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LINEAR FUNCTIONALS IN SLM-SPACES

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Abstract: This article deals with linear functionals defined on statistical linear spaces in Menger's sense (SLM-spaces). The main aim is to describe all continuous linear functionals defined on a SLM-space $(S,\mathcal{J},T)$ as a SLM-space, too. For these purposes we shall define a statistical norm of a linear functional which in a simple way characterizes continuous linear functionals.

Key words: Statistical metric space, statistical linear space, ε-η-topology, t-norm.

Classification: 60B99

Let a SLM-space $(S,\mathcal{J},T)$ be given. Let S^* be a vector space of all linear functionals defined on $(S,\mathcal{J},T)$, let S' be a linear subset $S'\subset S^*$ of all linear functionals continuous in the ε-η-topology. The basic properties of the ε-η-topology are given in [1], [2]. A special case of the dual space to a SLM-space is studied in [3].

Definition 1. Let a SLM-space $(S,\mathcal{J},T)$ be given, let $f\in S^*$, $f\neq 0$. A function $F_f(\cdot)$ defined by

$$F_f(u)=1-\sup_{\{x:f(x)\neq 0\}}\{F_x(\frac{|f(x)|}{u})+\omega F_x(\frac{|f(x)|}{u})\}\quad\text{for } u>0$$

$$F_f(u)=0 \text{ for } u\leq 0,$$

($\omega F_x(u)$ is the jump of $F_x(\cdot)$ at u), will be called a statistical norm of the functional f. For $f\equiv 0$ on S we put $F_0(u)=H(u)$ where $H(u)=0$ for $u\leq 0$ and $H(u)=1$ otherwise.

Properties of the statistical norm:

1. Let $0<u_1\leq u_2$ then $\frac{|f(x)|}{u_1}\geq\frac{|f(x)|}{u_2}$ for every $x\in S$. It implies that for every x with $f(x)\neq 0$

$$1-\{F_x(\frac{|f(x)|}{u_1})+\omega F_x(\frac{|f(x)|}{u_1})\}\leq 1-\{F_x(\frac{|f(x)|}{u_2})+\omega F_x(\frac{|f(x)|}{u_2})\}$$

and hence $F_f(u_1) \leq F_f(u_2)$. The statistical norm of $f \in S^*$ is a non-decreasing function in reals. Further, it is evident that $0 \leq F_f(u) \leq 1$ for every $u \in \mathcal{R}_1$.

2. The function $F_f(\cdot)$ has at most a countable number of discontinuity points and at every point the limits at the left and at the right exist.

3. In general, it is not true that $\lim_{u \to \infty} F_f(u) = 1$. In every case, of course, $\lim_{u \to \infty} F_f(u)$ exists and $\lim_{u \to \infty} F_f(u) \leq 1$.

4. If $F_f(u) = H(u)$ for every $u \in \mathcal{R}_1$, then $f(x) = 0$ for every $x \in S$.

5. In case of such a SLM-space $(S, \mathcal{J}, T)$ where $\omega F_x(0) = 0$ for every $x \neq 0$ the statistical norm F_f can be expressed in the form

$$F_f(u) = 1 - \sup_{x \neq 0} \{F_x(\frac{|f(x)|}{u}) + \omega F_x(\frac{|f(x)|}{u})\}, \text{ too.}$$

Definition 2. A functional $f \in S^*$ is said to be bounded with respect to the statistical norm if

$$\lim_{u \to \infty} F_f(u) > 0.$$

Theorem 1. A functional $f \in S^*$ is bounded with respect to the statistical norm if and only if f is continuous in the ε-η-topology.

Proof. Let $f \in S^*$ and let f be bounded with respect to the statistical norm. As f is linear it is sufficient to prove its continuity at the null vector in S. Assuming $\lim_{u \to \infty} F_f(u) = \varepsilon_0 > 0$ then

$\lim_{u \to \infty} \sup_{\{x: f(x) \neq 0\}} \{F_x(\frac{|f(x)|}{u}) + \omega F_x(\frac{|f(x)|}{u})\} = 1 - \varepsilon_0$ and hence for every $x, |f(x)| > 0$, $\lim_{u \to \infty} \{F_x(\frac{|f(x)|}{u}) + \omega F_x(\frac{|f(x)|}{u})\} \leq 1 - \varepsilon_0$. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence in S, $x_n \neq 0$ for every $n \in \mathcal{N}$ and $x_n \to 0$ in the ε-η-topology. It is clear that for every $n \in \mathcal{N}$

$$\lim_{u \to \infty} \{F_{x_n}(\frac{|f(x_n)|}{u}) + \omega F_{x_n}(\frac{|f(x_n)|}{u})\} = \omega F_{x_n}(0) \leq 1 - \varepsilon_0.$$

Let us suppose that $|f(x_n)| \not\to 0$. Then there exist such an $\varepsilon_1 > 0$ and such a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ that

$$|f(x_{n_k})| \geq \varepsilon_1 \text{ for every } k \in \mathcal{N}.$$

Hence

$$F_{x_{n_k}}(\frac{|f(x_{n_k})|}{u}) + \omega F_{x_{n_k}}(\frac{|f(x_{n_k})|}{u}) \geq F_{x_{n_k}}(\frac{\varepsilon_1}{u}) + \omega F_{x_{n_k}}(\frac{\varepsilon_1}{u})$$

also for every $k \in \mathcal{N}$ and it implies that for every $u>0$

$$\lim_{k\to\infty}\{F_{x_{n_k}}(\frac{|f(x_{n_k})|}{u})+\omega F_{x_{n_k}}(\frac{|f(x_{n_k})|}{u})\}=1 \text{ because } x_{n_k}\to 0$$

in the ε-η- topology.

But as follows from the properties of the supremum

$$\sup_{\{x:f(x)\neq 0\}}\{F_x(\frac{|f(x)|}{u})+\omega F_x(\frac{|f(x)|}{u})\}\geq F_{x_{n_k}}(\frac{|f(x_{n_k})|}{u})+\omega F_{x_{n_k}}(\frac{|f(x_{n_k})|}{u})$$

for every $k \in \mathcal{N}$ and therefore

$$\sup_{\{x:f(x)\neq 0\}}\{F_x(\frac{|f(x)|}{u})+\omega F_x(\frac{|f(x)|}{u})\}=1 \text{ for every } u>0.$$

This last equality is contrary to the assumption that

$$\lim_{u\to\infty}\sup_{\{x:f(x)\neq 0\}}\{F_x(\frac{|f(x)|}{u})+\omega F_x(\frac{|f(x)|}{u})\}=1-\varepsilon_0<1.$$

This result implies that $f\in S^*$ must be continuous in the ε-η-topology.

Let us suppose, on the contrary, that $f\in S'$ is not bounded with respect to the statistical norm, i.e. for every $u>0$

$$\sup_{\{x:f(x)\neq 0\}}\{F_x(\frac{|f(x)|}{u})+\omega F_x(\frac{|f(x)|}{u})\}=1.$$

As f is a linear functional, Definition 1 implies that for arbitrarily chosen $k>0$

$$F_f(u)=1-\sup_{\{x:|f(x)|=k\}}\{F_x(\frac{k}{u})+\omega F_x(\frac{k}{u})\}, \text{ too.}$$

Further, f is continuous and hence $|f(x)|\leq k_0$ in an ε-η-neighborhood $\mathcal{O}(\varepsilon_0,\eta_0)$. Now, let $u_n\nearrow+\infty$, $\varepsilon_n\searrow 0$. Then for every $n\in\mathcal{N}$ there exists $y_n\in S$ where $|f(y_n)|=k$ and therefore $y_n \not\to 0$ in the ε-η-topology but

$$1-\varepsilon<\sup_{\{x:f(x)\neq 0\}}\{F_x(\frac{|f(x)|}{u_n})+\omega F_x(\frac{|f(x)|}{u_n})\}\leq F_{y_n}(\frac{|f(y_n)|}{u_n})+$$

$$+\omega F_{y_n}(\frac{|f(y_n)|}{u_n})+\varepsilon_n\leq\varepsilon_n+F_{y_n}(\frac{k}{u_n})+\omega F_{y_n}(\frac{k}{u_n})\leq$$

$$\leq\varepsilon_n+F_{y_n}(\frac{k}{u_n}+\sigma_n) \text{ where } \sigma_n\searrow 0.$$

It implies that $1-(\varepsilon+\varepsilon_n)<F_{y_n}(\frac{k}{u_n}+\sigma_n)$, i.e. $y_n\in\mathcal{O}(\varepsilon+\varepsilon_n,\frac{k}{u_n}+\sigma_n)$ (for every $n\in\mathcal{N}$) and we have proved that $y_n\to 0$ in the ε-η-topology. This result, of course, is in contradiction to the continuity

of the functional f at the null vector in S. Q.E.D.

Let a SLM-space $(S,\mathcal{J},T)$ be given. Let $a\in\langle 0,1)$ and let us define $n_a(x)=\inf\{\lambda>0:F_x(\lambda)>a\}$. If $x=0$ then $n_a(0)=0$ for every $a\in\langle 0,1)$. On the contrary, if $n_a(x)=0$ for every $a\in\langle 0,1)$ then $x=0$ in S because $x=0$ if and only if $F_x(u)=H(u)$ for every $u\in\mathcal{R}_1$. At the first sight it is clear that $n_a(\lambda x)=|\lambda|n_a(x)$ for every $\lambda\in\mathcal{R}_1$ and $x\in S$. Unfortunately, it is not true that $n_a(x+y)\leq\leq n_a(x)+n_a(y)$ for every pair $x,y\in S$ in $(S,\mathcal{J},T)$ besides the strongest t-norm $T(a,b)=\min(a,b)$. Nevertheless, we can define for every $f\in S^*$ and every $a\in\langle 0,1)$

$$\|f\|_a=\sup\{|f(x)|:n_a(x)\leq 1\}.$$

Let us denote $\mathcal{O}_a=\{x\in S:n_a(x)\leq 1\}$. From the definition of $n_a(\cdot)$ it follows that when $a\leq b$, then $n_a(x)\leq n_b(x)$ for every $x\in S$ and hence $\mathcal{O}_a\supset\mathcal{O}_b$. Further, we immediately obtain that $\|f\|_a\geq\|f\|_b$ if $a\leq b$. We also see that for every real λ

$$\|\lambda f\|_a=|\lambda|\ \|f\|_a \text{ for every } a\in\langle 0,1) \text{ and}$$

every $f\in S^*$. We can prove, in an easy way, the triangular inequality

$$\|f+g\|_a\leq\|f\|_a+\|g\|_a$$

for every $f,g\in S^*$ and every $a\in\langle 0,1)$ because we know that $\sup_x\{|f(x)+g(x)|\}\leq\sup_x\{|f(x)|\}+\sup_x\{|g(x)|\}$. If $\mathcal{O}\in S^*$ is the null functional in S ($\mathcal{O}(x)=0$ for every $x\in S$), then surely $\|\mathcal{O}\|_a=0$ for every $a\in\langle 0,1)$. On the contrary, let us suppose that $\|f\|_a=0$ for every $a\in\langle 0,1)$. This assumption implies that $f(x)=0$ for every $x\in\mathcal{O}_0=\{x\in S:n_0(x)\leq 1\}$. Since for every $x\in S$ there exists such a vector $y\in\mathcal{O}_a$, $y=\lambda x$, we obtain that $f(x)=0$ for every $x\in S$. We can prove a stronger statement even that $\|f\|_a=0$ implies $f(x)=0$ for every $x\in S$. The assumption $\|f\|_a=0$ gives that $f(x)=0$ for every $x\in\mathcal{O}_a=\{x\in S:n_a(x)\leq 1\}$. Let $x_0\in S$, $n_a(x_0)\geq 1$. So, $y_0=\frac{x_0}{n_a(x_0)}\in\mathcal{O}_a$ and hence $f(y_0)=0$. It implies that also $f(x_0)==0$ and it yields together that $f(x)=0$ for every $x\in S$. The proved results lead us to the formulation of the following definition.

<u>Definition 3</u>. Let a SLM-space $(S,\mathcal{J},T)$ be given. Let f be a linear functional in $(S,\mathcal{J},T)$, let $a\in\langle 0,1)$. Then the number

$$\|f\|_a=\sup\{|f(x)|:n_a(x)\leq 1\}$$

where $n_a(x)=\inf\{\lambda>0:F_x(\lambda)>a\}$ will be called a conjugate norm to $n_a(\cdot)$.

The conjugate norm $\|f\|_a$ can assign the infinite value, too. $\|f\|_a$ is defined in $\langle 0,1)$, is nonincreasing and we put $\|f\|_1=$ $=\inf\{\|f\|_a : a<1\}$. As for every $x\in S$ the corresponding probability distribution function F_x is left continuous, then for every $x\in S$ $n_a(x)$ as a function in the argument a in $\langle 0,1)$ is right continuous.

Theorem 2. Let f be a linear functional defined in a SLM-space $(S,\mathcal{J},T)$. f is continuous in the ε-η-topology if and only if there exists $a_0\in\langle 0, 1)$ such that

$$\|f\|_{a_0}<\infty .$$

Proof. Let us suppose that $\|f\|_{a_0}<+\infty$ for $a_0\in\langle 0,1)$. As $\|f\|_a$ is nonincreasing in $\langle 0,1)$, then $\|f\|_a<+\infty$ for every $a\in\langle a_0,1\rangle$, $\|f\|_1=\inf_{a<1}\|f\|_a$. From the definition of the conjugate norm $\|f\|_a$ it follows that for every $x\in\mathcal{O}_{a_0}=\{x:n_{a_0}(x)\leq 1\}$

$|f(x)|\leq\|f\|_{a_0}$. Since $n_{a_0}(x)<1$ iff $F_x(1)>a_0$, we see that the

functional $f(\cdot)$ is bounded in the ε-η-neighborhood $\mathcal{O}(a_0,1)$ and hence f is continuous in the ε-η-topology.

On the contrary, let us suppose that f is a continuous linear functional in the ε-η-topology. Let us suppose that $\|f\|_a=+\infty$ for every $a\in\langle 0,1)$. This assumption implies that for every $n\in\mathcal{N}$ there exists $x_n\in S$ such that $|f(x_n)|>n$ and $x_n\in\mathcal{O}_{a_n}$, $a_n\nearrow 1$. If we put $y_n=\frac{x_n}{n}$, then $|f(y_n)|=\frac{|f(x_n)|}{n}>1$ for every n and $y_n\in\frac{1}{n}\mathcal{O}_{a_n}=$ $=\frac{1}{n}\{x\in S:n_{a_n}(x)\leq 1\}=\{x\in S:n_{a_n}(x)\leq\frac{1}{n}\}$ and hence $y_n\longrightarrow 0$ in the ε-η-topology although $|f(y_n)|>1$. It is impossible because we assumed continuity of the functional f at the null vector in S. Q.E.D.

At the beginning of our considerations we defined the statistical norm of a linear functional defined in a SLM-space $(S,\mathcal{J},T)$. At this situation a natural question arises about the relation between the statistical norm F_f and the conjugate norm $\|f\|_a$ in case of a continuous linear functional defined in S.
For this purpose let us put $a_0=\inf\{a:\|f\|_a<+\infty\}$ in case of a continuous functional f and $\|f\|_1=\inf_{a<1}\|f\|_a$. By these relations we

defined a nonincreasing function $\|f\|_a$ in the interval $\langle a_0,1\rangle$ with finite values in $(a_0,1\rangle$. It is clear that $|||f|||_a = \|f\|_{1-a}$, $a \in \langle 0,1-a_0\rangle$ is a nondecreasing function in $\langle 0,1-a_0\rangle$.

Now, let $\lambda \geq 0$ and let us define

$$\tilde{F}_f(\lambda) = \inf\{a>0:\ |||f|||_a \geq \lambda\} \quad \text{if } \{a>0:\ |||f|||_a \geq \lambda\} \neq \emptyset$$

$$\tilde{F}_f(\lambda) = 1 \quad \text{if } \{a>0:\ |||f|||_a \geq \lambda\} = \emptyset .$$

In this way we obtain a nondecreasing function defined in $\langle 0,+\infty)$ which is left continuous, $\lim_{\lambda\to\infty} \tilde{F}_f(\lambda) = 1-a_0$. Let us put $\varepsilon_f = \lim_{\lambda\to\infty} \tilde{F}_f(\lambda)$.

Theorem 3. For every continuous linear functional f defined in a SLM-space $(S,\mathcal{J},T)$ the function $\tilde{F}_f$ defined above is a nondecreasing left continuous real valued function in $\langle 0,\infty)$ with $\lim_{\lambda\to\infty} \tilde{F}_f(\lambda) = 1-a_0 \leq 1$ and $\tilde{F}_f(0)=0$.

Proof. As $|||f|||_a = \|f\|_{1-a}$ in $\langle 0,1-a_0\rangle$ is a nondecreasing function then $\{a>0:\ |||f|||_a \geq \lambda_1\} \supset \{a>0:\ |||f|||_a \geq \lambda_2\}$ for every pair $\lambda_1 \leq \lambda_2$ and hence $\tilde{F}_f(\lambda_1) \leq \tilde{F}_f(\lambda_2)$. Let $\lambda > 0$ be fixed and let us consider $\lambda_n \nearrow \lambda$; surely $\sup_n \tilde{F}_f(\lambda_n) \leq \tilde{F}_f(\lambda)$. From the definition of $\tilde{F}_f(\lambda)$ we know that for every $\varepsilon > 0$ there exists $a_n > 0$ such that $\tilde{F}_f(\lambda_n) + \varepsilon > a_n$ and $|||f|||_{a_n} \geq \lambda_n$ for every $n \in \mathcal{N}$. Since $\lambda_n \leq \lambda_{n+1}$ for every $n \in \mathcal{N}$ we can choose a_n in the same way, $a_n \leq a_{n+1}$, and hence $\lim_{n\to\infty} a_n = a_+$ exists. Surely $\lim_{n\to\infty} \tilde{F}_f(\lambda_n) \geq a_+ - \varepsilon$. The function $|||f|||_a$ is nondecreasing, hence $\lim_{n\to\infty} |||f|||_{a_n} \leq |||f|||_a$, then $|||f|||_{a_+} \geq \lambda$ which implies that $\tilde{F}_f(\lambda) \leq a_+$. In this way we have proved[+] that $\lim_{n\to\infty} \tilde{F}_f(\lambda_n) = \tilde{F}_f(\lambda)$ and hence $\tilde{F}_f(\cdot)$ is left contin uous in $(0,+\infty)$ at those points $\lambda \in \langle 0,+\infty)$ where $\{a:\ |||f|||_a \geq \lambda\} \neq \emptyset$. It lasts to prove the left continuity at that $\lambda \in (0,+\infty)$ where $\{a:\ |||f|||_a \geq \lambda\} = \emptyset$. Let $\lambda_n \nearrow \lambda$ and $\{a:\ |||f|||_a \geq \lambda\} = \emptyset$. If, at least for one $n_0 \in \mathcal{N}$ $\{a:\ |||f|||_a \geq \lambda_{n_0}\}$ is empty, too, then by the definition of $\tilde{F}_f(\cdot)$ $\tilde{F}_f(\lambda_n)=1$ and hence $\tilde{F}_f(\cdot)$ is left continuous at λ . Let us suppose that for every $n \in \mathcal{N}$ $\{a:\ |||f|||_a \geq \lambda_n\}$ is nonempty, i.e. for every λ_n there exists $a_n \in (0,1-a_0)$ such that $|||f|||_{a_n} \geq \lambda_n$. Since $|||f|||_a$ is nondecreasing in $(0,1-a_0)$ we can choose $\{a_n\}$ as a nondecreasing sequence, too; $\lim_{n\to\infty} a_n = a_+$. Hence $\lim_{n\to\infty} |||f|||_{a_n} \leq |||f|||_{a_+}$ and $|||f|||_{a_+} \geq \lambda$ but it means that the set $\{a:\ |||f|||_a \geq \lambda\}$ is nonempty which is contrary to the assumption. So, a number $n_0 \in \mathcal{N}$ must exist such that $\{a:\ |||f|||_a \geq \lambda_{n_0}\} = \emptyset$

and $\widetilde{F}_f(\cdot)$ is left continuous at λ.   Q.E.D.

Theorem 4. Let f be a linear continuous functional defined in a SLM-space $(S,\mathcal{J},T)$. Then the statistical norm $F_f(\cdot)$ and $\widetilde{F}_f(\cdot)$ are equal at all points.

Proof. First we shall prove the implication

$$F_f(u) < a \Rightarrow |||f|||_a \geq u.$$

Let $a \in \langle 0,1)$ and $u > 0$ be such that $F_f(u) < a$. By the definition $F_f(u) < a$ implies

$$\sup_{\{x:|f(x)|\neq 0\}} \{F_x(\frac{|f(x)|}{u}) + \omega F_x(\frac{|f(x)|}{u})\} > 1-a.$$

It means there exists $x_0 \in S$ with $f(x_0) \neq 0$ such that

$$F_{x_0}(\frac{|f(x_0)|}{u}) + \omega F_{x_0}(\frac{|f(x_0)|}{u}) > 1-a.$$

Then we can state by means of $n_{1-a}(x_0) = \inf\{\lambda > 0 : F_{x_0}(\lambda) > 1-a\}$ that

$$n_{1-a}(x_0) \leq \frac{|f(x_0)|}{u}$$

Now if we put $z_0 = \frac{ux_0}{|f(x_0)|}$ then $n_{1-a}(z_0) \leq 1$, $|f(z_0)| = u$ and hence

$$\|f\|_{1-a} = \sup\{|f(z)| : n_{1-a}(z) \leq 1\} \geq u.$$

It proves: if $F_f(u) < a$ then $|||f|||_a \geq u$. This implication can be expressed in the following form

$$\{a : F_f(u) < a\} \subset \{a : |||f|||_a \geq u\}.$$

Now, let us prove the opposite implication

$$F_f(u) \geq a \Rightarrow |||f|||_a \leq u.$$

Let $a \in \langle 0,1)$ and $u > 0$ be such that $F_f(u) \geq a$, i.e.

$$\sup_{\{x: f(x)\neq 0\}} \{F_x(\frac{|f(x)|}{u}) + \omega F_x(\frac{|f(x)|}{u})\} \leq 1-a.$$

This implies that $F_x(\frac{|f(x)|}{u}) \leq 1-a$ if $f(x) \neq 0$.

The definition of $n_{1-a}(\cdot)$ and the monotony of F_x give

$$\frac{|f(x)|}{u} \leq n_{1-a}(x).$$

The last inequality holds for $f(x)=0$ of course, too. It means the inequality $|f(x)| \leq u$ must hold for every $x \in S$ satisfying $n_{1-a}(x) \leq 1$. The definition of $\|f\|_{1-a}$ gives immediately that

$$\|f\|_{1-a} = |||f|||_a \leq u.$$

We proved the implications

$$\{a: |||f|||_a > u\} \subset \{a: F_f(u) < a\} \subset \{a: |||f|||_a \geq u\}.$$

Further, if ε is any positive number, then

$\{a: F_f(u) < a\} \subset \{a: |||f|||_a \geq u\} \subset \{a: |||f|||_a > u-\varepsilon\} \subset \{a: F_f(u-\varepsilon) < a\}$.

Now, by means of the definition of $\tilde{F}_f$ we obtain

$$F_f(u-\varepsilon) \leq \tilde{F}_f(u) \leq F_f(u)$$

and the left semicontinuity of F_f gives that

$$F_f(u) = \tilde{F}_f(u).$$

In case $\{a: |||f|||_a \geq u\} = \emptyset$ we have also $\{a: F_f(u) < a\} = \emptyset$ and thus $F_f(u) = \tilde{F}_f(u) = 1$. Q.E.D.

We have not so far mentioned the existence of a nontrivial continuous linear functional in a SLM-space $(S,\mathcal{J},T)$. In every SLM-space $(S,\mathcal{J},T)$ the trivial continuous linear functional 0 exists, $0(x)=0$ for every $x \in S$. The existence of a nontrivial continuous functional is closely connected with the strongest locally convex topology which is weaker than the ε-η-topology. The collection of all convex circled neighborhoods of 0 in the ε-η-topology forms a base for such a locally convex topology. In case of a SLM-space $(S,\mathcal{J},T)$ with t-norm $M(a,b)=\min(a,b)$ every ε-η-neighborhood is convex and circled and hence the topological dual space S' is sufficiently rich in continuous linear functionals. In case of the space $(S,\mathcal{J},M)$ we know, further, that for every $a \in \langle 0,1)$ the number

$$n_a(x) = \inf\{\lambda > 0: F_x(\lambda) > a\}$$

is a seminorm in S and in case of continuity at 0 of F_x for every $x \neq 0$ $n_a(\cdot)$ is a norm even for every $a \in (0,1)$. But without any assumption about a form of t-norm T in a SLM-space $(S,\mathcal{J},T)$ we can prove that the conjugate norm

$$\|f\|_a = \sup\{|f(x)|; n_a(x) \leq 1\}, \quad a \in \langle 0,1)$$

has properties similar to a norm because $\|0\|_a = 0$ for every $a \in \langle 0,1)$, if $\|f\|_a = 0$ then $f=0$ in S, $\|\lambda f\|_a = |\lambda|\,\|f\|_a$ for any $\lambda \in \mathcal{R}_1$ if $\|f\|_a < +\infty$ and $\|f+g\|_a < \|f\|_a + \|g\|_a$ for every $a \in \langle 0,1)$ if $\|f\|_a < +\infty$, $\|g\|_a < +\infty$. Using the conjugate norm we constructed the function $\tilde{F}_f$ for every continuous linear functional f in

S where $\tilde{F}_f(\cdot)$ is defined in $\langle 0,+\infty)$, nondecreasing and left continuous with $\lim_{u\to\infty} \tilde{F}_f(u)=\varepsilon_f$, $\varepsilon_f \in (0,1\rangle$. Let us construct a mapping

$$\mathcal{J}': S' \to \mathcal{F}' \quad , \quad \mathcal{J}'(f)(u)=F_f'(u)= \begin{cases} 0 & u \le 0 \\ \tilde{F}_f(u) & \text{for } u>0 \end{cases}$$

where S' is the topological dual space of S, $\mathcal{F}'$ is the set of all left continuous condecreasing functions defined in $\mathcal{R}_1$ with non-negative values less or equal to 1.

If f=0, then $\|f\|_a=0$ for every $a\in\langle 0,1)$ and $|||f|||_a=0$ for $(0,1\rangle$, too, which implies that $F_0'(u)=H(u)$ for every u. If $\tilde{F}_f(u)=1$ for every $u>0$, $\|f\|_a<+\infty$ for $a\in\langle 0,1-a_0)$, and therefore $\tilde{F}_f(u)<1-a_0$ but it is impossible. It implies that $|||f|||_a<+\infty$ in $\langle 0,1)$. Let us suppose that for every $u>0$ there exists $a_0\in\langle 0,1)$ such that $|||f|||_{a_0}\ge u$. As follows from the definition of $\tilde{F}_f(u)$ in this case $\tilde{F}_f(u)\le a_0<1$, and it is also impossible. It means that $\{a: |||f|||_a\ge u>0\}$ is empty and the only possibility is that $|||f|||_a=$ $=0$. This fact implies that f=0 in S. Let λ be any real number and f any continuous linear functional in S. Then for every $a\in$ $\in\langle 0,1)$ with $|||f|||_a<+\infty$ $|||\lambda f|||_a=|\lambda|\ |||f|||_a$ and for $\lambda\neq 0$

$$\{a: |||\lambda f|||_a\ge u\}=\{a: |||f|||_a\ge \frac{u}{|\lambda|}\} \text{ and hence } F_{\lambda f}'(u)=F_f'(\frac{u}{|\lambda|}).$$

In case $\lambda=0$ we have $\lambda f=0$ and $F_{\lambda f}'(u)=H(u)$ and if we put $F_f'(\frac{u}{0})=$ $=H(u)$ for every $u>0$ then $F_f'(\frac{u}{|0|})=H(u)$ for every $u>0$. Let us prove the generalized triangular inequality given by the t-norm $T(a,b)=\min(a,b)$, i.e.

$$F_{f+g}'(u+v)\ge \min(F_f'(u),F_g'(v)).$$

Surely, it is possible to consider the case $u>0$, $v>0$ only. The functionals f, g are continuous and for f there exists such a number $\varepsilon_f>0$ that $|||f|||_a<+\infty$ in $\langle 0,\varepsilon_f)$, similarly for g, $|||g|||_a<$ $<+\infty$ in $\langle 0,\varepsilon_g)$. It follows that for every

$$a\in\langle 0,\min(\varepsilon_f,\varepsilon_g))$$

$$|||f+g|||_a\le |||f|||_a+|||g|||_a.$$

By the definition

$$F_f'(u)=\inf\{a: |||f|||_a\ge u\}$$

$$F_g'(v)=\inf\{a: |||g|||_a\ge v\}$$

$$\{a: |||f|||_a\ge u\}\neq\emptyset\neq\{a: |||g|||_a\ge v\}$$

and $\{a: |||f+g|||_a \geq u+v\} \subset \{a: |||f|||_a + |||g|||_a \geq u+v\}$ as well. Now, let us suppose that

$$F'_{f+g}(u+v) < \min(F'_f(u), F'_g(v)).$$

It means that there exists such a number $a_\varepsilon \geq 0$ that

$a_\varepsilon \in \{a: |||f+g|||_a \geq u+v\}$ $a_\varepsilon - \varepsilon < F'_{f+g}(u+v) < a_\varepsilon < \min(F'_f(u), F'_g(v))$.

Then for every $a \geq \min(\inf\{a: |||f|||_a \geq u\}, \inf\{a: |||g|||_a \geq v\})$

$$a_\varepsilon < a.$$

It means that $|||f|||_{a_\varepsilon} < u$, $|||g|||_{a_\varepsilon} < v$, which together gives

$$|||f|||_{a_\varepsilon} + |||g|||_{a_\varepsilon} < u+v.$$

As for a_ε $|||f+g|||_{a_\varepsilon} \geq u+v$, then this fact is contrary to the conclusion that

$$|||f|||_{a_\varepsilon} + |||g|||_{a_\varepsilon} < u+v.$$

This proves the inequality

$$F'_{f+g}(u+v) \geq \min(F'_f(u), F'_g(v))$$

must hold.

Now, we must consider the case $F'_f(u)=1$, $F'_g(v)=\inf\{a: |||g|||_a \geq v\}$. It means that $\{a: |||f|||_a \geq u\} = \emptyset$ and $\{a: |||g|||_a \geq v\} \neq \emptyset$. In case if $\{a: \|f+g\|_a \geq u+v\} \neq \emptyset$ $F'_{f+g}(u+v) = \inf\{a: \|f+g\|_a \geq u+v\}$. Now, let us suppose the contrary again, i.e.

$F'_{f+g}(u+v) < \min(F'_f(u), F'_g(v))$; then for some $a_\varepsilon \in \{a: |||f+g|||_a \geq u+v\}$

$a_\varepsilon - \varepsilon < F'_{f+g}(u+v) < a_\varepsilon < \min\{F'_g(v), 1\}$. It means, of course, that $|||g|||_{a_\varepsilon} < v$, $|||f|||_a < u$ for every $a \in \langle 0,1)$ and hence $|||g|||_{a_\varepsilon} + |||f|||_{a_\varepsilon} < u+v$. As $|||f+g|||_{a_\varepsilon} \geq u+v$ then $|||f|||_{a_\varepsilon} + |||g|||_{a_\varepsilon} \geq u+v$, which is impossible and the generalized inequality must hold. Now, suppose that $\{a: \|f+g\|_a \geq u+v\} = \emptyset$. Then, by the definition $F'_{f+g}(u+v) = 1$ and the generalized triangular inequality holds in a trivial way.

The last possibility is the case $\{a: \|f+g\|_a \geq u+v\} \neq \emptyset$ but $\{a: \|f\|_a \geq u\} = \{a: \|g\|_a \geq v\} = \emptyset$. Then $F'_f(u)=1$, $F'_f(v)=1$, too. Let us suppose $F'_{f+g}(u+v) < 1$. Then there exists $a_\varepsilon < 1$ such that $F'_{f+g}(u+v) < a_\varepsilon < 1$. As we suppose $\{a: \|f+g\|_a \geq u+v\}$ is nonempty then $\|f+g\|_{a_\varepsilon} \geq u+v$ which implies either $\|f\|_{a_\varepsilon} \geq u$ or $\|g\|_{a_\varepsilon} \geq v$. This conclusion is of course impossible and the generalized triangular inequality holds in this case, too.

We have proved that to every $f \in S'$ it is possible to assign a function F'_f such that $f=0$ iff $F'_f=H$,

$F'_{\lambda f}(u)=F'_f(\frac{u}{|\lambda|})$ for every $u \in \mathcal{R}_1$ and every $\lambda \in \mathcal{R}_1$

and the generalized triangular inequality

$F'_{f+g}(u+v) \geq \min(F'_f(u),F'_g(v))$

holds for every $f,g \in S'$ and $u,v \in \mathcal{R}$.

In general, F'_f need not be a probability distribution function because $\lim_{n\to\infty} F_f(u)= \varepsilon_f$ need not be equal to one. This fact leads us to the following definition.

Definition 4. Let S be a linear space, let T be a t-norm, let $\mathcal{F}'$ be the set of all real valued nondecreasing functions defined in reals which are left continuous and $\lim_{u\to\infty} F(u)=0$, $\lim_{u\to\infty} F(u) \leq 1$ for every $F \in \mathcal{F}'$. If $\mathcal{Y}'$ is a mapping $\mathcal{Y}':S \to \mathcal{F}'$ such that

1. $(x=0) \Leftrightarrow (\mathcal{Y}'(x)=H)$ where $H(0)=0$, $H(u)=1$ for every $u>0$ $\mathcal{Y}'(x)[0]=0$
2. $\mathcal{Y}'(\lambda x)[u]=\mathcal{Y}'(x)[\frac{u}{|\lambda|}]$ for every $x \in S$ and every $\lambda \in \mathcal{R}_1$
3. $\mathcal{Y}'(x+y)[u+v] \geq T(\mathcal{Y}'(x)[u],\mathcal{Y}'(y)[v])$ for every $x,y \in S$ and $u,v \in \mathcal{R}_1$

then the triple $(S,\mathcal{Y}',T)$ is called a generalized statistical linear space in the sense of Menger (GSLM-space).

The definition 4 is nonempty because every SLM-space is a GSLM-space, of course, and the dual space $(S',\mathcal{Y}',\min)$ to every SLM-space $(S,\mathcal{Y},T)$ is a GSLM-space, too.

Theorem 5. Let a SLM-space $(S,\mathcal{Y},T)$ be given. Then its topological dual space S' can be understood as a GSLM-space $(S',\mathcal{Y}',\min)$ where

$$\mathcal{Y}'(f)=F'_f(.) \text{ for } f \in S'.$$

The proof of this Theorem 5 was given before. We shall try to use the mapping $\mathcal{Y}'$ in the dual space S' to introduce an analogical topology to the ε-η-topology. Similarly, as for the ε-η-topology, we shall define a family of neighborhoods which forms a base of a topology. Let $\varepsilon \in (0,1\rangle$, $\eta > 0$, then the subset in S'

$$\sigma'(f_0,\varepsilon,\eta)=\{f \in S':F'_{f-f_0}(\eta)>1-\varepsilon\}$$

will be called an ε-η-neighborhood of f_0 in S'. It is clear that the family $\{\mathcal{U}=\{\sigma'(f_0,\varepsilon,\eta), \varepsilon \in (0,1\rangle, \eta>0\}, f_0 \in S'\}$ forms a base for a topology which we shall call the ε-η-topology, too. It is clear that for every $\sigma'(f_0, \varepsilon, \eta)$ $f_0 \in \sigma'(f_0, \varepsilon, \eta)$ because $F'_{f_0-f_0}(u)=H(u)=1$ for $u>0$. For any pair $\sigma'(f_0, \varepsilon_i, \eta_i)$, $i=1,2$ there exists such an $\sigma'(f_0, \varepsilon_0, \eta_0)$ that

$$\sigma'(f_0, \varepsilon_0, \eta_0) \subset \sigma'(f_0, \varepsilon_1, \eta_1) \cap \sigma'(f_0, \varepsilon_2, \eta_2).$$

It is sufficient to put $\varepsilon_0=\min(\varepsilon_1, \varepsilon_2)$, $\eta_0=\min(\eta_1, \eta_2)$. Further, if $\sigma'(f_0, \varepsilon_0, \eta_0)$ is given then for every $\varepsilon \leq \varepsilon_0$, $\eta \geq \eta_0$ $\sigma'(f_0, \varepsilon, \eta) \subset \sigma'(f_0, \varepsilon_0, \eta_0)$; similarly, for every $\varepsilon \geq \varepsilon_0$, $\eta \leq \eta_0$ $\sigma'(f_0, \varepsilon, \eta) \supset \sigma'(f_0, \varepsilon_0, \eta_0)$. If $f_1 \in \sigma'(f_0, \varepsilon_0, \eta_0)$, i.e. $F'_{f_1-f_0}(\eta_0)>1-\varepsilon_0$, then there exists $\sigma'(f_1, \varepsilon^*, \eta^*)$ such that

$$\sigma'(f_1, \varepsilon^*, \eta^*) \subset \sigma'(f_0, \varepsilon_0, \eta_0).$$

As the function $F'_{f_1-f_0}(\eta_0)$ is left continuous at η_0 there exist $\varepsilon<\varepsilon_0$, $\eta<\eta_0$ such that

$$F'_{f_1-f_0}(\eta)>1-\varepsilon>1-\varepsilon_0.$$

Let $0<\eta^*<\eta_0-\eta$, $\varepsilon^*=\varepsilon$ and consider the ε-η-neighborhood $\sigma'(f_1, \varepsilon^*, \eta^*)=\{f \in S' : F'_{f-f_1}(\eta^*)>1-\varepsilon^*\}$. Let $f \in \sigma'(f_1, \varepsilon^*, \eta^*)$ then $F'_{f-f_0}(\eta_0)=F'_{f-f_0}(\eta_0-\eta+\eta) \geq \min(F'_{f-f_1}(\eta^*), F'_{f_1-f_0}(\eta)) \geq$ $\geq \min(1-\varepsilon^*, 1-\varepsilon)>1-\varepsilon_0$ hence $f \in \sigma'(f_0, \varepsilon_0, \eta_0)$.

We have proved that the system of the ε-η-neighborhoods in S' defines a topology. This topology will be called also the ε-η-topology and thanks to the generalized triangular inequality $F'_{f+g}(u+v) \geq \min(F'_f(u), F'_g(v))$ it is no problem to prove that every net $\{f_\alpha\}_\alpha$ in S' has at most one limit point because $F'_f=H$ if and only if $f=0$ in S'. This fact proves that the ε-η-topology is a Hausdorffian topology. The generalized triangular inequality enables us to prove also that

if $f_\alpha \to f$ and $g_\alpha \to g$ then $f_\alpha + g_\alpha \to f+g$.

Unfortunately, it is not true that $\lambda_\alpha f \to 0$, in general, in this ε-η-topology if $\lambda_\alpha \to 0$ in reals because if $\varepsilon_f<1$ then

$$\lim_{\lambda_\alpha \searrow 0} F'_{\lambda_\alpha f}(u)=\lim_{\lambda_\alpha \to 0} F'_f\left(\frac{u}{|\lambda_\alpha|}\right)=\varepsilon_f<1 \text{ for every } u>0.$$

This fact says that the ε-η-topology in S' is not a linear topology, i.e. the operation of $\lambda \cdot f$ need not be continuous in $\mathcal{R} \times S'$.

Theorem 6. The ε-η-topology in the dual space $(S', \mathcal{Y}', \min)$ of a SLM-space $(S, \mathcal{J}, T)$ is a linear topology if and only if $\varepsilon_f = 1$ for every $f \in S'$.

Proof. The proof is very simple. If $\varepsilon_f = 1$ for every $f \in S'$, then for every $\lambda_\alpha \to 0$ of reals and every $f \in S'$

$$\lim_{\lambda_\alpha \searrow 0} F'_{\lambda_\alpha f}(u) = \lim_{\lambda_\alpha \searrow 0} F'_f\left(\frac{u}{|\lambda_\alpha|}\right) = \varepsilon_f = 1$$

for every $u > 0$ and hence $\lambda_\alpha f \to 0$ in the ε-η-topology.

If there exists, at least, one $f_0 \in S'$ with $\varepsilon_f < 1$ then $\lambda_\alpha f_0 \not\to 0$ in the ε-η-topology which cannot be a linear topology in such a case. Q.E.D.

Theorem 7. The ε-η-topology in the dual space $(S', \mathcal{Y}', \min)$ of a SLM-space $(S, \mathcal{Y}, T)$ is metrizable.

Proof. The mapping $\mathcal{Y}'(f)$ is constructed using the conjugate norm $\|f\|_a = \sup\{|f(x)| : n_a(x) \le 1\}$, $a \in \langle 0,1)$, $f \in S'$. For our purposes we have put $|||f|||_a = \|f\|_{1-a}$ for $a \in (0,1\rangle$ and $\varepsilon_f = \sup\{a : |||f|||_a < +\infty\}$. Now, we use $|||f|||_a$ for the definition of a metric in the dual space S'. Let us define for every $f, g \in S'$

$$\mathcal{N}_a(f-g) = \frac{|||f-g|||_a}{1 + |||f-g|||_a} \quad \text{for } a \in \langle 0, \varepsilon_{f-g})$$

$$\mathcal{N}_a(f-g) = 1 \qquad \text{for } a \in \langle \varepsilon_{f-g}, 1).$$

Using the inequality $\varepsilon_{f+g} \ge \min(\varepsilon_f, \varepsilon_g)$ we can immediately prove that for every $a \in \langle 0,1)$ $\mathcal{N}_a(\cdot)$ is a metric defined in S'. Since $\mathcal{N}_a(\cdot) \le 1$ for every $a \in \langle 0,1)$ then the integral

$$\rho(f;g) = \int_0^1 \mathcal{N}_a(f-g)\, da$$

exists and $\rho(f;g)$ is also a metric in S'. Let $\{f_n\}$ be a sequence in S' such that $\rho(0;f_n) \xrightarrow[n\to\infty]{} 0$. As

$$\rho(0,f_n) = \int_0^1 \mathcal{N}_a(f)\,da = \int_0^{\varepsilon_{f_n}} \frac{|||f_n|||_a}{1 + |||f_n|||_a}\, da + (1 - \varepsilon_{f_n}) \text{ for every } n \in \mathcal{N},$$

it is clear that $\varepsilon_{f_n} \to 1$ and $\int_0^{\varepsilon_{f_n}} \frac{|||f_n|||_a}{|||f_n|||_a + 1}\, da \to 0$ if $n \to \infty$. $|||f|||_a$ is a nondecreasing function in $\langle 0,1)$ hence $\mathcal{N}_a(f)$ is also a nondecreasing function in $\langle 0,1)$ and the convergence $\rho(0,f_n) \to 0$ implies that $\mathcal{N}_a(f_n) \to 0$ for every $a \in \langle 0,1)$ hence $|||f_n|||_a \to 0$ if $n \to \infty$ for every $a \in \langle 0,1)$.

Now, let u be any positive real number, then according to the definition of $F'_f(u)$

$$F'_{f_n}(u)=\inf\{a: |||f_n|||_a \geq u\}$$

or

$$F'_{f_n}(u)=1 \text{ if } \{a: |||f_n|||_a \geq u\}=\emptyset.$$

We proved that $|||f_n|||_{a_0} \longrightarrow 0$ for $a_0 \in \langle 0,1)$, i.e. for every $a_0 \in \langle 0,1)$ and every $u_0 > 0$ there exists a natural n_0 such that for every $n \geq n_0$

$$|||f_n|||_{a_0} < u_0.$$

It means that $F'_{f_n}(u_0) \geq a_0$ for every $n \geq n_0$. The arbitrariness of u_0 and of a_0 implies immediately that

$$\lim_{n\to\infty} F'_{f_n}(u_0)=1.$$

This fact proves the convergence of $\{f_n\}_{n=1}^{\infty}$ to the null functional in S' with respect to the ε-η-topology.

Now, on the contrary, let a sequence $\{f_n\}_{n=1}^{\infty}$ converge to 0 in S' with respect to the ε-η-topology, i.e.

$$\lim_{n\to\infty} F'_{f_n}(u)=1$$

for every $u > 0$. We have for every $\varepsilon > 0$ and every $u > 0$ there exists a natural n_0 such that for every $n \geq n_0$

$$F'_{f_n}(u) > 1-\varepsilon.$$

As follows from the definition of $F'_f(\cdot)$ either $\{a: |||f_n|||_a \geq u\}=\emptyset$ or $\inf\{a: |||f_n|||_a \geq u\} > 1-\varepsilon$. It implies that

$$\{a: |||f_n|||_a < u\} \supset \langle 0,1-\varepsilon)$$

Then $\lambda\{a: |||f_n|||_a < u\} \geq 1-\varepsilon$ (λ is the Lebesgue measure) for every $u > 0$ and this proves that $|||f_n|||_a \to 0$ if $n \to \infty$ for every $a \in \langle 0,1)$. As $\eta_a(f_n) \leq 1$ for every $n \in \mathcal{N}$, thus

$$\rho(0,f_n)=\int_0^1 \eta_a(f_n)da \longrightarrow 0$$

where $n \to \infty$ and Theorem 7 is proved. Q.E.D.

Theorem 8. Let a SLM-space $(S,\mathcal{Y},\min)$ be given. Let $(S',\mathcal{Y}',\min)$ be its dual space. Then the ε-η-topology in $(S,\mathcal{Y},\min)$ is normable if and only if

$$\inf_{f\in S'} \varepsilon_f > 0.$$

Proof. Let $(S,\mathcal{Y},\min)$ be given and let the ε-η-topology in

S be normable. Then there exists such a convex neighborhood K which is ε-η-bounded. It means that the set K must be bounded with respect to every seminorm $n_a(\cdot)$, $a \in \langle 0,1)$; in other words, for every $a \in \langle 0,1)$ there exists K_a such that for every $x \in K$, $n_a(x) \le K_a < +\infty$. Let f be any continuous linear functional defined in S. The continuity of f implies that $\sup_{x \in K} |f(x)| \le K_f < +\infty$. Further, since K forms a neighborhood in the ε-η-topology in S, there exists $\sigma(\varepsilon_0, \eta_0)$ in S such that $\sigma(\varepsilon_0, \eta_0) \subset K$, $\varepsilon_0 > 0$, $\eta_0 > 0$. It means that for every $x \in \sigma(\varepsilon_0, \eta_0)$ $|f(x)| \le K_f$, too.

As $\sigma(\varepsilon_0, \eta_0) = \{x: n_{1-\varepsilon_0}(x) < \eta_0\} = \eta_0\{x: n_{1-\varepsilon_0}(x) < 1\}$ then for every $x \in \{x: n_{1-\varepsilon_0}(x) < 1\}$ and $f \in S'$

$$\sup \{|f(x)| : x \in \{x: n_{1-\varepsilon_0}(x) < 1\}\} \le \frac{K_f}{\eta_0} < +\infty .$$

Further, f is continuous and by the aid of Definition 3 we obtain

$$\|f\|_{1-\varepsilon_0} = \sup \{|f(x)| : x \in \sigma_{1-\varepsilon_0}\} = \sup \{|f(x)| : n_{1-\varepsilon_0}(x) \le 1\} \le \frac{K_f}{\eta_0}$$

which implies that $|||f|||_{\varepsilon_0} < +\infty$ for every $f \in S'$. It says that $\varepsilon_f \ge \varepsilon_0 > 0$ for every $f \in S'$, i.e. $\inf\{\varepsilon_f : f \in S'\} > 0$.

Let us suppose, vice versa, that $\inf_{f \in S'} \varepsilon_f = \varepsilon_0 > 0$. It means that for every $a \in \langle 0, \varepsilon_0)$ and every $f \in S'$ $|||f|||_a < +\infty$ and $|||f|||_a$ is a norm in S'. As for any $a \in \langle 0, \varepsilon_0)$

$$|||f|||_a = \|f\|_{1-a} = \sup \{|f(x)| : n_{1-a}(x) \le 1\} < +\infty$$

then $\{x: n_{1-a}(x) \le 1\}$ must be ε-η-bounded. Further, $\{x: n_{1-a}(x) \le 1\}$ is an absolutely convex neighborhood of 0 in the ε-η-topology as was shown in [1].

This ε-η-boundedness proves that the ε-η-topology is normable by a norm

$$\|x\| = \inf\{\lambda > 0 : n_{1-a}(x) \le \lambda\} = n_{1-a}(x). \qquad \text{Q.E.D.}$$

<u>Theorem 9</u>. Let B be a Banach space and B' its topological dual space. Then $B = (B, \mathcal{Y}, \min)$ where $\mathcal{Y}(x)[u] = H(u - \|x\|)$ and $B' = (B', \mathcal{Y}', \min)$ where $\mathcal{Y}'(f)[u] = H(u - \|f\|)$.

Proof. First, we must verify all the requirements which are put on $\mathcal{Y}, \mathcal{Y}'$. If x=0 in B, then $\|x\| = 0$ and $\mathcal{Y}(0)[u] = H(u)$. As $\|\lambda x\| = |\lambda| \, \|x\|$, then $H(u - \|\lambda x\|) = H(u - |\lambda| \, \|x\|) = H(\frac{u}{|\lambda|} - \|x\|)$ and therefore $\mathcal{Y}(\lambda x)[u] = \mathcal{Y}(x)[\frac{u}{|\lambda|}]$. If $H(u - \|x\|) = H(u)$ for every

$u > 0$ then it is possible only if $x=0$ because $\|x\|$ is a norm. Thanks to the triangular inequality $\|x+y\| \leq \|x\| + \|y\|$ it holds that

$$H(u+v-\|x+y\|) \geq \min[H(u-\|x\|), H(v-\|y\|)].$$

The same properties can be proved for the mapping $\mathcal{Y}'$. The mapping $\mathcal{Y}'$ can be defined using the statistical norm of $f \in S'$, i.e.

$$\mathcal{Y}'(f)[u] = 1 - \sup_{x \neq 0} \{F_x(\frac{|f(x)|}{u}) + \omega F_x(\frac{|f(x)|}{u})\} =$$

$$= 1 - \sup_{x \neq 0} \{H(\frac{|f(x)|}{u} - \|x\|) + \omega H(\frac{|f(x)|}{u} - \|x\|)\} =$$

$= H(u - \|f\|)$ because for every $x \in B$ $|f(x)| \leq \|x\| \|f\|$.

Q.E.D.

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