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A REMARK ON DISTANCES IN PRODUCTS OF GRAPHS

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Abstract: We estimate the diameter of the direct product of a family of graphs.

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1. This remark concerns the categorical (direct) product of simple graphs. In the paper [1] the formula for the distance between vertices in the product of bipartite graphs is given and the diameter of this product is also determined. In [2] these questions are examined for non-bipartite graphs. We show a simple approach to the solution of these questions based on the notion of the odd and even distance between vertices.

The graphs we consider are undirected and have no multiple edges and no loops; we call these graphs simple. Let us recall that the (direct) product $\prod_{i\in I} G_i$ of a family $\{G_i; i\in I\}$ of simple graphs is defined by

$$V(\prod_{i\in I} G_i)=\prod_{i\in I} V(G_i)$$

and

$$E(\prod_{i\in I} G_i)=\{\{x,y\};\{x_i,y_i\}\in E(G_i) \text{ for every } i\in I\}$$

where x_i denotes the i-th coordinate of the vertex x in $\prod_{i\in I} G_i$. In general, graphs under consideration and the set I can be infinite.

The diameter of a connected graph G is denoted by d(G). A sequence S in G is any finite series $(v_0,v_1,\dots,v_n)$ of vertices in G such that consecutive vertices are connected by an edge. The vertices v_0 and v_n are called the endpoints of S; we also write $S=S(v_0,v_n)$. The sequence is closed iff $v_0=v_n$. The number n is the length of the sequence. We denote the length of the sequence S by $\ell(S)$ and say that S is odd or even if $\ell(S)$ has this property. If $S_1=(v_0,v_1,\dots,v_n)$ and $S_2=(v_n,v_{n+1},\dots,v_{n+k})$ are two sequences with a common endpoint, we define $S_1+S_2=(v_0,v_1,\dots,v_{n+k})$.

2. Let G be a simple graph and x, y be vertices in G. We define the odd and the even distance between vertices x and y by

$$d^1(x,y)=\min\{\ell(S);\ S \text{ is an odd sequence between } x \text{ and } y\}$$

and

$$d^2(x,y)=\min\{\ell(S);\ S \text{ is an even sequence between } x \text{ and } y\}.$$

Here we put $d^1(x,y)=\infty$ ($d^2(x,y)=\infty$) if there is no odd (even) sequence between x and y.

For $i\in\{1,2\}$ we define $d^i(G)=\sup\{d^i(x,y);x,y\in V(G)\}$.

Observation:

(1) $d(x,y)=\min(d^1(x,y),d^2(x,y))$

(2) G is bipartite if and only if for every pair of vertices x, y either $d^1(x,y)=\infty$ or $d^2(x,y)=\infty$.

3. Let $\{G_i;i\in I\}$ be a fixed family of graphs without isolated vertices and x, y fixed vertices of $\prod_{i\in I} G_i$. Let us denote $d_i^1=d^1(x_i,y_i)$, $d_i^2=d^2(x_i,y_i)$, $d_i=d(x_i,y_i)$.

Proposition 1: $d(x,y)=\min(\sup_{i\in I} d_i^1,\sup_{i\in I} d_i^2)$

Proof: It can easily be shown that $d^1(x,y)=\sup_{i\in I} d_i^1$ and $d^2(x,y)=\sup_{i\in I} d_i^2$, hence the proposition follows from the observation (1).

This proposition has a number of immediate corollaries.

Corollary 1 (see [1]): Let all graphs G_i, $i\in I$, be bipartite. Then $d(x,y)<\infty$ if and only if the set $\{d_i;i\in I\}$ is bounded and either all d_i's are odd or all d_i's are even. If $d(x,y)<\infty$ then $d(x,y)=\sup_{i\in I} d_i$. Consequently, every component of $\prod_{i\in I} G_i$ has diameter equal to $\sup_{i\in I} d(G_i)$.

Proof: If $d(x,y)<\infty$ then either $\sup_{i\in I} d_i^1<\infty$ or $\sup_{i\in I} d_i^2<\infty$.

By Observation (2), in the first case $d_i^1=d_i$ for every i and in the second case $d_i^2=d_i$ for every i. Therefore all d_i's are odd in the first case and all d_i's are even in the second case. In both cases $d(x,y)=\sup_{i\in I} d_i$ by Proposition 1. The converse direction is similar.

Further we shall need the following simple proposition.

Proposition 2: Let G, H be connected graphs (with at least one edge), H non-bipartite. Then $G\times H$ is connected.

Now let us formulate the main theorem (compare [2]).

Theorem: Let all graphs G_i, $i\in I$, be non-bipartite. Then $\prod_{i\in I} G_i$ is connected if and only if

(1) $J=\{i\in I; d(G_i)=\infty\}$ is finite

and

(2) $\{d(G_i); i\notin J\}$ is bounded.

Moreover $\sup_{i\in I} d(G_i)\leqq d(\prod_{i\in I} G_i)\leqq 2\cdot\sup_{i\in I} d(G_i)$ and these estimates are the best possible.

The proof is an immediate consequence of the Propositions 1 and 2 and the following lemma.

Lemma: Let G be a non-bipartite graph with the finite diameter d. Then $d^1(G)\leqq 2d+1$ and $d^2(G)\leqq 2d$.

Proof: We shall prove the second inequality (the proof of the first one is similar). Obviously $d^2(x,y)<\infty$. Let $S=S(x,y)$ be the shortest sequence of even length between vertices x and y. Let us suppose that its length is equal to 2d+2k, $k>0$, and denote $S=S_1(x,u)+(u,z,v)+S_2(v,y)$ where S_1 and S_2 are sequences of length d+k-1. There are paths P_1 and P_2 between vertices u, y and x, v respectively, of lengths $d-\ell_1$ and $d-\ell_2$ where $0\leqq \ell_1,\ell_2\leqq d$. Then there are sequences S_1+P_1 and P_2+S_2 between x and y of lengths $(2d-1)+(k-\ell_i)<2d+2k$, hence the numbers $k-\ell_i$ are both even according to the choice of S. Then $\ell_1+\ell_2$ is even and therefore there exists a sequence $P_2+(v,z,u)+P_1$ of even length $2d-(\ell_1+\ell_2)+2$ between x and y. But $2d-(\ell_1+\ell_2)+2<2d+2k$ (if $\ell_1=\ell_2=0$ then $k\geqq 2$ because $k-\ell_i$ are even) contradicting the choice of S.

Proof of Theorem: By Proposition 2 it suffices to show that $\prod_{i \in I \setminus J} G_i$ is connected. But if $d(G_i) \leqq d$ for $i \notin J$ then, by Lemma, $\sup d^2(x_i, y_i) \leqq 2d$ for every pair of vertices x, y in $\prod_{i \in I \setminus J} G_i$. We conclude $d(x,y) \leqq 2d$ by Proposition 1.

4. In this section we show that the estimates in Theorem are the best possible. The notation $[n,m]$ for the set of all integers k such that $n \leqq k \leqq m$ will be used. By $C(0,1,\ldots,k)$ we shall denote the circuit $([0,k]; \{\{i,i+1\};\ i \in [0,k-1]\} \cup \{\{k,0\}\})$ of length k+1. The path $(r, r+1, \ldots, s)$ for $r < s$ or $(r, r-1, \ldots, s)$ for $r > s$ is denoted by $P(r,s)$.

Let G be a non-bipartite graph and G^n its n-th square. Then by Theorem

$$d(G) \leqq d(G^n) \leqq 2 \cdot d(G).$$

Moreover, the following proposition holds.

Proposition: For every $d \geqq 2$ and $i \in [0,d]$ there exists a graph G such that $d(G)=d$ and $d(G^n)=d+i$.

Proof: We reformulate the proposition in the following way:

For every $k \geqq 1$ and $i \in \left[0, \frac{k}{2}\right]$ there exists a graph G such that $d(G^n)=k$ and $d(G)=k-i$.

Since $d(G^n)=\min(d^1(G), d^2(G))$ the proof can be carried out in the following two steps:

(1) For every $k \geqq 1$ and $i \in [0,k]$ there exists a graph G such that $2k=d^2(G) < d^1(G)$ and $d(G)=2k-i$.

(2) For every $k \geqq 1$ and $i \in [0,k]$ there exists a graph G such that $2k+1=d^1(G) < d^2(G)$ and $d(G)=(2k+1)-i$.

Case (1): Let $k \geqq 1$, $i \in [0,k]$. First we construct a graph G such that $d(G)=d^2(G)=2k$ and $d^1(G)=2k+1$.

Let $C_{4k}=C(0,1,\ldots,4k-1)$ be the circuit of length 4k. Let us define

$$G=([0,4k-1],\ E(C_{4k}) \cup \{\{0,2k\}\}).$$

Then clearly $d(G)=2k$, $d^2(k,k+1)=2k$, $d^1(k,k)=2k+1$.
We show that $d^2(m,n) \leqq 2k$ and $d^1(m,n) \leqq 2k+1$ for every $m,n \in [0,4k-1]$.

We can suppose without loss of generality that $m \in [0,k]$.

We consider three cases.

(a) $n \in [0,2k]$.

Then both m and n lie on the circuit of odd length 2k+1 and the proposition is clear.

(b) $n \in [3k,4k]$.

We can suppose without loss of generality that $m=d(m,0) \leq d(n,0)=n$. Then $P(m,n)$ and $P(m,0)+(0,2k)+P(2k,n)$ are paths between m and n of lengths $\leq 2k+1$, one of them even and the other odd, hence both numbers $d^1(m,n)$ and $d^2(m,n)$ are less than 2k+2.

(c) $n \in [2k,3k]$.

We can suppose that $m=d(m,0) \leq d(n,2k)=n-2k$ and the proof can be finished using the same paths as in the case (b).

Now we consider the circuit $C_{4k-2i}=C(0,1,\dots,4k-2i-1)$ of length 4k-2i and define

$$G=([0,4k-2i-1],\ E(C_{4k-2i}) \cup \{\{0,2k\}\}).$$

Then $d^2(G)=2k$ and $d^1(G)=2k+1$ (the proof is analogous to the case i=0) and $d(G)=2k-i$.

Case (2): Let $C_{2k+1}=C(0,1,\dots,2k)$ and $C'_{2\ell+1}=C(0',1',\dots,(2\ell)')$ be the circuits of lengths 2k+1 and $2\ell+1$ with disjoint sets of vertices.

For $i \in [0,2k]$, $j \in [0,2\ell]$ let us define the graph $G^{k,\ell}_{i,j}$ by

$$V(G^{k,\ell}_{i,j})=[0,2k] \cup [0,2\ell]' \text{ where}$$

$[0,2\ell]'$ denotes the set $\{i';i \in [0,2\ell]\}$ and

$$E(G^{k,\ell}_{i,j})=E(C_{2k+1}) \cup E(C'_{2\ell+1}) \cup \{0,0'\} \cup \{i,j'\}.$$

Let $L^{k,\ell}_{i,j}=\max\{i,j,2k+1-i,2\ell+1-j\}$. It is easy to show that

$$(1) \qquad d(G^{k,\ell}_{i,j}) \leq L^{k,\ell}_{i,j}+1.$$

Further we define the graph $G_0=([0,2k] \cup [0,2\ell]',\ E(C_{2k+1}) \cup E(C'_{2\ell+1}) \cup \{0,0'\})$ and denote $L=\max(2k+1,2\ell+1)$. Then

$$(2) \qquad d^1(G_0) \leq L \text{ and } d^2(G_0) \leq L+1$$

For this, let x, y be the vertices of G_0. We show that there exists a path of even length $\leq L+1$ and also a path of odd length $\leq L+1$ between x and y. If $\{x,y\} \subseteq [0,2k]$ or $\{x,y\} \subseteq [0,2\ell]'$ then the proposition is obvious. Hence we can suppose that $x=i \in [0,k]$, $y=j' \in [0,\ell]'$ and $i \leq j$. Then $P(i,0)+(0,0')+P(0',j')$ and

$P(i,0)+(0,0')+P(0',j')$ are the paths to be found.

Now let us define a graph G^* by

$V(G^*)=[0,2k]\,\dot{\cup}\,[0,2k]'$

and

$E(G^*)=E(C_{2k+1})\cup E(C'_{2k+1})\cup\{\{i,i'\};i\in[0,2k]\}$.

<u>Lemma 1</u>: $d^1(G^*)=2k+1$ and $d^2(G^*)=2k+2$.

<u>Proof</u>: By (2) $d^1(G^*)\leqq 2k+1$ and $d^2(G^*)\leqq 2k+2$. We show that $d^1(0,0)=2k+1$ and $d^2(0,0')=2k+2$. Let S be a sequence of odd length from 0 to 0. Let f be a mapping defined by $f(i)=f(i')=i$. Clearly S contains an even number of edges of type $\{i,i'\}$. Therefore the image f(S) of S is the closed sequence from 0 to 0 of odd length in the graph $C(0,1,\dots,2k)$. Hence, f(S) contains every edge of $C(0,1,\dots,2k)$ and the length of S is at least 2k+1. Now, let S be an even sequence from 0 to 0'. Then f(S) is the closed sequence from 0 to 0 of odd length, hence S contains at least 2k+1 edges of type $\{i,i+1\}$ or $\{i',(i+1)'\}$. But S must contain some edge of type $\{i,i'\}$, hence the length of S is at least 2k+2.

Next we consider graphs $G^{k,k}_{i,i}$, $1\leqq i\leqq k$, and denote $G^{k,k}_{i,i}$ simply by G_i. An immediate consequence of (2) and Lemma 1 is the following:

<u>Lemma 2</u>: $d^1(G_i)=2k+1$ and $d^2(G_i)=2k+2$.

<u>Lemma 3</u>: $d(G_i)=(2k+1)-(i-1)$ for $1\leqq i\leqq k$.

<u>Proof</u>: By (1) $d(G_i)\leqq L^{k,k}_{i,i}+1=(2k+1-i)+1$.

If i is odd then $d(0,\frac{2k+1+i}{2})=d(i,\frac{2k+1+i}{2})=\frac{2k+1-i}{2}$, hence $d(\frac{2k+1+i}{2},(\frac{2k+1+i}{2})')=\frac{2k+1-i}{2}+1+\frac{2k+1-i}{2}=(2k+1-i)+1$.

If i is even then $d(0,\frac{2k+i}{2})=\frac{2k-i+2}{2}=\frac{2k-i}{2}+1=d(i,\frac{2k+i}{2})+1$ and $d(i',(\frac{2k+i+2}{2})')=\frac{2k-i+2}{2}=\frac{2k-i}{2}+1=d(0',(\frac{2k+i+2}{2})')+1$, hence $d(\frac{2k+i}{2},(\frac{2k+i+2}{2})')=\frac{2k-i}{2}+(\frac{2k-i}{2}+1)+1=(2k+1-i)+1$.

By Lemma 2 and Lemma 3 the proof of Case (2) will be completed if we construct the graph G such that $2k+1=d^1(G)<d^2(G)$ and $d(G)=k+1$. But G^* has these properties as follows from Lemma 1 and the following Lemma 4.

Lemma 4:

(1) $d^{G^*}(i,j')=\min(|i-j|,2k+1-|i-j|)+1$

(2) $d(G^*)=k+1$.

References

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