COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

28,1 (1987)

LIOUVILLE TYPE CONDITION AND THE INTERIOR REGULARITY OF QUASILINEAR PARABOLIC SYSTEM (THE CASE OF BMO-SOLUTIONS)

J. STARÁ, J. DANĚČEK, O. JOHN

Abstract: The Liouville property of quasilinear parabolic system implies the regularity of this system also in case that both the Liouville condition and the regularity of the system are formulated for the weak solutions with finite BMO-seminorms. On the other hand (again in the framework of BMO) from the slightly modified condition of the regularity of the system it follows that this system has the Liouville property.

Key words: Quasilinear parabolic systems, BMO-seminorms, regularity, Liouville property.

Classification: 35K55

---

1. Introduction. The aim of this note is to extend the study of the connection between the Liouville type condition and the interior regularity of the weak solutions of quasilinear parabolic systems. Meanwhile in the papers [1],[2] we dealt with the bounded solutions, here we are concerned in the case of the solutions belonging to the space BMO. In contrast with the papers just mentioned, also the equivalence of the Liouville type condition with the regularity is studied more in details.

In general, it is not known how to prove the boundedness of the solutions of the initial-boundary value problems for quasilinear systems. On the other hand, Daněček [3] gave (in elliptic case) an example of the class of quasilinear systems for which each weak solution of the Dirichlet problem belongs to the space BMO. This fact stimulated our interest in the theme described above. (In the elliptic case it was proved by Daněček in his Thesis that the Liouville type condition implies the regularity in the framework of BMO.)

In the bibliography, we restrict ourselves just to the items we need for references. For the more representative list of articles see [2].

2. Preliminaries. Let $Q=\Omega\times\mathbb{R}$ where Ω be a domain in $\mathbb{R}^n$, $n\geqq 2$. Denote $z=[x,t]$, $x=[x_1,\dots,x_n]$, $u=[u^1,\dots,u^m]$, $m\geqq 1$. Let us consider the system

$$\frac{\partial u^i}{\partial t}-\frac{\partial}{\partial x_\alpha}\left(a^{ij}_{\alpha\beta}(z,u)\frac{\partial u^j}{\partial x_\beta}\right)=-f^i(z,u,D_xu),\ i,j=1,\dots,m;\ \alpha,\beta=1,\dots,n,$$

which we rewrite as

(1) $$u_t-\operatorname{div}(A(z,u)D_xu)=-f(z,u,D_x u).$$

Together with (1) we deal with the systems

(1*) $$u_t-\operatorname{div}(A(z_0,u)D_xu)=0,\ z_0\in Q.$$

Let the following assumptions on the coefficients $A=\{a^{ij}_{\alpha\beta}\}$ and the right hand side functions $f=\{f^i\}$ be satisfied:

(2) A is uniformly continuous and bounded on $Q\times\mathbb{R}^m$.

(3) There exists $\lambda>0$ such that

$(A(z,u)\xi,\xi)\geqq\lambda|\xi|^2$, $\forall\xi\in\mathbb{R}^{mn}$, $[z,u]\in Q\times\mathbb{R}^m$.

(4) $\lim_{|u|\to\infty}A(z,u)=d(z)$ uniformly in Q (with respect to z).

(5) $f(z,u,p)$ is continuous on $Q\times\mathbb{R}^m\times\mathbb{R}^{mn}$.

(6) $|f(z,u,p)|\leqq c|p|^\gamma$ where $\gamma<2$.

The function $u\in W^{1,0}_{2,loc}(Q)$ is said to be a weak solution of (1) in Q if for each $\varphi\in C_0^\infty(Q)$ holds

(7) $$\int_Q[u\varphi_t-A(z,u)D_xuD_x\varphi]\,dz=\int_Q f(z,u,D_xu)\varphi\,dz.$$

For $z_0=[x_0,t_0]$ and $R>0$ define $Q(z_0,R)=B(x_0,R)\times(t_0-R^2,t_0+R^2)$, i.e., the open ball in $\mathbb{R}^{n+1}$ with the centre z_0 and radius R with respect to the parabolic metric. Denote further

(8) $$u_{z_0,R}=\frac{1}{\mu Q(z_0,R)}\int_{Q(z_0,R)}u(z)\,dz$$

(9) $$U(z_0,R)=\frac{1}{R^{n+2}}\int_{Q(z_0,R)}|u(z)-u_{z_0,R}|^2\,dz.$$

Define now $BMO(\mathbb{R}^{n+1})$ as the class of all measurable functions u on $\mathbb{R}^{n+1}$ for which the set $U=\{U(z_0,R);\ z_0\in\mathbb{R}^{n+1},R>0\}$ is bounded, putting

(10) $$|||u|||_{BMO(\mathbb{R}^{n+1})}=\sup U.$$

In a similar way the class BMO(Q) can be introduced. In this case we take for U the set of all $U(z_0,R)$ with $z_0 \in Q$ and $R < \mathrm{dist}(z_0, \partial Q)$.

3. Main results

Definition 1. The system (1) is said to be (L) (to have Liouville property) if for each $z_0 \in Q$ and each weak solution u of (1*) in $\mathbb{R}^{n+1}$ holds the following: $u \in BMO(\mathbb{R}^{n+1})$ implies that u is a constant vector function.

Definition 2. The system (1) is said to be (R) (regular) if each weak solution of this system which belongs to BMO(Q) is locally Hölder continuous (i.e., $u \in C^{0,\alpha,\alpha/2}_{loc}(Q)$, $\alpha \in (0,1)$).

Definition 3. The system (1) is said to be (UR)(uniformly regular) if for each $\mu > 0$ and each $K \subset Q$, K compact, there exists $C(K,\mu) > 0$ such that for each weak solution u of (1), $u \in BMO(Q)$, we have

$$|||u|||_{BMO(Q)} \leq \mu \implies |||u|||_{C^{0,\alpha,\alpha/2}(K)} \leq C(K,\mu).$$

(Here $|||u|||_{C^{0,\alpha,\alpha/2}(K)} = \sup\left\{\frac{|u(z)-u(\bar{z})|}{|x-\bar{x}|^{\alpha}+|t-\bar{t}|^{\alpha/2}};\ z,\bar{z} \in K,\ z \neq \bar{z}\right\}$.)

Definition 4. The system (1) is said to be (SUR)(strongly uniformly regular) if for each $z_0 \in Q$ the system (1*) is (UR) with respect to the domain Q(0,2).

Theorem 1. Let (1) be (SUR); then it is (L).

Theorem 2. Let (1) be (L); then it is (UR).

Remarks. It follows from Definition 1 that if the system (1) is (L), then each of systems (1*) is (L). Thus Theorem 2 yields: (L) $\implies$ (SUR). So we obtained the equivalence of (L) and (SUR). On the other hand, from Theorems 1, 2 and Definitions we get (SUR) $\implies$ (UR) $\implies$ (R).

Does (UR) $\implies$ (SUR) take place?

4. Proofs

Proof of Theorem 1. Let for some $z_0 \in Q$ the function u be a weak solution of the system (1*) in $\mathbb{R}^{n+1}$ which belongs to $BMO(\mathbb{R}^{n+1})$. Denote $\mu = |||u|||_{BMO(\mathbb{R}^{n+1})}$.

For $R>0$ put $\varsigma = [\xi, \tau] = \left[\frac{x}{R}, \frac{t}{R^2}\right]$ and define $u_R(\varsigma)=u(R\xi, R^2\tau)$. As the functions u_R are again the solutions of (1*) in $\mathbb{R}^{n+1}$, they are also the solutions of (1*) in $Q(0,2)$. Obviously, $|||u_R|||_{BMO(Q(0,2))} \leq \mu$ and so (according to Definition 4) there is a constant C such that

$$\forall_{R>0} \; |||u_R|||_{C^{0,\alpha,\alpha/2}(B(0,1))} \leq C.$$

Let now $z \in \mathbb{R}^{n+1}$. We have $|u(z)-u(0)| = |u_R(\frac{x}{R}, \frac{t}{R^2})-u_R(0)| \leq$ $\leq C \frac{|x|^{\alpha}+|t|^{\alpha/2}}{R^{\alpha}}$ if only $R > |x|+|t|^{1/2}$.

Passing to the limit with $R \to +\infty$ we obtain that $u(z)=u(0)$ and so u is a constant vector function.

<u>Proof of Theorem 2</u>. Let $K \subset Q$ be a given compact set and let $\mu > 0$. According to the partial regularity theory for parabolic systems (see e.g. [4]) it is sufficient to prove that

(11) $\lim_{R \to 0+} U(z,R)=0$ uniformly with respect to K and $\mathcal{U} = \{u;\ u$ is a weak solution of (1) in Q with $|||u|||_{BMO(Q)} \leq \mu\}$. x)

Suppose that (11) is false, i.e. that:

(12) There exist some compact $K \subset Q$, two positive numbers μ and ε and the sequences $\{z_h\} \subset K$, $\{R_h\} \subset \mathbb{R}$, $R_h \searrow 0$ and $\{u_h\}$ (u_h is a weak solution of (1) in Q for which $|||u_h|||_{BMO(Q)} < \mu$), such that $U_h(z_h,R_h) \geq \varepsilon$, $h=1,2,\dots$

In what follows we shall prove that (12) leads to the contradiction with (L).

Put

(13)
$$\varsigma = [\xi, \tau] = \left[\frac{x-x_h}{R_h}, \frac{t-t_h}{R_h^2}\right],$$
$$v_h(\varsigma) = u_h(x_h+R_h\xi, \; t_h+R_h^2\tau)-(u_h)_{z_h,R_h}.$$

From (12) and (13) we obtain (for an arbitrary constant function q)

(14)
$$\varepsilon \leq U_h(z_h,R_h)=(R_h)^{-n-2} \int_{Q(z_h,R_h)} |u_h(z)-(u_h)_{z_h,R_h}|^2\, dz = \int_{Q(0,1)} |v_h(\varsigma)|^2\, d\varsigma \leq \int_{Q(0,1)} |v_h(\varsigma)-q|^2\, d\varsigma .$$

----

x) It is not difficult to prove that the crucial lemma 8 of the paper [4] is valid - under our assumptions on the coefficients A - even in case when the boundedness of $u_{z,R}$ be substituted by the condition that $u \in BMO(Q)$.

The functions v_h solve the systems

$$(15)\qquad \int_{O_{R_h}} [v_h \varphi_\tau - A(x_h+R_h\xi, t_h+R_h^2\tau, v_h(\xi)+(u_h)_{z_h,R_h}) D_\xi v_h D_\xi \varphi] d\xi =$$

$$= \int_{O_{R_h}} R_h^2 f(x_h+R_h\xi, t_h+R_h^2\tau, v_h(\xi)+(u_h)_{z_h,R_h}, R_h^{-1} D_\xi v_h)\varphi\, d\xi ,$$

where O_{R_h} is the image of Q in the mapping ξ defined by (13).

For each $T>0$ there exists $h(T)\in\mathbb{N}$ such that for $h\geq h(T)$ the inclusion $Q(0,T)\subset\subset O_{R_h}$ holds. So each $v_h (h\geq h(T))$ solves (15) in $Q(0,T)$. It follows from (2), (3) that the coefficients $A_h(\xi)= =A(x_h+R_h\xi , t_h+R_h^2\tau , v_h(\xi)+(u_h)_{z_h,R_h})$ are measurable and equibounded with respect to h on $Q(0,T)$ and satisfy the ellipticity condition with the same constant λ.

The assumptions (5) and (6) yield

$$R_h^2 | f(x_h+R_h\xi, t_h+R_h^2\tau, v_h(\xi)+(u_h)_{z_h,R_h}, R_h^{-1} D_\xi v_h(\xi))| \leq$$

$$\leq C\, R_h^{2-\gamma} |D_\xi v_h(\xi)|^\gamma ,$$

so the right hand sides of (15) have a controllable growth, uniform with respect to h.

From these facts we can conclude that there exists a constant $C(T)>0$ for which

$$(16)\qquad \|v_h\|^2_{W_2^{1,1/2}(Q(0,T/2))} \leq c(T)\, \|v_h\|^2_{L_2(Q(0,T))},\quad h\geq h(T)$$

Further we need to show that the right hand side in (16) can be estimated by some absolute constant, depending only on T. For this purpose we give the estimate of $\|v_h\|_{L_2(Q(0,T))}$ by $|||u_h|||_{BMO(Q)}$. This fact can be derived as a particular case of Lemma 3.III in [5]. For the convenience of the reader we give here the proof. Omitting index h, we have

$$(17)\qquad \|v\|^2_{L_2(Q(0,T))} = \int_{Q(0,T)} |v(\xi)|^2\, d\xi = R^{-n-2} \int_{Q(z,RT)} |u(\bar z)-$$

$$-u_{z,R}|^2 d\bar z \leq 2\, \{ T^{n+2}(TR)^{-n-2} \int_{Q(z,RT)} |u(\bar z)-u_{z,RT}|^2 d\bar z +$$

$$+ R^{-n-2} \int_{Q(z,RT)} |u_{z,RT}-u_{z,R}|^2 d\bar z\} \leq 2T^{n+2}\ \{ |||u|||^2_{BMO(Q)} +$$

$$+ \varkappa |u_{z,RT}-u_{z,R}|^2 \},$$

where $\varkappa$ is the volume of $Q(0,1)$.

To estimate $|u_{z,RT}-u_{z,R}|^2$ we restrict ourselves to the case that $T=2^i$.$^{x)}$ Putting $\wp=TR$ we estimate at first $|u_{z,\wp}-u_{z,\wp/2}|^2 \leqq 2\{|u_{z,\wp}-u(\bar z)|^2+|u_{z,\wp/2}-u(\bar z)|^2\}$. After the integration over $Q(z,\wp/2)$ we get

$$\varkappa(\wp/2)^{n+2}|u_{z,\wp}-u_{z,\wp/2}|^2 \leqq 2\left\{\int_{Q(z,\wp)}|u_{z,\wp}-u(\bar z)|^2 d\bar z + \right.$$
$$\left. + \int_{Q(z,\wp/2)}|u(\bar z)-u_{z,\wp/2}|^2 dz\right\},$$

and from here

$$|u_{z,\wp}-u_{z,\wp/2}|^2 \leqq \frac{2}{\varkappa}(2^{n+2}+1)\,|||u|||^2_{BMO(Q)}.$$

Iterating this estimate, we have

$$|u_{z,R}-u_{z,RT}|^2=|u_{z,\wp/2^i}-u_{z,\wp}|^2 \leqq i\,[\,|u_{z,\wp}-u_{z,\wp/2}|^2+\ldots+$$
$$+|u_{z,\wp/2^{i-1}}-u_{z,\wp/2^i}|^2] \leqq \frac{2i}{\varkappa}\,|||u|||^2_{BMO(Q)}(2^{n+2}+1).$$

Substituting to (17) we obtain finally:

(18) $$\|v_h\|^2_{L_2(Q(0,T))} \leqq c(T)\,|||u_h|||^2_{BMO(Q)} \leqq c(T)\,\mu\,,$$

which together with (16) gives

(19) $$\|v_h\|^2_{W_2^{1,1/2}(Q(0,T/2))} \leqq c(T)\,\mu, \quad (h \geqq h(T)).$$

The estimate (19) together with the compactness of imbedding of $W_2^{1,1/2}(Q(0,T/2))$ into $L_2(Q(0,T/2))$ enable us to assert (using the diagonal process) that there exists a subsequence (we use the same notation for it) such that

(20) $$z_h \longrightarrow z_0 \in K,$$

(21) $$v_h \longrightarrow v \text{ in } L_2(Q(0,T)) \text{ for each } T>0,$$

(22) $$D_\xi v_h \longrightarrow D_\xi v \text{ weakly in } L_2(Q(0,T)) \text{ for each } T>0,$$

(23) $$v_h \longrightarrow v \text{ almost everywhere in } \mathbb{R}^{n+1},$$

(24) $$\text{either } (u_h)_{z_h,R_h} \longrightarrow p \in \mathbb{R}^m \text{ or } \lim_{h\to\infty} |(u_h)_{z_h,R_h}|=+\infty.$$

Further, it is easy to check that

(25) $$v \in BMO(\mathbb{R}^{n+1}),\quad |||v|||_{BMO(\mathbb{R}^{n+1})} \leqq \mu\,.$$

----

x) The case of general T can be easily derived from here using the standard technique.

Taking in (15) the fixed test-function φ we pass to the limit with $h \to \infty$ and (using (20) - (24)) we obtain that v is a weak solution either of the system

(26) $\int_{\mathbb{R}^{n+1}} [v\varphi_\tau - A(z_0, v(\zeta)+p) D_\zeta v\, D_\zeta \varphi]\, d\zeta = 0$

or of the system

(27) $\int_{\mathbb{R}^{n+1}} [v\varphi_\tau - d(z_0) D_\zeta v\, D_\zeta \varphi]\, d\zeta = 0.$

In case of (26), v is a constant vector-function because of (25) and the assumption (L). In case of (27), v is a constant function again because it is a weak solution of the system with constant coefficients which belongs to $BMO(\mathbb{R}^{n+1})$.

Coming back to (14) and putting v for q we obtain

$$\varepsilon \le \int_{Q(0,1)} |v_h(\zeta) - v|^2 d\zeta \,.$$

But the last integral tends to the zero as $h \to \infty$ and so we get the contradiction.

References

[1] O. JOHN: The interior regularity and the Liouville property for the quasilinear parabolic systems, Comment.Math. Univ.Carolinae 23(1982), 685-690.

[2] O. JOHN, J. STARÁ: On the regularity of the weak solution of Cauchy problem for nonlinear parabolic systems via Liouville property, Comment.Math.Univ.Carolinae 25 (1984), 445-457.

[3] J. DANĚČEK: Regularity for nonlinear elliptic systems of second order , Comment.Math.Univ.Carolinae 27(1986), 755-764.

[4] M. GIAQUINTA, E. GIUSTI: Partial regularity for solutions to nonlinear parabolic systems, Annali di Matematica pura ed appl. 97(1973), 253-266.

[5] S. CAMPANATO: Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{2,\theta}(\Omega,\delta)$, Annali di Matematica pura ed appl. 73 (1966), 55-102.

Matematicko-fyzikální fakulta, Karlova universita, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(Oblatum 29.9. 1986)