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A REMARK ON RADICAL-SEMISIMPLE CLASSES OF FULLY ORDERED GROUPS

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Abstract: It is shown that a non-trivial radical-semisimple class of fully ordered groups cannot determine a hereditary upper radical or a homomorphically closed semisimple class.

Key words: Radical-semisimple class, fully ordered groups.

Classification: 06F25

The study of radical and semisimple classes of fully ordered groups was initiated by Chehata and Wiegandt [1]. For references to the subsequent work on this topic, the references of Gardner [2] can be consulted. The radical theory of this class of groups has some peculiar properties; the mentioned two papers can be consulted. We will show here that a non-trivial radical-semisimple class of fully ordered groups (such classes do exist) can never have a hereditary upper radical or a homomorphically closed semisimple class. This result is based on two results from Gardner [2] and the theory of complementary radicals [3].

Let us firstly agree on some notation and conventions. Fully ordered groups (f.o. groups) are not necessarily abelian. If I is a convex normal subgroup of G, it will be denoted by $I \triangleleft G$. A class of f.o. groups $\mathcal{M}$ is <u>hereditary</u> if $I \triangleleft G \in \mathcal{M}$ implies $I \in \mathcal{M}$ and <u>homomorphically closed</u> if any 0-homomorphic image of a member from $\mathcal{M}$ is also in $\mathcal{M}$. We will also use the following two conditions that $\mathcal{M}$ may satisfy:

(*) $0 \neq A \triangleleft B$ and $A \in \mathcal{M}$ implies $B \in \mathcal{M}$.

(**) $0 \neq A/B \in \mathcal{M}$ implies $A \in \mathcal{M}$.

As usual, $\mathcal{U}$ and $\mathcal{S}$ will denote the upper radical and semisimple operators respectively. The next two assertions have been proved by Gardner [2] for fully ordered abelian groups. They remain true for arbitrary f.o. groups.

Let $\mathcal{R}$ be a radical class of f.o. groups, $\mathcal{S}$ the corresponding semisimple class. Then

(1) $\mathcal{R}$ is hereditary iff $\mathcal{S}$ satisfies the condition $(*)$.

(2) $\mathcal{S}$ is homomorphically closed iff $\mathcal{R}$ satisfies the condition $(**)$.

We shall also need the following: A radical class $\mathcal{R}$ of f.o. groups is a <u>complementary radical class</u> if $\mathcal{R} \cup \mathcal{S}\mathcal{R}$ is the class of all f.o. groups. A semisimple class $\mathcal{S}$ is a <u>complementary semisimple class</u> if $\mathcal{U}\mathcal{S}$ is a complementary radical class. In [3] it was shown that there are no non-trivial complementary radical or semisimple classes in the class of all f.o. groups.

We can now state and prove our main result:

<u>Theorem</u>. Let $\mathcal{R} \neq 0$ be a radical-semisimple class of f.o. groups. The following are equivalent:

(i) $\mathcal{U}\mathcal{R}$ is hereditary

(ii) $\mathcal{S}\mathcal{R}$ is homomorphically closed

(iii) $\mathcal{R}$ is the class of all f.o. groups.

<u>Proof</u>. Clearly only (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) need a verification. Firstly, assume $\mathcal{U}\mathcal{R}$ is hereditary. From (1) above, it follows that $\mathcal{S}\mathcal{U}\mathcal{R} = \mathcal{R}$ must satisfy the condition $(*)$. Since $\mathcal{R}$ is a radical class, Proposition 2.2 in [3] yields $\mathcal{R}$ a complementary radical class. But such classes are only the trivial ones (Example 5 in [3]) and we conclude that $\mathcal{R}$ must be the class of all f.o. groups. If $\mathcal{S}\mathcal{R}$ is homomorphically closed, then from (2) above $\mathcal{U}\mathcal{S}\mathcal{R} = \mathcal{R}$ must satisfy the condition $(**)$. But any semisimple class which satisfies the condition $(**)$ must be a complementary semisimple class in view of Proposition 2.2* in [3]. As above, we conclude that $\mathcal{R}$ is the class of all f.o. groups.

References

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