j+1}) - F(w_j)}{s_{j+1} - s_j} - \frac{F(w_j) - F(w_{j-1})}{s_j - s_{j-1}}, y^* \right\rangle \leq \\ & \leq \langle Q(j+1, j) - Q(j, j-1), y^* - \sigma_j(y^*) - \frac{q}{r} \cdot t_j \rangle + \langle Q(j+1, j), \frac{q}{r} \cdot t_{j+1} \rangle - \\ & - \langle Q(j, j-1), \frac{q}{r} \cdot t_{j-1} \rangle \leq \|Q(j+1, j) - Q(j, j-1)\| \cdot (1 + q + \frac{qm}{r}) + \\ & + \langle Q(j+1, j), \frac{q}{r} \cdot t_{j+1} \rangle - \langle Q(j, j-1), \frac{q}{r} \cdot t_{j-1} \rangle. \end{aligned}$$

It is easy to see that $z^* \in Z^*$ iff there exists $\tilde{y}^* \in Y$ such that $\tilde{y}^* = z^*$ on Z .

Since y^* was an arbitrary functional with $\|y^*\| = 1$ then

$$\begin{aligned} & \left\| \frac{F(w_{j+1}) - F(w_j)}{s_{j+1} - s_j} - \frac{F(w_j) - F(w_{j-1})}{s_j - s_{j-1}} \right\| \leq \\ & \leq (1 + q + \frac{qm}{r}) \|Q(j+1, j) - Q(j, j-1)\| + \langle Q(j+1, j), \frac{q}{r} \cdot t_{j+1} \rangle - \\ & - \langle Q(j, j-1), \frac{q}{r} \cdot t_{j-1} \rangle. \end{aligned}$$

Taking the sum over $j=1, 2, \dots, k$ we get

$$\begin{aligned} & \sum_{j=1}^k \left\| \frac{F(w_{j+1}) - F(w_j)}{s_{j+1} - s_j} - \frac{F(w_j) - F(w_{j-1})}{s_j - s_{j-1}} \right\| \leq \\ & \leq (1 + q + \frac{mq}{r}) \cdot \sum_{j=1}^k \|Q(j+1, j) - Q(j, j-1)\| + \sum_{j=2}^{k+1} \langle Q(j, j-1), \frac{q}{r} \cdot t_j \rangle - \\ & - \sum_{j=0}^{k-1} \langle Q(j+1, j), \frac{q}{r} \cdot t_j \rangle = (1 + q + \frac{mq}{r}) \sum_{j=1}^k \|Q(j+1, j) - Q(j, j-1)\| - \\ & - \sum_{j=2}^{k-1} \langle Q(j+1, j) - Q(j, j-1), \frac{q}{r} \cdot t_j \rangle + \langle Q(k+1, k), \frac{q}{r} \cdot t_{k+1} \rangle + \\ & + \langle Q(k, k-1), \frac{q}{r} \cdot t_k \rangle - \langle Q(2, 1), \frac{q}{r} \cdot t_1 \rangle - \langle Q(1, 0), \frac{q}{r} \cdot t_0 \rangle \leq \\ & \leq (1 + q + \frac{2mq}{r}) \sum_{j=1}^k \|Q(j+1, j) - Q(j, j-1)\| + \frac{4mq}{r} \text{Lip}(c) \leq \\ & \leq (1 + q + \frac{2mq}{r}) \cdot \mathcal{K}(c, S) + \frac{4mq}{r} \text{Lip}(c). \end{aligned}$$

Then by Definition 1 we have

$$\mathcal{K}(F \circ c, S) \leq a \cdot \mathcal{K}(c, S) + b \cdot \text{Lip}(c)$$

where $a = 1 + q + \frac{2mq}{r}$ and $b = \frac{4mq}{r}$. The theorem is proved.

Remark. Problem 1.1 from [1], whether it is possible to write " σ -convex fragments" instead of "CFC-fragments" in Theorem,

is still open.

R e f e r e n c e

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