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Abstract: We give a simple proof of Hurewicz s theorem that if X is a metrizable space, then every closed subspace of X is Baire iff the rationals do not embed as a closed subspace into X.

Key words: Baire, closed subspace, rationals.

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Call a space crowded if it has no isolated points. In this note we give a simple proof of the following result.

Theorem. Let X be a first countable regular space. Then every closed subspace of X is a Baire space iff X has no countable closed crowded subspace.

In view of the well-known fact that up to homeomorphism the space of rationals is the only countable first countable crowded regular space, [S], this is a small generalization of the theorem of Hurewicz, [H], mentioned in the abstract. The proof was found in 1975 or 1976; at the occasion of the Sixth Prague Topology Symposium I have been urged to finally publish it. At the Symposium G. Debs also announced the Theorem.

We proceed to the proof. Necessity is clear. To prove the sufficiency it suffices to prove the following:

Claim. Let Y be a first countable regular crowded space. If $\mathcal{G}$ is a countable collection of dense open sets in Y then Y has a countable crowded subspace K such that $\overline{K}\setminus K \subseteq \bigcap\mathcal{G}$.

For $y \in Y$, if $\mathcal{A}$ is a collection of subsets of Y we say that $\mathcal{A}$ converges to y if for every neighborhood U of y one has $A \subseteq U$ for all but finitely many $A \in \mathcal{A}$, and if A is a subset of Y we

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say A <u>converges</u> to y if $\{\{a\}: a \in A\}$ converges to y.

We prove the Claim: Enumerate $\mathcal{G}$ as $\langle G_n : n \in \omega \rangle$. Construct a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of pairwise disjoint open collections in Y and a sequence $\langle K_n : n \in \omega \rangle$ of countable subsets of Y as follows:

$\mathcal{U}_0 = \{Y\}$. At stage $n+1$, for each $U \in \mathcal{U}_n$, choose $k_n(U) \in U \cap G_n$ and choose an infinite pairwise disjoint collection $\mathcal{V}_n(U)$ of nonempty open sets that converges to $k_n(U)$ and satisfies

(1) $\overline{\bigcup \mathcal{V}_n(U)} \subseteq U \cap G_n$.

Let

$\mathcal{U}_{n+1} = \bigcup \{\mathcal{V}_n(U) : U \in \mathcal{U}_n\}$ and $K_n = \mathrm{ran}(k_n)$.

Note that

(2) $K_n \subseteq \bigcup \mathcal{U}_n$ and $\bigcup \mathcal{U}_{n+1} \subseteq \bigcup \mathcal{U}_n \setminus K_n$ and $\bigcup \mathcal{U}_{n+1} \subseteq G_n$.

This completes the construction of $\langle K_n : n \in \omega \rangle$ and $\langle \mathcal{U}_n : n \in \omega \rangle$.

Of course, the subspace $K = \bigcup_{n \in \omega} K_n$ of Y is countable. It remains to show K is crowded and satisfies $\overline{K} \setminus K \subseteq \bigcap \mathcal{G}$.

For each $n \in \omega$ and each $U \in \mathcal{U}_n$ the subset $\{k_{n+1}(V) : V \in \mathcal{V}_n(U)\}$ of K_{n+1} converges to $k_n(U)$ since $\mathcal{V}_n(U)$ converges to $k_n(U)$, and it does not contain $k_n(U)$ since $k_n(U) \notin K_{n+1}$. Hence K is crowded.

For the proof that K is closed we point out that since the $\mathcal{U}_n$'s are pairwise disjoint, it follows from (2) that

(3) $\forall j \in \omega \ \ \forall U \in \mathcal{U}_j : U \cap K \subseteq \{k_j(U)\} \cup \bigcup \mathcal{V}_j(U)$.

To see this consider any $j \in \omega$, $U \in \mathcal{U}_j$, and $s \in \omega$. We have

$s > j \Rightarrow K_s \subseteq \bigcup \mathcal{U}_s \subseteq \ldots \subseteq \bigcup \mathcal{U}_{j+1} \Rightarrow K_s \cap U \subseteq \bigcup \{V \in \mathcal{U}_{j+1} : V \subseteq U\} =$
$\qquad = \bigcup \mathcal{V}_j(U)$;

$s = j \Rightarrow k_j(U) \in K_s \cap U = \{k_j(V) : V \cap U \neq \emptyset\} = \{k_j(U)\}$; and

$s < j \Rightarrow K_s \cap U \subseteq K_s \cap \bigcup \mathcal{U}_{j+1} \subseteq K_s \cap \bigcup \mathcal{U}_{s+1} = \emptyset$.

The crux of the matter is that (1) and (3) imply

(4) $\forall j \in \omega \ \ \forall U \in \mathcal{U}_j : \overline{K \cap U} \subseteq U$.

Now consider any $x \in \overline{K} \setminus K$. Since $\forall j \in \omega\, [\bigcup \mathcal{U}_{j+1} \subseteq G_j]$, by (2), we prove $x \in \bigcap \mathcal{G}$ if we show that there is a sequence $\langle U_j : j \in \omega \rangle$ with $\forall j \in \omega\, [x \in U_j \in \mathcal{U}_j]$: Let $U_0 = Y$ (recall $\mathcal{U}_0 = \{Y\}$). Next, consider any $j \in \omega$ and assume U_j known. As $x \in \overline{K}$, but $x \neq k_j(U_j)$, and as $\mathcal{V}_j(U_j)$ converges to $k_j(U_j)$, we see from (3) that there is

$U_{j+1} \in \mathcal{V}_j(U_j)$ with $x \in \overline{U_{j+1} \cap K}$. Then $x \in U_{j+1}$ because of (4). This completes the construction of $\langle U_n : n \in \omega \rangle$.

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