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ON THE DIRICHLET PROBLEM FOR A DEGENERATE ELLIPTIC EQUATION

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Abstract: We study the Dirichlet problem for an elliptic equation in a bounded domain $Q \subset R_n$ with the boundary data in $L^2(\partial Q)$. It is assumed that the ellipticity degenerates at every point of the boundary ∂Q. We prove the existence of a solution in a weighted Sobolev space $W^{1,2}(Q)$.

Key words: Degenerate elliptic equation, the Dirichlet problem.

Classification: 35D05, 35J25

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1. Introduction. In this paper we investigate the Dirichlet problem for a degenerate elliptic equation

(1) $$(L+\lambda)u = -\sum_{i,j=1}^{n} D_i(\rho(x)a_{ij}(x)D_j u) + \sum_{i=1}^{n} a_i(x)D_i u + (a_0(x)+\lambda) = f(x) \quad \text{in } Q,$$

(2) $$u = \Phi \quad \text{on } \partial Q.$$

In a bounded domain $Q \subset R_n$ with a smooth boundary ∂Q, where λ is a real parameter, a boundary data Φ is in $L^2(\partial Q)$ and $\rho(x)$ is a C^2-function on $\bar{Q}$ equivalent to the distance $d(x,\partial Q)$ for $x \in \bar{Q}$ and its properties are described in Section 2.

Throughout this paper we make the following assumptions

(A) The coefficients a_{ij}, a_i and a_0 $(i,j=1,\dots,n)$ are in $C^\infty(R_n)$ $a_{ij}=a_{ji}$ $(i,j=1,\dots,n)$

(B) There exists a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \le \gamma|\xi|^2$$

for all $x \in \bar{Q}$ and $\xi \in R_n$. Moreover there exists a constant $\beta > 0$ such that $a_0(x) \ge \beta$ on $\bar{Q}$.

(C) $f \in L^2(Q)$.

Since the elliptic equation (1) degenerates on ∂Q, the theory of second-order equations with non-negative characteristic form asserts that the boundary condition is to be imposed on a certain subset of ∂Q, which can be described with the aid of the so called Fichera function (see p. 17 in [10]). In our situation the Fichera function is reduced to $z(x)=\sum_{i=1}^{m} a_i(x)D_i\varphi(x)$. Consequently following the terminology of [10], the boundary condition (2) should be imposed on

$$\Sigma_2=\{x\in\partial Q: \sum_{i=1}^{m} a_i(x)D_i\varphi(x)>0\}.$$

Throughout this work it is assumed that

(D) $\sum_{i=1}^{m} a_i(x)D_i\varphi(x)>0$ on ∂Q,

therefore $\Sigma_2=\partial Q$.

The main difficulty encountered in constructing a solution of the Dirichlet problem with L^2-boundary data arises from the fact that functions in $L^2(\partial Q)$ are not, in general, traces of functions from the Sobolev space $W^{1,2}(Q)$. Consequently the Dirichlet problem (1),(2) cannot be reduced to the problem in $\overset{\circ}{W}{}^{1,2}(Q)$. It is also clear that the boundary condition (2) requires a proper formulation.

The purpose of this note is to establish the existence of solutions to the problem (1),(2). We construct a solution by approximating ϕ and f in $L^2(\partial Q)$ and $L^2(Q)$, respectively, by sequences of smooth functions. Then we can use the recent results of [7] in which the existence of solutions in $C(\bar{Q})\cap C^2(Q)$ has been established as well as some estimates near the boundary of the gradient of a solution. In Section 2 we find the uniform bound for this approximating sequence of solutions in a Sobolev space $\tilde{W}^{2,2}(Q)$. The space $\tilde{W}^{2,2}(Q)$, defined in Section 2, appears to be the right Sobolev space to study the Dirichlet problem (1),(2) with $\phi\in L^2(\partial Q)$. Section 3 is devoted to the main existence result. In the final Section 4 we make some comments on the existence of solutions in the case when (D) is replaced by a weaker condition $\sum_{i=1}^{m} a_i(x)D_i\varphi(x)\geq 0$ on ∂Q.

The methods employed in this paper are not new and have appeared in [1],[2] and [9]. The degenerate Dirichlet problem has an extensive literature (see for example [4],[5],[7],[10] and the references given there). The case where $\sum_{i=1}^{m} a_i(x)D_i\varphi(x) < 0$ on ∂Q is more complex and in general the boundary condition is irrelevant (see [4]). Finally we point out that the case $\sum_{i=1}^{m} a_i(x)D_i\varphi(x) > \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(x)D_i\varphi(x)D_j\varphi(x)$ on ∂Q has been considered in [5] but with zero boundary data.

2. Preliminaries. Let $r(x)=\mathrm{dist}(x,\partial Q)$ for $x \in \bar{Q}$. It follows from the regularity of the boundary ∂Q that there is a number σ_0 such that for $\sigma \in (0,\sigma_0]$ the domain $Q_\sigma = Q \cap \{x: \min_{y\in\partial\Omega} |x-y| > \sigma\}$ with the boundary ∂Q_σ possesses the following property: to each $x_0 \in \partial Q$ there is a unique point $x_\sigma(x_0) \in \partial Q_\sigma$ such that $x_\sigma(x_0)=x_0 - \sigma\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The above relation gives a one-to-one mapping at least of class C^2, of ∂Q onto ∂Q_σ. The inverse mapping of $x_0 \longrightarrow x_\sigma(x_0)$ is given by the formula $x_0 = x_\sigma + \sigma\nu_\sigma(x_\sigma)$, where $\nu_\sigma(x_\sigma)$ is the outward normal to ∂Q_σ at x_σ.

Now let $x_0 \in \partial Q$, $0 < \sigma < \sigma_0$ and let $\bar{x}_\sigma$ be given by $\bar{x}_\sigma = x_\sigma(x_0) = x_0 - \sigma\nu(x_0)$. Let

$$A_\varepsilon = \partial Q_\sigma \cap \{x_\sigma; |x_\sigma - \bar{x}_\sigma| < \varepsilon\},$$

$$B_\varepsilon = \{x;\ \tilde{x} = x_\sigma + \sigma\nu_\sigma(\tilde{x}_\sigma),\ \tilde{x}_\sigma \in A_\varepsilon\},$$

and

$$\frac{dS_\sigma}{dS_0} = \lim_{\varepsilon\to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denotes the n-1 dimensional Hausdorff measure of a set A. Mikhailov [9] proved that there is a positive number γ_0 such that

$$\gamma_0^{-2} \le \frac{dS_\sigma}{dS_0} \le \gamma_0^2 \tag{3}$$

and

$$\lim_{\sigma\to 0} \frac{dS_\sigma}{dS_0} = 1 \tag{4}$$

uniformly on ∂Q, and moreover $\frac{dS_\sigma}{dS_0}$ is at least C^1-function on $\partial Q \times [0,\sigma_0]$ (see formula (16) in [9].

According to Lemma 1 in [3] p. 382, the distance $r(x)$ belongs to $C^2(\bar{Q}-Q_{\delta_0})$ if δ_0 is sufficiently small. Denote by $\varrho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties $\varrho(x)=r(x)$ for $x \in \bar{Q}-Q_{\delta_0}$, $\varrho \in C^2(\bar{Q})$, $\varrho(x) \geq \frac{3\delta_0}{4}$ in Q_{δ_0}, $\gamma_1^{-1} r(x) \leq \varrho(x) \leq \gamma_1 r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta = \{x; \varrho(x)=\delta\}$ for $\delta \in (0,\delta_0]$ and finally $\partial Q = \{x; \varrho(x)=0\}$.

The following result is an immediate consequence of Theorem 2.3 in [7].

Theorem 1. Let $f \in W^{\ell,\infty}(Q)$ with $\ell \geq 1$. Then there exists $0 < \mathcal{H} < 1$ with $\mathcal{H} < \inf_{\partial Q} \sum_{i=1}^{m} a_i(x) D_i \varrho(x)$ such that any solution u in $C^2(Q) \cap C(\bar{Q})$ of (1), (2) with $\Phi = 0$ on ∂Q satisfies the estimate

$$(5) \qquad \| \varrho^{1-\mathcal{H}} Du \|_{L^\infty(Q)} \leq C(\ell) \| f \|_{W^{\ell,\infty}(Q)},$$

where $C(\ell)$ is a constant.

To construct a solution of (1), (2) in $W^{2,2}_{loc}(Q)$ we need

Lemma 1. Let Φ_m and f_m be sequences in $C^2(\partial Q)$ and $C^1(\bar{Q})$, respectively, such that

$$\lim_{m\to\infty} \int_{\partial Q} [\Phi_m(x)-\Phi(x)]^2 dS_x = 0 \text{ and } \lim_{m\to\infty} \int_Q [f_m(x)-f(x)]^2 dx = 0.$$

Let u_m be a solution of (1) with $f=f_m$ in $C^2(Q) \cap C(\bar{Q})$ satisfying the boundary condition

$$(2m) \qquad u_m = \Phi_m \text{ on } \partial Q.$$

Then there exist positive constants λ_0 and C, independent of m, such that

$$(6) \qquad \int_Q |D^2 u_m|^2 \varrho^3 dx + \int_Q |Du_m|^2 \varrho dx + \int_Q u_m^2 dx \leq$$
$$\leq C \left(\int_Q f_m^2 dx + \int_{\partial Q} \Phi_m^2 ds_x \right),$$

for all $m=1,2,\ldots$ and $\lambda \geq \lambda_0$.

Proof. According to Theorem 1 and Theorem 2.3 in [7] for each m there exists a solution u_m of (1), (2_m) in $C^2(Q) \cap C(\bar{Q})$ with $\varrho^{1-\mathcal{H}} Du_m \in L^\infty(Q)$ provided $\lambda \geq 0$. Multiplying (1) by u_m and integrating by parts we obtain

$$(7) \qquad \int_{\partial Q_\delta} \delta \sum_{i,j=1}^{m} a_{ij} D_i u_m \cdot u_m D_j \varrho dS_x + \int_{Q_\delta} \varrho \sum_{i,j=1}^{m} a_{ij} D_i u_m D_j u_m dx +$$
$$+ \int_{Q_\delta} \sum_{i=1}^{m} a_i D_i u_m \cdot u_m dx + \int_{Q_\delta} a_0 u_m^2 dx + \lambda \int_{Q_\delta} u_m^2 dx = \int_{Q_\delta} f_m \cdot u_m dx.$$

The first integral can be estimated using Young's inequality

(8) $$\left| \int_{\partial Q_\delta} \delta \sum_{i,j=1}^{m} a_{ij} D_i u_m u_m D_j \, dS \right| \le C_1 \delta^2 \int_{\partial Q_\delta} |Du_m|^2 dS + \int_{\partial Q_\delta} u_m^2 dS,$$

where C_1 is independent of δ. Integrating by parts the third integral we get

(9) $$\int_{Q_\delta} \sum_{i=1}^{m} a_i D_i u_m \cdot u_m dx = \frac{1}{2} \int_{Q_\delta} \sum_{i=1}^{m} a_i D_i (u_m^2) dx =$$

$$= -\frac{1}{2} \int_{\partial Q_\delta} \sum_{i=1}^{m} a_i D_i \varrho u_m^2 dS - \frac{1}{2} \int_{Q_\delta} \sum_{i=1}^{m} D_i a_i u_m^2 dx.$$

Combining (7),(8) and (9) with the ellipticity condition we arrive at the estimate

$$\gamma^{-1} \int_{Q_\delta} \varrho |Du_m|^2 dx + \int_{Q_\delta} (\lambda - \frac{1}{2} + a_0 - \frac{1}{2} \sum_{i=1}^{m} D_i a_i) u_m^2 dx \le$$

$$\le C_1 \delta^2 \int_{\partial Q_\delta} |Du_m|^2 dS + \int_{\partial Q_\delta} (\frac{1}{2} \sum_{i=1}^{m} a_i D_i \varrho + 1) u_m^2 dS + \frac{1}{2} \int_{Q_\delta} f_m^2 dx.$$

Since $\varrho^{1-\varkappa} Du_m \in L^\infty(Q)$, $\lim_{\delta \to 0} \delta^2 \int_{\partial Q_\delta} |Du_m|^2 dS_x = 0$.

Consequently taking λ sufficiently large, say $\lambda \ge \lambda_0$, and letting $\delta \to 0$, we get

(10) $$\int_Q \varrho |Du_m|^2 dx + \int_Q u_m^2 dx \le C_2 \left(\int_{\partial Q} \Phi_m^2 dS + \int_Q f_m^2 dx \right)$$

for all m, where C_2 is independent of m. To estimate $\int_Q |D^2 u_m|^2 \varrho^3 dx$, we first observe that, if v is a $W^{2,2}$-function with compact support in Q, then

$$\int_Q \varrho \sum_{i,j=1}^{m} a_{ij} D_i u_m D_{jk}^2 v dx + \int_Q \sum_{i=1}^{m} a_i D_i u_m D_k v dx + \int_Q (a_0 + \lambda) u_m D_k v dx =$$

$$= \int_Q f_m D_k v dx.$$

Integrating by parts the first integral we get

$$\int_Q D_k \varrho \sum_{i,j=1}^{m} a_{ij} D_i u_m D_j v \, dx + \int_Q \varrho \sum_{i,j=1}^{m} D_k a_{ij} D_i u_m D_j v \, dx +$$

$$+ \int_Q \varrho \sum_{i,j=1}^{m} a_{ij} D_{ki}^2 u_m D_j v \, dx - \int_Q \sum_{i=1}^{m} a_i D_i u_m D_k v \, dx -$$

$$- \int_Q (a_0 + \lambda) u_m D_k v \, dx = - \int_Q f \, D_k v \, dx.$$

Letting $v = D_k u_m (\varrho - \delta)^2$ in Q_δ and $v = 0$ on $Q - Q_\delta$ we deduce from the last equation

(11) $$\int_{Q_\delta} D_k \varrho \sum_{i,j=1}^{m} a_{ij} D_i u_m D_{jk}^2 u_m (\varrho - \delta)^2 +$$

$$+ 2 \int_{Q_\delta} D_k \varrho \sum_{i,j=1}^{m} a_{ij} D_i u_m D_k u_m D_j \varrho (\varrho - \delta) dx +$$

$$+\int_{Q_\delta}\rho\sum_{i,j=1}^{m} D_k a_{ij} D_i u_m D_{jk}^2 u_m(\rho-\delta)^2 dx + 2\int_{Q_\delta}\rho\sum_{i,j=1}^{m} D_k a_{ij} D_i u_m D_k u_m(\rho-\delta) D_j\rho\, dx +$$

$$+\int_{Q_\delta}\rho\sum_{i,j=1}^{m} a_{ij} D_{ki}^2 u_m D_{kj}^2 u_m(\rho-\delta)^2 dx + 2\int_{Q_\delta}\rho\sum_{i,j=1}^{m} a_{ij} D_{ki}^2 u_m D_k u_m(\rho-\delta) D_j\rho\, dx -$$

$$-\int_{Q_\delta}\sum_{i=1}^{m} a_i D_i u_m D_{kk}^2 u_m(\rho-\delta)^2 - 2\int_{Q_\delta}\sum_{i=1}^{m} a_i D_i u_m D_k u_m(\rho-\delta) D_k\rho\, dx -$$

$$-\int_{Q_\delta}(a_0+\lambda) u_m D_{kk}^2 u_m(\rho-\delta)^2 dx - 2\int_{Q_\delta}(a_0+\lambda) u_m D_k u_m(\rho-\delta) D_k\rho\, dx =$$

$$= -\int_{Q_\delta} f\, D_{kk}^2 u_m(\rho-\delta)^2 dx - 2\int_{Q_\delta} f\, D_k u_m(\rho-\delta) D_k\rho\, dx.$$

Let us denote the integrals on the left side of (11) by $J_1,\dots,J_{10}$ Estimation of these integrals can be obtained as follows

(12) $$J_5 \ge \gamma^{-1}\int_{Q_\delta}\sum_{j=1}^{m} |D_{jk}u_m|^2\,\rho(\rho-\delta)^2 dx.$$

Using the Young inequality we get

(13) $$|J_1+J_2+J_3+J_4| \le C_3(\varepsilon)\int_{Q_\delta}|Du_m|^2(\rho-\delta)dx + \varepsilon\int_{Q_\delta}\sum_{j=1}^{m}|D_{kj}u_m|^2(\rho-\delta)^3 dx.$$

Similarly we have

(14) $$|J_6+J_7| \le C_4\left[\int_{Q_\delta}\rho|Du_m|^2 dx + \int_{Q_\delta}|Du_m|^2(\rho-\delta)dx\right] + \varepsilon\left[\int_{Q_\delta}\sum_{j=1}^{m}|D_{kj}^2 u_m|^2\rho(\rho-\delta)^2 dx + \int_{Q_\delta}\sum_{j=1}^{m}|D_{kj}^2 u_m|^2(\rho-\delta)^3 dx\right],$$

(15) $$|J_9| + \left|\int_{Q_\delta} f D_{kk}^2 u_m(\rho-\delta)^2 dx\right| \le C_5\left(\int_{Q_\delta} u_m^2 dx + \int_{Q_\delta} f^2 dx\right) + \varepsilon\int_{Q_\delta}\sum_{j=1}^{m}|D_{kj}^2 u|^2(\rho-\delta)^3 dx$$

and finally

(16) $$|J_8+J_{10}| \le C_6\left[\int_{Q_\delta}|Du_m|^2(\rho-\delta)dx + \int_{Q_\delta} u_m^2 dx\right],$$

where C_i are independent of δ and $\varepsilon > 0$ is to be determined. We deduce from (11) - (16) that

$$\int_{Q_\delta}[(\gamma^{-1}-\varepsilon)\rho(\rho-\delta)^2 - 3\varepsilon(\rho-\delta)^3]\sum_{j=1}^{m}|D_{jk}^2 u_m|^2 dx \le$$

$$\le C_7\left(\int_{Q_\delta}|Du_m|^2(\rho-\delta)dx + \int_{Q_\delta}|Du_m|^2\rho dx + \int_{Q_\delta} f^2 dx + \int_{Q_\delta} u_m^2 dx\right),$$

where $C_7>0$. Since

$$(\gamma^{-1}-\varepsilon)\rho(\rho-\delta)^2 - 3\varepsilon(\rho-\delta)^3 = (\rho-\delta)^2\,[(\gamma^{-1}-\varepsilon)\rho - 3\varepsilon(\rho-\delta)] =$$

$$= (\rho-\delta)^2 \, [(\gamma^{-1}-\varepsilon)(\rho-\delta)+\delta(\gamma^{-1}-\varepsilon)-3\varepsilon(\rho-\delta)] =$$

$$=(\rho-\delta)^2 \, [(\gamma^{-1}-4\varepsilon)(\rho-\delta)+\delta(\gamma^{-1}-\varepsilon)] > (\rho-\delta)^3(\gamma^{-1}-4\varepsilon)$$

for ε sufficiently small, say $\varepsilon = \frac{\gamma^{-1}}{5}$, the last two inequalities yield

$$(17) \quad \int_{Q_\delta} \sum_{j=1}^{m} |D^2_{jk}u_m|^2 (\rho-\delta)^3 dx \le 5\gamma C_7 \, [\int_{Q_\delta} |Du_m|^2 (\rho-\delta) dx +$$

$$+ \int_{Q_\delta} |Du_m|^2 \rho \, dx + \int_{Q_\delta} f^2 \, dx + \int_{Q_\delta} u_m^2 dx].$$

Letting $\delta \to 0$ in (17) and combining the resulting inequality with (10) we easily arrive at (6).

Lemma 1 shows that a possible solution to the problem (1),(2) lies in the space $\tilde{W}^{2,2}(Q)$ defined by

$$\tilde{W}^{2,2}(Q) = \{u; u \in W^{2,2}_{loc}(Q) \text{ and } \int_Q |D^2u(x)|^2 \rho(x)^3 dx +$$

$$+ \int_Q |Du(x)|^2 \rho(x) dx + \int_Q u(x)^2 dx < \infty\}$$

and equipped with the norm

$$||u||^2_{\tilde{W}^{2,2}} = \int_Q |D^2u(x)|^2 \rho(x)^3 dx + \int_Q |Du(x)|^2 \rho(x) dx + \int_Q u(x)^2 dx.$$

The proof that u_m converges weakly in $\tilde{W}^{2,2}(Q)$ to a solution of (1),(2) will be given in Section 4.

3. <u>Traces in</u> $\tilde{W}^{2,2}(Q)$. To proceed further we need some properties of the space $\tilde{W}^{2,2}(Q)$.

<u>Lemma 2</u>. <u>If</u> $u \in \tilde{W}^{2,2}(Q)$ <u>then</u> $\delta^2 \int_{\partial Q_\delta} |Du|^2 ds$ <u>is continuous on</u> $[0,\delta_0]$ <u>and moreover</u>

$$\lim_{\delta \to 0} \delta^2 \int_{\partial Q_\delta} |Du|^2 dS_x = 0.$$

Proof. Let $0 < \delta < \delta_0$, then

$$\int_{Q_\delta - Q_{\delta_0}} \rho |D_i u|^2 dx = \int_{\delta}^{\delta_0} \mu d\mu \int_{\partial Q_\mu} [D_i u(x)]^2 ds =$$

$$= \int_{\delta}^{\delta_0} \mu d\mu \int_{\partial Q} [D_i u(x(x_0))]^2 \frac{dS_\mu}{dS_0} dS_0 = \frac{\delta_0^2}{2} \int_{\partial Q} [D_i u(x_{\delta_0}(x_0))]^2 \frac{dS_{\delta_0}}{dS_0} dS_0 -$$

$$- \frac{\delta^2}{2} \int_{\partial Q} [D_i u(x(x_0))]^2 \frac{dS_\delta}{dS_0} dS_0 -$$

$$- \int_{\delta}^{\delta_0} \mu^2 \int_{\partial Q} \Big[\sum_{j=1}^{n} D^2_{ji} u(x_\mu(x_0)) D_i u(x_\mu(x_0)) \frac{\partial x_\mu}{\partial \mu} \frac{dS_\mu}{dS} + $$

$$+ \big[D_i u(x\ (x_0)) \big]^2 \frac{\partial}{\partial \mu} \Big(\frac{dS_\mu}{dS_0} \Big) \Big] \, dS_0 .$$

From this identity we can compute

$$\delta^2 \int_{\partial Q} [D_i u(x_\delta(x_0))]^2 \frac{dS_\delta}{dS_0} \, dS_0$$

and express this integral in terms of other integrals which are continuous on $[0,\delta_0]$, since $u \in \tilde{W}^{2,2}(Q)$. On the other hand $\frac{dS_\delta}{dS_0} \to 1$, as $\delta \to 0$, uniformly on ∂Q, therefore the continuity of the integral $\delta^2 \int_{\partial Q_\delta} |Du|^2 dS$ easily follows. Assuming that $\lim_{\delta \to 0} \delta^2 \int_{\partial Q_\delta} |Du|^2 dS > 0$, we would have

$$\delta^2 \int_{\partial Q_\delta} |Du|^2 dS > a \text{ on } (0, \delta_1]$$

for some positive constants a and δ_1 and this would imply that

$$\int_{Q - Q_{\delta_1}} \rho |Du|^2 dx = \int_0^{\delta_1} \mu d\mu \int_{\partial Q_\mu} |Du|^2 dS = \infty$$

and we get a contradiction.

<u>Lemma 3</u>. <u>Let</u> $u \in \tilde{W}^{2,2}(Q)$ <u>be a solution of</u> (1), <u>then</u> $\int_{\partial Q_\delta} u^2 dS$ <u>is bounded on</u> $(0,\delta_0]$.

Proof. Multiplying (1) by u and integrating over Q_δ we obtain

$$\frac{1}{2} \int_{\partial Q_\delta} u^2 \sum_{i=1}^{n} a_i D_i \rho dS_x = - \frac{1}{2} \int_{Q_\delta} \sum_{i=1}^{n} D_i a_i u^2 dx + \int_{Q_\delta} \rho \sum_{i,j=1}^{n} a_{ij} D_i u D_j u \, dx +$$

$$+ \delta \int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i u \cdot u D_j \rho dS_x + \int_{Q_\delta} (a_0 + \lambda) u^2 dx - \int_{Q_\delta} f u dx .$$

We may assume that

$$a = \inf_{Q - Q_{\delta_0}} \sum_{i=1}^{n} a_i(x) D_i \rho(x) > 0$$

taking δ_0 sufficiently small, if necessary. Since by Young's inequality

$$\delta \int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i u \cdot u D_j \rho dS_x \le C \delta^2 \int_{\partial Q_\delta} |Du|^2 dS_x + \frac{a}{2} \int_{\partial Q_\delta} u^2 dS_x ,$$

where C is a positive constant depending on n, a and $\|a_{ij}\|_{L^\infty}$ the result follows easily from Lemma 2.

In order to prove the existence of a trace of a solution $u\in\tilde{W}^{2,2}(Q)$ of (1) we introduce an auxiliary function $x^{\delta}:\bar{Q}\to\bar{Q}_{\delta/2}$ defined in the following way.

For $\delta\in(0,\frac{\delta_0}{2}]$ we define the mapping $x^{\delta}:\bar{Q}\to\bar{Q}_{\delta/2}$ by

$$x^{\delta}(x)=\begin{cases} x & \text{for } x\in Q_{\delta},\\ \dfrac{x+y_{\delta}(x)}{2} & \text{for } x\in\bar{Q}-Q_{\delta},\end{cases}$$

where $y_{\delta}(x)$ denotes the closest point on ∂Q_{δ} to $x\in\bar{Q}-Q_{\delta}$. Thus $x^{\delta}(x)=x_{\delta/2}(x)$ for each $x\in\partial Q$, moreover x^{δ} is Lipschitz.

We are now in a position to prove the main result of this section.

<u>Theorem 2.</u> <u>Let</u> $u\in\tilde{W}^{2,2}(Q)$ <u>be a solution of</u> (1), <u>Then there exists a function</u> $\Phi\in L^2(\partial Q)$ <u>such that</u>

$$\lim_{\delta\to 0}\int_{\partial Q}[u(x_{\delta}(x))-\Phi(x)]^2\,dS_x=0.$$

Proof. Since by Lemma 3, $\int_{\partial Q}u(x_{\delta}(x))^2dS_x$ is bounded, there exists a sequence $\delta_m\to 0$, and a function $\Phi\in L^2(\partial Q)$ such that

$$\lim_{m\to\infty}\int_{\partial Q}u(x_{\delta_m}(x))g(x)dS_x=\int_{\partial Q}\Phi(x)g(x)dS_x$$

for each $g\in L^2(\partial Q)$. We prove that the above relation remains valid if the sequence $\{\delta_m\}$ is replaced by the parameter δ.

Since $\int_{\partial Q}u(x_{\delta}(x))g(x)dS_x$ is continuous on $(0,\delta_0]$ it suffices to prove the existence of the limit at 0 and with g replaced by $\Psi\in C^1(\bar{Q})$. Integration by parts yields

$$\int_{\partial Q_{\delta}}\sum_{i=1}^{m}a_iD_i\varrho\,\Psi\,u\,dS_x=-\int_{Q_{\delta}}\sum_{i=1}^{m}D_i(a_i\Psi)u\,dx+\int_{Q_{\delta}}(a_0+\lambda)\Psi u\,dx+$$

$$+\int_{Q_{\delta}}\varrho\sum_{i,j=1}^{m}a_{ij}D_iu\cdot D_j\Psi\,dx+\delta\int_{\partial Q_{\delta}}\sum_{i,j=1}^{m}a_{ij}D_iuD_j\varrho\,\Psi\,dS-\int_{Q_{\delta}}f\Psi\,dx.$$

Using Lemma 2, the continuity of the left side easily follows. Letting $\delta\to 0$, we deduce from the last identity that

$$(18)\quad\int_{\partial Q}\Phi\Psi\sum_{i=1}^{m}a_iD_i\varrho\,dS_x=-\int_{Q}\sum_{i=1}^{m}D_i(a_i\Psi)u\,dx+$$

$$+\int_{Q}(a_0+\lambda)\Psi u\,dx+\int_{Q}\varrho\sum_{i,j=1}^{m}a_{ij}D_iuD_j\Psi\,dx-\int_{Q}f\Psi\,dx=\int_{Q}F(\Psi)dx.$$

It is clear that this relation continues to hold for $\Psi \in W^{1,2}(Q)$. Now taking $\Psi(x)=u(x^{\delta}(x))$ we get

$$(19)\quad \int_{\partial Q}\Phi(x)u(x^{\delta}(x))\sum_{i=1}^{m}a_i(x)D_i\rho(x)dS_x=\int_{Q_\delta}F(u(x))dx+ + \int_{Q-Q_\delta}F(u(x^{\delta}(x))dx.$$

We now prove that

$$(20)\quad \lim_{\delta\to 0}\int_{Q_\delta}F(u(x))dx=\lim_{\delta\to 0}\int_{\partial Q}u(x_\delta(x))^2\sum_{i=1}^{m}a_i(x)D_i\rho(x)dS_x$$

and

$$(21)\quad \lim_{\delta\to 0}\int_{Q-Q_\delta}F(u(x^{\delta}(x))dx=0.$$

Since $x^{\delta}(x)=x_{\frac{\delta}{2}}(x)$ on ∂Q, (19), (20) and (21) yield that

$$\int_{\partial Q}\Phi(x)^2dS_x=\lim_{\delta\to 0}\int_{\partial Q}u(x_\delta(x))^2\sum_{i=1}^{m}a_i(x)D_i\rho(x)dS_x$$

and the L^2-convergence follows from the uniform convexity of $L^2(\partial Q)$.

To show (20), observe that using the fact that u is a solution to (1) we get

$$\int_{Q_\delta}F(u(x))sx=-\int_{Q_\delta}\sum_{i=1}^{m}D_i(a_iu)u\,dx-\int_{Q_\delta}\sum_{i=1}^{m}a_iD_iu\cdot u\,dx-$$

$$-\delta\int_{\partial Q_\delta}\sum_{i,j=1}^{m}a_{ij}D_iu\cdot uD_j\rho dS=\int_{\partial Q_\delta}u^2\sum_{i=1}^{m}a_iD_i\rho dS-\int_{\partial Q_\delta}\sum_{i=1}^{m}a_{ij}D_iu\cdot uD_j\rho dS$$

and this claim follows from Lemma 2. Finally

$$\left|\int_{Q-Q_\delta}F(u(x^{\delta}))dx\right|\le \mathrm{Const}\Big[\int_{Q-Q_\delta}|f(x)||u(x^{\delta})|dx+$$

$$+\int_{Q-Q_\delta}(x)|Du(x)||Du(x^{\delta})|dx+\int_{Q-Q_\delta}|u(x)||u(x^{\delta})|dx+$$

$$+\int_{Q_\delta}|Du(x^{\delta})||u(x)|dx\Big].$$

Now Lemma 2 from [1] implies that the first and third integrals converge to 0 as $\delta\to 0$. The convergence to 0 of the second and fourth integral follows from Lemmas 5 and 3 of [2] respectively.

4. Existence of solution to the problem (1) - (2). Theorem 2 of Section 3 suggests the following approach to the Dirichlet problem (1), (2).

Let $\Phi \in L^2(\partial Q)$. A solution u of (1) in $\tilde{W}^{2,2}(Q)$ is a solution of the Dirichlet problem with the boundary condition (2) if

(22) $\lim_{\delta\to 0}\int_{\partial Q}[u(x_\delta(x))-\Phi(x)]^2 dS_x=0.$

<u>Theorem 3.</u> Let $\lambda\ge\lambda_0$ <u>(where λ_0 is a constant from Lemma 1). Then for every $\Phi\in L^2(\partial Q)$ there exists a unique solution $u\in\tilde W^{2,2}(Q)$ of the problem (1), (2).</u>

Proof. Let u_m be a sequence of solutions of the problem (1), (2m) constructed in the proof of Lemma 1. By the estimate (6) there exists a subsequence, which we relabel as u_m, converging weakly to a function u in $\tilde W^{1,2}(Q)$. According to Theorem 4.11 in [8], $\tilde W^{1,2}(Q)$ is compactly embedded in $L^2(Q)$, therefore we may assume that u_m tends to u in $L^2(Q)$ and a.e. on Q. It is evident that u satisfies (1). By virtue of Theorem 2 there exists a trace $\xi\in L^2(\partial Q)$ of u in the sense of L^2-convergence. We have to show that $\xi=\Phi$ a.e. on ∂Q. As in the proof of Theorem 1, for every $\Psi\in C^1(Q)$ we derive the following identities

$$\int_{\partial Q}\sum_{i=1}^{m} a_i D_i\varrho\,\xi\,\Psi\,dS_x=\int_Q \varrho\sum_{i,j=1}^{m} a_{ij}D_iuD_j\Psi\,dx+\int_Q(a_0+\lambda)u\Psi\,dx-$$
$$-\int_Q\sum_{i=1}^{m} D_i(a_i\Psi)u\,dx-\int_Q f\Psi\,dx=\int_Q F(\Psi)dx$$

and similarly for u_m we have

$$\int_{\partial Q}\sum_{i=1}^{m} D_i\varrho\,\Phi_m\Psi\,dS_x=\int_Q\varrho\sum_{i,j=1}^{m} a_{ij}D_iu_mD_j\Psi\,dx+$$
$$+\int_Q(a_0+\lambda)u_m\Psi\,dx-\int_Q\sum_{i=1}^{m} D_i(a_i\Psi)u_m dx-\int_Q f\Psi\,dx=\int_Q F_m(\Psi)dx.$$

Since $\lim_{m\to\infty}\int_Q F_m(\Psi)dx=\int_Q F(\Psi)dx$, we have that

$$\int_{\partial Q}\Phi\,\Psi\sum_{i=1}^{m} a_iD_i\varrho\,dS_x=\int_{\partial Q}\xi\,\Psi\sum_{i=1}^{m} a_iD_i\varrho\,dS_x$$

for any $\Psi\in C^1(\bar Q)$ and consequently $\Phi=\xi$ a.e. on ∂Q. The uniqueness of solution of (1), (2) can be deduced from the following energy estimate

$$\int_Q|D^2u(x)|^2\varrho(x)^3dx+\int_Q|Du(x)|^2\varrho(x)dx+\int_Q u(x)^2dx\le$$
$$\le C\left[\int_Q f(x)^2dx+\int_{\partial Q}\Phi(x)^2dS_x\right]$$

which is valid for any $u\in\tilde W^{2,2}(Q)$ satisfying (1), (2) with $\lambda\ge\lambda_0$ and the proof of which is a slight modification of the proof of (6). We only use Lemma 2 in place of Theorem 1.

<u>Remark 1.</u> If $\Phi\in L^\infty(\partial Q)$, we may assume that $\lambda=0$. Indeed,

we approximate Φ by a sequence of C^1-functions Φ on ∂Q, which is uniformly bounded in m. The corresponding estimate (6) from Lemma 1 takes the form

$$\int_Q |D^2u_m|^2 \varrho^3 dx + \int_Q |Du_m|^2 \varrho\, dx \le \text{Const}\, \Big[\int_Q f_m^2 dx +$$
$$+ \int_{\partial Q} \Phi_m^2 dS_x + \int_Q u_m^2 dx \Big].$$

It follows from [7] p. 283 that the sequence u_m is uniformly bounded in m and our claim easily follows.

5. Case $\sum_{i=1}^{m} a_i D_i \varrho \ge 0$ on ∂Q.

In this section we assume that $\sum_{i=1}^{m} a_i D_i \varrho \ge 0$ on ∂Q. For each $\varepsilon > 0$ we consider the Dirichlet problem

$$(1^\varepsilon) \quad (L^\varepsilon + \lambda)u = -\sum_{i,j=1}^{m} D_i(\varrho a_{ij} D_j u) + \sum_{i=1}^{m} (a_i + \varepsilon D_i \varrho) D_i i + (a_0 + \lambda)u = f \text{ on } Q,$$

with the boundary condition (2), where $\Phi \in L^2(\partial Q)$.

Inspection of the proof of Theorem 2 shows that there exists λ_0 such that for each $0 < \varepsilon < 1$ there exists a solution $u_\varepsilon \in \tilde{W}^{2,2}(Q)$ of the problem (1^ε), (2).

Theorem 4. Let $\Phi \in L^2(\partial Q)$ and suppose that $\sum_{i=1}^{m} a_i(x) D_i \varrho(x) \not\equiv 0$ on ∂Q. Then there exists a solution u in $\tilde{W}^{2,2}(Q)$ of (1) such that

$$\lim_{\delta \to 0} \int_{\partial Q_\delta} u(x)\Psi(x) \sum_{i=1}^{m} a_i(x) D_i \varrho(x) dS_x = \int_{\partial Q} \Phi(x)\Psi(x) \sum_{i=1}^{m} a_i(x) D_i \varrho(x) dS_x$$

for each $\Psi \in C^1(\bar{Q})$.

Proof. Observe that $\sum_{i=1}^{m} a_i(x) D_i \varrho(x) + \varepsilon |D\varrho(x)|^2 > 0$ on ∂Q.

Hence multiplying (1^ε) by u^ε and integrating by parts over Q_δ and then letting $\delta \to 0$, we obtain that

$$\int_Q \varrho \sum_{i,j=1}^{m} a_{ij} D_i u_\varepsilon D_j u_\varepsilon dx + \int_Q \Big[\lambda + a_0 - \frac{1}{2} \sum_{i=1}^{m} (D_i a_i + \varepsilon D^2_{ii} \varrho)\Big] u_\varepsilon^2 \, dx =$$
$$= \frac{1}{2} \int_{\partial Q} \Big[\sum_{i=1}^{m} a_i D_i \varrho + \varepsilon (D_i \varrho)^2\Big] \Phi^2 dS_x = \int_Q f\, u_\varepsilon \, dx.$$

As in the final part of the proof of Theorem 1 we get

$$\int_Q |D^2 u_\varepsilon|^2 \varrho^3 dx \le C_1 \Big(\int_Q |Du_\varepsilon|^2 \varrho dx + \int_Q u_\varepsilon^2 dx + \int_Q f^2 dx \Big),$$

where $C_1 > 0$ is a constant independent of ε . Combining these two relations we obtain

$$\int_Q |D^2u_\varepsilon|^2\rho^3 dx+\int_Q |Du_\varepsilon|^2\rho dx+\int_Q u_\varepsilon^2 dx \le C_2\left(\int_Q f^2dx+\int_{\partial Q}\Phi^2 dS_x\right),$$

for each $\varepsilon > 0$ and $\lambda \geq \lambda_0$, where λ_0 can be chosen independently of ε . It is clear that there exists $\varepsilon_m \longrightarrow 0$ such that $u_{\varepsilon_m} \longrightarrow u$ weakly in $\tilde{W}^{2,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q and that u is a solution of (1). Taking $\Psi \in C^1(\bar{Q})$ we find out by integration by parts that

$$\int_{Q_\delta}\rho\sum_{i,j=1}^{m} a_{ij}D_iuD_j\Psi dx-\delta\int_{\partial Q_\delta}\sum_{i,j=1}^{m} a_{ij}D_iuD_i\rho\,\Psi dS_x+$$
$$+\int_{Q_\delta}(\lambda+a_0-\sum_{i=1}^{m}D_i(a_i\Psi))udx=\int_{\partial Q_\delta}\sum_{i=1}^{m} a_iD_i\rho u\Psi dS_x+\int_{Q_\delta} fudx.$$

Lemma 2 and the Hölder inequality yield

$$\lim_{\delta\to 0}\delta\int_{\partial Q_\delta}\sum_{i,j=1}^{m} a_{ij}D_iu\cdot\Psi dS_x=0$$

and consequently

(23) $$\lim_{\delta\to 0}\delta\int_{\partial Q_\delta}\sum_{i=1}^{m} a_iD_i\rho u\Psi dS_x=\int_Q\rho\sum_{i,j=1}^{m} a_{ij}D_iuD_j\Psi dx+$$
$$+\int_Q[\lambda+a_0-\sum_{i=1}^{m}D_i(a_i\Psi)]\,udx-\int_Q fudx.$$

Similarly, using the fact that $u_{\varepsilon_m}(x_\delta)$ converges to Φ in $L^2(\partial Q)$, we get that

$$\int_Q\rho\sum_{i,j=1}^{m} a_{ij}D_iu_{\varepsilon_m}D_j\Psi dx+\int_Q[\lambda+a_0-\sum_{i=1}^{m}D_i(a_i+\varepsilon_mD_i\rho)\Psi]\,u_{\varepsilon_m}dx=$$
$$=\int_{\partial Q}\Phi\left[\sum_{i=1}^{m} a_iD_i\rho+\varepsilon_m|D\rho|^2\right]\Psi dS_x+\int_Q fu_{\varepsilon_m}dx.$$

Letting $\varepsilon_m \longrightarrow 0$, we deduce from the last identity that

(24) $$\int_Q\rho\sum_{i,j=1}^{m} a_{ij}D_iuD_j\Psi dx+\int_Q[\lambda+a_0-\sum_{i=1}^{m}D_i(a_i\Psi)]\,u\,dx=$$
$$=\int_{\partial Q}\Phi\Psi\sum_{i=1}^{m} a_iD_i\rho\,dS_x+\int_Q fu\,dx.$$

Comparing (23) and (24) we obtain that

(25) $$\lim_{\delta\to 0}\int_{\partial Q_\delta}\left(\sum_{i=1}^{m} a_iD_i\rho\right)u\Psi dS_x=\int_{\partial Q}\Phi\Psi\sum_{i=1}^{m} a_iD_i\rho dS_x.$$

Remark 2. Assume that $\sum_{i=1}^{m} a_i(x)D_i\varphi(x)=0$ on ∂Q. Inspection of the proof of Theorem 3 shows that there exists a solution $u\in\tilde{W}^{2,2}(Q)$ of (1) such that

(26) $$\lim_{\delta\to 0}\int_{\partial Q_\delta}\Big(\sum_{i=1}^{m} a_iD_i\varphi\Big)\,u\,\Psi\,dS_x=0$$

for each $\psi\in C^1(Q)$. The relation (26) shows that the boundary data Φ is irrelevant. A natural question arises whether a solution u, understood as a limit of a sequence u_ε from Theorem 3, is independent of the choice of Φ. We are only able to give an affirmative answer provided $\Phi\in L^\infty(Q)$.

Indeed, let Φ_1 and Φ_2 belong to $L^\infty(\partial Q)$. Let us denote the corresponding sequences of solutions by u^1_ε and u^2_ε, respectively. Since $u^1_\varepsilon-u^2_\varepsilon$ satisfies the homogeneous equation (1), by Theorem 2.1 in [7], we may assume that $u^1_\varepsilon-u^2_\varepsilon$ is bounded independently of ε. Set

$$\lim_{\varepsilon\to 0} u^1_\varepsilon=u^1 \text{ and } \lim_{\varepsilon\to 0} u^2_\varepsilon=u^2,$$

where the limits are understood weakly in $\tilde{W}^{2,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. It is clear that u^1-u^2 belongs to $\tilde{W}^{2,2}(Q)\cap L^\infty(Q)$. As in Theorem 3 we arrive at the following identity

$$\int_Q \varphi\sum_{i,j=1}^{m} a_{ij}D_i(u^1-u^2)D_j(u^1-u^2)dx+\int_Q\Big(\lambda_0+a_0-\frac{1}{2}\sum_{i=1}^{m}D_ia_i\Big)(u^1-u^2)^2dx=0$$

for $\lambda\ge\lambda_0$, and consequently $u^1=u^2$ a.e. on Q, provided λ_0 is sufficiently large. To establish this identity we have used a relation

$$\lim_{\delta\to 0}\delta\int_{\partial Q_\delta}\sum_{i,j=1}^{m} a_{ij}D_i(u^1-u^2)D_j\varphi(u^1-u^2)dS_x=0,$$

which follows from Lemma 2 provided $u^1-u^2\in L^\infty(Q)$.

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