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ON THE STRUCTURE OF COMMUTING ISOMETRIES  
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Abstract: We give two examples disproving a Słociński's conjecture about the structure of two commuting isometries.

Key words: Commuting isometries, Wold decomposition, unilateral shift.

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Let  $V$  be an isometry acting on a separable (complex) Hilbert space  $H$ . By the well-known Wold theorem  $H$  can be decomposed into the orthogonal sum  $H = H_1 \oplus H_2$  where  $H_1$  and  $H_2$  reduce  $V$ ,  $V|_{H_1}$  is unitary, and  $V|_{H_2}$  is a unilateral shift. For a pair of commuting isometries the situation is much more complicated. This was studied in a series of papers [6], [9], [12], [1], [11], [7], [2] but satisfactory results were obtained only in the case of a pair  $V_1, V_2 \in B(H)$  of doubly commuting isometries ( $V_1V_2 = V_2V_1$ ,  $V_1V_2^* = V_2^*V_1$ , see [10], [7]). In this case space  $H$  can be decomposed into the orthogonal sum of four subspaces

$$(1) \quad H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss}$$

such that all the summands reduce both  $V_1$  and  $V_2$ ,  $V_1|_{H_{uu} \oplus H_{us}}$  and  $V_2|_{H_{uu} \oplus H_{su}}$  are unitary,  $V_1|_{H_{su} \oplus H_{ss}}$  and  $V_2|_{H_{us} \oplus H_{ss}}$  are unilateral shifts.

The more detailed structure of these subspaces is described

in [2].

In [11], Słociński suggested to study pairs of commuting isometries satisfying the following property (we call such isometries compatible).

Definition. Let  $V_1, V_2$  be commuting isometries on a separable Hilbert space  $H$ . We say that  $V_1$  and  $V_2$  are compatible if  $P_1(m)$  commutes with  $P_2(n)$  for every positive integers  $m, n$ , where  $P_i(m)$  is the orthogonal projection onto the range of  $V_i^m$  ( $i = 1, 2$ ).

From further description of summands in the Wold-type decomposition (1) of a pair of doubly commuting isometries it is easy to see that

$$P_1(m) P_2(n) = P_2(n) P_1(m) = P(m, n)$$

for any positive integers  $m, n$ , where  $P(m, n)$  is the orthogonal projection onto the range of  $V_1^m V_2^n$ . This means that any two doubly commuting isometries are compatible but the converse is not true:

Example 1. Let  $S \subset \mathbb{Z} \times \mathbb{Z}$  be a non-void set of pairs of integers such that  $(i, j) \in S$  implies  $(i+1, j) \in S$  and  $(i, j+1) \in S$ . Let  $H_S$  be a Hilbert space with an orthonormal basis  $\{e_s : s \in S\}$ . Define isometries  $V_1(S), V_2(S) \in B(H_S)$  by  $V_1(S)e_{ij} = e_{i+1, j}$ ,  $V_2(S)e_{ij} = e_{i, j+1}$ .

Clearly,  $V_1(S)$  and  $V_2(S)$  are compatible isometries but in general they are not doubly commuting. If for example  $(0, 1) \in S$ ,  $(1, 0) \in S$  and  $(0, 0) \notin S$  then  $V_2(S)^* V_1(S)e_{01} = e_{10}$  and  $V_1(S)V_2(S)^* e_{01} = 0$ .

As the property of compatibility means some sort of orthogonality, the preceding example suggests the possibility of some model for compatible isometries. In [11] Słociński conjectured

that for any two compatible isometries  $V_1, V_2 \in B(H)$  the space  $H$  can be decomposed into the orthogonal sum  $H = \bigoplus_{i=1}^{\infty} H_i$  of subspaces reducing both  $V_1$  and  $V_2$  such that  $V_1|_{H_i}$  and  $V_2|_{H_i}$  are unitarily equivalent to a pair  $V_1(S), V_2(S)$  for some  $S$  (see Example 1).

The aim of this note is to disprove the Słociński's conjecture. We exhibit two examples showing difficulties which arise in the study of compatible isometries. Although the Słociński's conjecture is not true we hope that these two examples will enable to construct some canonical model for compatible isometries similar to the theory of multiplicity for normal operators.

Let  $S, V_1(S)$  and  $V_2(S)$  be as in Example 1. Suppose that both  $V_1(S)$  and  $V_2(S)$  are unilateral shifts, i.e. they contain no unitary part. Let  $(i, j) \in S$ . Note that  $x = e_{ij}$  has the following properties:

(2) For every  $k \geq 0$  there exists  $n_k \geq 0$  such that

$$V_2(S)^k x \in V_1(S)^{n_k} H_S \oplus V_1(S)^{n_k+1} H_S$$

(in fact  $n_k$  is the integer satisfying  $(i-n_k, j+k) \in S$  and  $(i-n_k-1, j+k) \notin S$ ),

(3) if  $V_2(S)^k x \in V_1(S)^r H_S$  for some  $k > 0, r \geq 0$  then

$$(x, V_1(S)^{*r} V_2(S)^k x) = 0.$$

Analogously for  $V_1(S)^k x \in V_2(S)^r H_S$ . These properties will be used later.

Example 2. Let  $M$  be a separable Hilbert space and  $U \in B(M)$  be a unitary operator which contains no bilateral shift (i.e. there is no subspace which reduces  $U$  to a bilateral shift). Put  $H = \bigoplus_{i=0}^{\infty} M_i, M_i = M (i \geq 0)$ , and define isometries  $V_1, V_2 \in B(H)$  by

$$V_1(x_0, x_1, \dots) = (0, x_0, x_1, \dots),$$

$$V_2(x_0, x_1, \dots) = (0, Ux_0, Ux_1, \dots).$$

Clearly,  $V_1$  and  $V_2$  are commuting unilateral shifts which are compatible as  $V_1^n H = V_2^n H$  for every  $n \geq 0$ .

Suppose that there exists a subspace  $H' \subset H$  reducing both  $V_1$  and  $V_2$  such that the pair  $(V_1|_{H'}, V_2|_{H'})$  is unitarily equivalent to  $(V_1(S), V_2(S))$  for some  $S$ . Then there exists  $x \in H' \subset H$ ,  $x \neq 0$ , with property (3). In particular,  $(x, V_2^{*n} V_1^n x) = 0$  and  $(x, V_1^{*n} V_2^n x) = 0$  for every  $n > 0$ . Taking  $H'' = \sqrt{\{ \dots, V_2^{*n} V_1^n x, x, V_1^{*n} V_2^n x, V_1^{*2n} V_2^{2n} x, \dots \}}$  and using the relations

$$V_1 V_1^* = V_2 V_2^*, \quad V_1^* V_2 V_1^* = V_1^{*2} V_2$$

we find that  $V_1^* V_2|_{H''}$  is a bilateral shift. On the other hand,  $V_1^* V_2(x_0, x_1, \dots) = (Ux_0, Ux_1, \dots)$ , hence  $V_1^* V_2$  is an orthogonal sum of countably many copies of  $U$  which was supposed not to contain a bilateral shift. By theory of multiplicity (see [3])  $V_1^* V_2$  does not contain a bilateral shift as well, a contradiction.

Example 3. Let  $S = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : j \geq 0\}$ . For  $x \in \langle 0, 1 \rangle$  let  $d_k(x)$  be the binary digits of  $x = \sum_{k=1}^{\infty} d_k(x) 2^{-k}$  (for the sake of uniqueness we exclude the case  $d_n(x) = d_{n+1}(x) = \dots = 1$  for some  $n$ ). For  $(i, j) \in S$  define

$$B_{ij} = \{x \in \langle 0, 1 \rangle : i + \sum_{k=1}^j d_k(x) \geq 0\}.$$

Let  $H$  be the Hilbert space of all matrices  $f = (f_{ij})_{(i,j) \in S}$  of functions  $f_{ij} \in L^2(\langle 0, 1 \rangle)$ ,  $\text{supp } f_{ij} \subset B_{ij}$ , with the norm  $\|f\|^2 = \sum_{(i,j) \in S} |f_{ij}|^2$ . As usual, we identify functions which differ only on a set of zero Lebesgue measure  $m$ , and all the inclusions are to be understood in this way (for example  $\text{supp } f_{ij} \subset B_{ij}$  means that  $m(\{x \in \langle 0, 1 \rangle : x \notin B_{ij}, f(x) \neq 0\}) = 0$ ). For  $f = (f_{ij}) \in H$  define

$$(V_1 f)_{ij} = f_{i-1,j}, \quad (V_2 f)_{ij} = f_{i,j-1}.$$

Obviously,  $V_1$  and  $V_2$  are commuting isometries. Further

$$(4) \quad \begin{aligned} V_1^n H &= \{(f_{ij})_{(i,j) \in S} : \text{supp } f_{ij} \subset B_{i-n,j}\}, \\ V_2^m H &= \{(f_{ij})_{(i,j) \in S} : \text{supp } f_{ij} \subset B_{i,j-m}\}, \end{aligned}$$

which easily gives that  $V_1$  and  $V_2$  are compatible, and

$$(5) \quad V_1^n H \ominus V_1^{n+1} H = \{(f_{ij}) : \text{supp } f_{ij} \subset B_{i-n,j} - B_{i-n-1,j}\}$$

where

$$B_{i-n,j} - B_{i-n-1,j} = \{x \in \langle 0,1 \rangle : i - n + \sum_{r=1}^j d_r(x) = 0\}.$$

Suppose that there exists a subspace  $H' \subset H$  reducing both  $V_1$  and  $V_2$  such that the pair  $(V_1|_{H'}, V_2|_{H'})$  is unitarily equivalent to  $(V_1(S), V_2(S))$  for some  $S$ . Then there exists  $x \in H' \subset H$ ,  $x \neq 0$ , with property (2). Let  $x = (f_{ij})_{(i,j) \in S}$  and  $i', j'$  be fixed indices such that  $f_{i'j'} \neq 0$ . For  $k \geq 1$  let  $n_k$  be such that

$$V_2^k x \in V_1^{n_k} H \ominus V_1^{n_k+1} H$$

(see (2)). Then (5) gives

$$\begin{aligned} \text{supp } f_{i'j'} &\subset B_{i'-n_k, j'+k} - B_{i'-n_k-1, j'+k} = \\ &= \{x \in \langle 0,1 \rangle : i' - n_k + \sum_{r=1}^{j'+k} d_r(x) = 0\}. \end{aligned}$$

This inclusion with the analogical condition for  $k+1$

$$\text{supp } f_{i'j'} \subset \{x \in \langle 0,1 \rangle : i' - n_{k+1} + \sum_{r=1}^{j'+k+1} d_r(x) = 0\}$$

gives the inclusion

$$\text{supp } f_{i'j'} \subset \{x \in \langle 0,1 \rangle : d_{j'+k+1}(x) = n_{k+1} - n_k\}.$$

Therefore

$$\text{supp } f_{i'j'} \subset \bigcap_{k=1}^{\infty} \{x \in \langle 0,1 \rangle : d_{j'+k+1}(x) = n_{k+1} - n_k\}$$

and  $m(\text{supp } f_{i'j'}) = 0$ , hence  $f_{i'j'} = 0$  a.e., a contradiction.

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