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ON A CERTAIN CLASS OF MULTIDIGRAPHS, FOR WHICH REVERSAL OF NO ARC DECREASES THE NUMBER OF THEIR CYCLES

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Abstract: We describe and construct a class of directed graphs with multiple arcs for which Ádám's conjecture does not hold, i.e. reversal direction of no arc of these multidigraphs decreases the number of their directed cycles.

Key words: Digraph with multiple arcs, directed cycle.

Classification: 05C20, 05C38

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1. Introduction. Let $G=(V,A)$ and $c(G)$ denote a digraph (i.e. directed graph without loops) and number of its directed cycles respectively (for unexplained notation we refer to [1]).

We consider the Ádám's conjecture [4]:
for every digraph $G=(V,A)$ with $c(G)>0$ there is an arc $\langle x,y\rangle \in A$ such that $c((V,(A-\{\langle x,y\rangle\})\cup\{\langle y,x\rangle\}))< c(G)$
i.e. for every digraph G with $c(G)>0$ there is an arc reversing which dicreases the number of directed cycles.

In 1976 E.J. Grinberg [2] gave the negative answer to Ádám's conjecture for multidigraphs (i.e. digraphs with multiple arcs). Generalizing the Grinberg example we give here an infinite family of counterexamples to Ádám's conjecture. Note that C. Thomassen independently found a class of counterexamples [5]. This paper is a part of the author's thesis [3].

Given a digraph $G=(V,A)$ and natural number $p>0$ we define the multidigraph $G^p=(V,A^p)$ which we obtain when replacing every arc $\langle x,y\rangle \in A$ by p parallel arcs, i.e. vertices x and y are connected in G^p by p arcs provided it was the case in G. We shall construct a class of digraphs $G_n=(V_n,A_n)$ such that for every $n\geq 4$ there is a p such that the multidigraph $G_n^p=(V_n,A_n^p)$ does not fulfil the Ádám's conjecture, i.e. reversing any arc $\langle x,y\rangle \in A_n^p$ (of course only one of parallel ones) the number of directed cycles will not decrease.

First we define digraphs G_n and prove two technical lemmas in which we evaluate the possible length of directed paths in digraphs G_n. Exploiting this for multidigraphs G_n^p we prove the main theorem.

2. <u>The construction and evaluations.</u> We consider digraphs $G_n=(V_n,A_n)$ for $n \geq 4$ with the set of vertices $V_n=\{[i,j]:i,j \in \{1,..,n\}\}$ and the set of arcs $A_n=\{\langle [a,b],[a,b+1]\rangle : a \in \{1,..,n\}, b \in \{1,..,n-1\}\} \cup \{\langle [a,b],[a+1,b]\rangle : a \in \{1,..,n-1\}, b \in \{1,..,n\}\} \cup \cup \{\langle [a,n],[1,a]\rangle : a \in \{1,..,n\}\} \cup \cup \{\langle [n,b],[b,1]\rangle : b \in \{1,..,n\}\}$ (drawn in the plane - see Fig. 1). A set of arcs $P=\{\langle x_1,x_2\rangle, \langle x_2,x_3\rangle,...,\langle x_{k-1},x_k\rangle\} \subseteq A_n$ such that $\forall i \neq j : x_i \neq x_j$ is said to be a (directed) path (shortly $P=\langle x_1,...,x_k\rangle$) of length $|P|=k-1$. If moreover $\langle x_k,x_1\rangle \in A_n$, then $C=P\cup\{\langle x_k,x_1\rangle\}$ is said to be a directed cycle of length k.

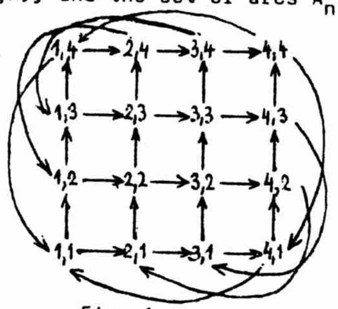

Fig. 1

By the definition of digraph G_n, the sum of coordinates of vertices is increasing by 1 (modulo n) in every step of a path. Hence it holds: $\forall [a,b],[c,d] \in V_n$:

$$|\langle [a,b],..,[c,d]\rangle| \in \{kn+(c-a+d-b) \bmod n : k \in \{1,..,n-1\}\}$$

Thus for $\langle x,y\rangle \in A_n$: $|\langle x,..,y\rangle| \in \{kn+1 : k \in \{1,..,n-1\}\}$
and $|\langle y,..,x\rangle| \in \{kn-1 : k \in \{2,..,n\}\}$.

<u>Lemma 1</u>. For every arc $\langle x,y\rangle \in A_n$ of G_n holds

$$|\langle x,..,y\rangle| \in \{1,n+1,2n+1\}, \quad |\langle y,..,x\rangle| \in \{n-1,2n-1\}.$$

Proof. We use the invariance of G_n with respect to the rotation around a diagonal (by isomorphism $[a,b] \longmapsto [b,a]$ and with respect to the translation in the direction of the diagonal (by isomorphism

$$[a,b] \longmapsto \begin{cases} [a+1,b+1] & \text{for } a \neq n \neq b \\ [1,a+1] & \text{for } b=n \\ [b+1,1] & \text{for } a=n \\ [1,1] & \text{for } a=n=b \end{cases} \Bigg) .$$

Then w.l.o.g. we can consider only arcs $\langle x,y\rangle$ of type $\langle [a,b],[a,b+1]\rangle$ or $\langle [a,b],[a+1,b]\rangle$ (connecting two "neighbouring" vertices "inside" the square $[1,1]..[n,1]..[n,n]..[1,n]..[1,1]$).

Corresponding paths contain two types of "short cut" arcs ($\langle [a,n],[1,a]\rangle$ and $\langle [n,b],[b,1]\rangle$). According to the number and type of "short cut" arcs in the path we can consider only the following cases:

(i) The path contains two successive "short cut" arcs of different type e.g. $\langle x,..[a_0,n], [1,a_0]..[1,a_1],[2,a_1]..$ $..[n,a_n];[a_n,1]..,y\rangle$ where $1 \le a_0 \le \dots \le a_n \le n$, $a_0 < a_n$.

Then it separates vertices of the square so that

$\{x,\dots,[a_0,n]\} \in \{[i,b_i] : b_i > a_i,\ i=1,\dots,n\}$ and

$\{[a_n,1],\dots,y\} \in \{[i,b_i] : b_i < a_{i-1},\ i=1,\dots,n\}$.

Thus $\langle x,y\rangle \notin A_n$, which is a contradiction.

(ii) The path contains three "short cut" arcs of the same type e.g. $\langle x,..[n,a_1];[a_1,1]..[n,a_2];[a_2,1]..[n,a_3];[a_3,1]..,y\rangle$.

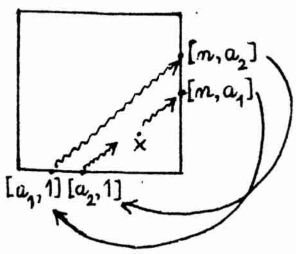

Fig. 2

Let $a_1 < a_2$, then necessarily $a_3 < a_2$ (see Fig. 2).
If $a_3 < a_1$ then $[n,a_1];[a_1,1]...$ $..[n,a_2];[a_2,1]$ separates vertices x and y (see (i)).
If $a_1 < a_3 < a_2$ then $[n,a_2];[a_2,1]..[n,a_3];[a_3,1]$ separates vertices x and y.
Thus again $\langle x,y\rangle \notin A_n$.
Analogously for $a_1 > a_2$.

(iii) Other cases:

- path with two "short cuts" of the same type
  e.g. $|\langle [a,b]..[n,a_1];[a_1,1]..[n,a_2];[a_2,1]..[c,d]\rangle| = 2n+c-a+d-b$
  and for $\langle x,y\rangle \in A_n$: $|\langle x,..,y\rangle| = 2n+1$, $|\langle y,..,x\rangle| = 2n-1$,
- path with only one "short cut" arc:
  e.g. $|\langle [a,b]..[n,a_1];[a_1,1]..[c,d]\rangle| = n+c-a+d-b$
  for $\langle x,y\rangle \in A_n$: $|\langle x,..,y\rangle| = n+1$, $|\langle y,..,x\rangle| = n-1$,
- path without any "short cuts"
  for $\langle x,y\rangle \in A_n$: $|\langle x,y\rangle| = 1$ and such path $\langle y,..,x\rangle$ does not exist.

Lemma 2. For every arc $\langle x,y\rangle \in A_n$ of G_n $(n \ge 4)$ there is a path $\langle x,\dots,y\rangle$ such that $|\langle x,\dots,y\rangle| = 2n+1$.

Proof. Using the above mentioned invariances w.l.o.g. we construct $\langle [a,b]..[a,n]..[n,n];[n,1]..[n,a+1];[a+1,1]..[a+1,b]\rangle$.

3. <u>Main theorem and remarks</u>. Let $G_n^p=(V_n,A_n^p)$ be a multidigraph, containing p parallel copies of every arc of G_n.

<u>Theorem</u>. For every $n\geq 4$ there is p such that Ádám's conjecture does not hold for G_n^p, i.e.

$$\forall\langle x,y\rangle\in A_n^p: c((V_n,(A_n^p-\{\langle x,y\rangle\})\cup\{\langle y,x\rangle\}))-c(G_n^p)\geq 0.$$

Proof. We denote by $(x,y)^G$ the number of all paths $\langle x,\dots,y\rangle$ of digraph G and by $(x,y)_i^G$ the number of paths $\langle x,\dots,y\rangle$ of length i (obviously $\sum_{i=1}^{|V|}(x,y)_i^G=(x,y)^G$).

We reformulate our theorem using the number of paths and we obtain $\forall\langle x,y\rangle\in A_n^p: c((V_n,(A_n^p-\{\langle x,y\rangle\})\cup\{\langle y,x\rangle\}))-c(G_n^p)=$

$$=(p-1)(y,x)^{G_n^p}+(x,y)^{G_n^p}-1-p(y,x)^{G_n^p}=(x,y)^{G_n^p}-(y,x)^{G_n^p}-1=$$

$$=\sum_{i=1}^{n^2}p^i(x,y)_i^{G_n}-\sum_{i=1}^{n^2}p^i(y,x)_i^{G_n}-1.$$

Using Lemma 1 we get the expression

$$(*)\quad p^{2n+1}(x,y)_{2n+1}^{G_n}-p^{2n-1}(y,x)_{2n-1}^{G_n}+p^{n+1}(x,y)_{n+1}^{G_n}-p^{n-1}(y,x)_{n-1}^{G_n}+p-1$$

and using Lemma 2 ($(x,y)_{2n+1}^{G_n}>0$) there is p such that the expression $(*)$ is non-negative. Q.E.D.

<u>Remark 1</u>. Studying the minimal value of p we can show:

(i) $(x,y)_{n+1}^{G_n}=(y,x)_{n-1}^{G_n}$ for every $\langle x,y\rangle\in A_n$ (according to one-to-one correspondence between paths

$\langle[a,b],[a,b+1]..[c,n]\ [1,c]..[1,d],[2,d]..[a+1,b-1],[a+1,b]\rangle$

and

$\langle[a+1,b]..[c+1,n-1],[c+1,n]..[d+1,n]\ [1,d+1]..[a,b]\rangle$).

(ii) Using still another invariance - converting the arc's direction followed by rotation around a secondary diagonal (with isomorphism $[a,b]\mapsto[n+1-b,n+1-a]$) - we may consider only $\left\lceil\frac{n+1}{2}\right\rceil$ classes of arcs of G_n.

Counting $(x,y)_{2n+1}^{G_n}$, $(y,x)_{2n-1}^{G_n}$ for arcs of digraph G_4 we obtain: for arcs from the class containing $\langle[1,1],[2,1]\rangle$:

$$(x,y)_9=14,\ (y,x)_7=27$$

and for arcs from the class containing $\langle[2,1],[3,1]\rangle$:

$$(x,y)_9=3,\ (y,x)_7=11.$$

Therefore $4\cdot(x,y)_9>(y,x)_7$, thus p=2 in $(*)$ is sufficient.

(iii) Similarly (counting paths in the considered classes) Ádám's conjecture does not hold already for multidigraphs G_n^2 when $n \leq 16$.

Remark 2. In reformulation of Ádám's conjecture for multidigraphs we can require to turn all copies of the arc $\langle x,y \rangle \in A_n$ in G_n^p. Then the answer easily follows from that one given above (we obtain it by replacing 1 by p in (*) and multiplying the whole expression by p).

Also leaving out any arc of digraph G_n, Ádám's conjecture already holds for all G_n^p.

Remark 3. In [3] we showed that Ádám's conjecture holds for some classes of digraphs.

Remark 4. For simple digraphs (without multiple arcs) Ádám's conjecture is still open (though it holds for $|V| \leq 8$).

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