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A CONTRIBUTION TO TOPOLOGY IN AST: COMPACTNESS

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Abstract: Compact π-relations S_c are introduced. A distinguished role of the relations S_c and $\{\overset{\circ}{=}_c\}$ is presented. Properties of compact relations being real classes are investigated. Obtained results are applied also to model theory and graph theory.

Key words: Nonstandard, compact, indiscernibility relation, π-class, real class, indiscernibles, independence of a graph.

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Introduction: The compactness of π-equivalences is expressed by the demand that in any infinite set there are two equivalent elements. (Remember that in AST every set is finite according to the Tarski's definition. The finiteness is defined by a nonstandard manner.) Such a compactness, which is investigated in the paper, is remarkable not only for σ-equivalences. Three theorems designated by the name A. Vencovská have the principal position in the paper. The first one (Th. 1.23) has been formulated by the author after a detailed analysis of the proof of the third one (Th. 1.26) given by A. Vencovská in the seminar on AST.

These theorems point out an outstanding position of the relations S_c (introduced by A. Vencovská to prove the third theorem) and $\{\overset{\circ}{=}_c\}$. These relations are the finest (the least) ones of all the relations described below. The theorems are substantially applicable for the orientation in compact relations as further theorems in the paper demonstrate. An interesting consequence for the model theory is that if two elements x_1, x_2 of a saturated nonstandard model of Peano's arithmetic have the same type then there are two infinite (*finite) sets of indiscernibles m_1, m_2 such that $x_1 \in m_1$ & $x_2 \in m_2$ & $m_1 \cap m_2 \neq 0$. On the other hand, it is proved that there are x_1, x_2 of the same type which cannot be simultaneously elements of an infinite set of indiscernibles. Some applications to graph theory, presented here, may be also found quite remarkable.

In the paper there are used ideas and assertions from the works quoted at the end of the paper. Therefore some of such assertions have only short proofs or hints with the quotations of works where more detailed proofs are described. In this sense the paper is readable (except some remarks concerning the generalization of the results) without the given quotation, too.

§ 1. Basic theorems. Remember some notions from [V] which we shall use in the paper.

Definition 1.1: 1) A class X is said to be set-theoretically definable (Sd(X)) iff there is a set formula $\varphi(x)$ (the appearance of set parameters is also allowed) such that $x \in X \equiv \varphi(x)$ holds.

2) If we admit the appearance of set parameters only from the class Y, we denote this $Sd_Y(X)$.

3) A class X is said to be a π-class iff it is an intersection of a countable amount of Sd classes.

Definition 1.2: 1) A class X is said to be revealed iff for every countable class Y we have $Y \subseteq X \Rightarrow (\exists y \subseteq X)(Y \subseteq y)$.

2) A class X is said to be fully revealed iff there is no normal formula φ (only the quantification of set variables is allowed) such that $FN = \{x; \varphi(x,X)\}$.

The axiom of prolongation asserts that for every countable class X there is a set function f such that X=f"FN.

Using this axiom and overspilling the following assertions may be easily proved (see [V] and [SV]).

Theorem 1.3: 1) Every fully revealed class is revealed.

2) Every class definable by a normal formula from a fully revealed class is a fully revealed class.

3) Every Sd class is fully revealed.

4) The intersection of a countable amount of revealed classes is a revealed class.

5) Especially, every π-class is a revealed class.

Remember that a linear ordering of V can be defined by a set formula (without parameters). This ordering is "well with respect to the Sd classes" and can be defined using the natural ordering of N and the Sd function $F:N \longleftrightarrow V$ defined recursively using the property $F(x)=\{F(t);$ the t-th element of the dyadic expansion of x is $1\}$. (For more details see [V].)

The theorems we shall give in the paper have a structural character and it would be possible to generalize them without changing ideas of their proofs. We mean that such general formulations would only darken the content of ideas here (we hope also that for some less advanced readers this way will be more convenient). We use therefore the less complicated version - namely Sd classes, π-classes and real classes (the definition of real classes is reminded below). A nice usage of the alternative topological point of view on more general systems of classes may be found in [Ve] or [ČVj2].

Definition 1.4: A reflexive and symmetric relation R is said to be compact iff it has the following property: $(\forall x \subseteq dom(R))(\neg Fin(x) \Rightarrow (\exists t,u \in x)(t \neq u \,\&\, \langle t,u\rangle \in R))$.

Definition 1.5: A class X is said to be R-net for a reflexive and symmetric relation R iff the following holds: $(\forall x,y \in X)(x \neq y \Rightarrow \neg\langle x,y\rangle \in R)$.

Theorem 1.6: A reflexive and symmetric relation R is compact iff $(\forall x \subseteq dom(R))(x$ is an R-net $\Rightarrow Fin(x))$.

Proof: Obvious.

Theorem 1.7: If R is a reflexive and symmetric Sd relation then R is compact iff $(\forall x \subseteq dom(R))(\neg Fin(x) \Rightarrow (\exists y \subseteq x)(\neg Fin(y) \,\&\, (\forall t,u \in y)(\langle t,u\rangle \in R)))$.

Let us prove at first two assertions.

Lemma 1.8: If R is a reflexive, symmetric and compact Sd relation then we have: 1) $(\exists k \in FN)(\forall z)(z$ is an R-net $\Rightarrow$ $\Rightarrow card(z) \leq k)$,

2) $(\forall x \subseteq dom(R))(\neg Fin(x) \Rightarrow (\exists t \in x) \neg Fin(R''\{t\} \cap x))$.

Proof: 1) Let us put $k=max(\{card(z);\ z$ is an R-net$\})$. (This is possible as Sd(R) and R is compact). We have $k \in FN$.

2) The compactness of R implies the compactness of $R \cap X^2$ for every X. Hence we prove the assertion only for the set relation $r=R \cap x^2$ and $x=dom(r)$. Let z be a maximal (in $\subseteq$) r-net.

Such a net exists and must be finite (due to 1)). We have $x=r''z$ (as z is maximal) and the infiniteness of x implies the existence of $t \in z$ such that $r''\{t\}$ is infinite.

Now we can easily prove Theorem 1.7.

Proof: Let t_1 be such that $\mathrm{card}(R''\{t_1\} \cap x)$ is the largest one (thus infinite). Put $x_2=R''\{t_1\} - \{t_1\}$ and choose t_2 analogously as t_1 but for x_2. Proceed further by recursion. It follows (by 2) of the previous lemma) that the recursion cannot stop after a finite number of steps. Hence we have $(\forall i,j)(\langle t_i, t_j \rangle \in R)$.

Remark: The reader may notice that we have not needed the prolongation axiom neither for Th. 1.7 nor for Lemma 1.8.

<u>Theorem 1.9</u>: The assertion from Th. 1.7 holds for any π-class R, too.

Proof: Let $R= \bigcap \{R_i ; i \in FN\}$ and let R_i be reflexive and symmetric Sd relations. Let us put $x=x_1$ and let $x_{i+1} \subseteq x_i$ be a set of the maximal cardinality such that $(\forall t,u \in x_{i+1})(\langle t,u \rangle \in R_i)$. As to the compactness of $R_i \cap x_i^2$ and due to the theorem 1.7 we obtain that x_{i+1} is infinite. Let us prolong the sequence x_i and use overspill. We obtain then an infinite set x_α such that $(\forall i \in FN)(x_\alpha \subseteq x_i)$ and hence $(\forall u,v \in x_\alpha)(\langle u,v \rangle \in R)$.

Remark: The reader may notice that if the sequence R_i is coded by an Sd class ($R_i=X''\{i\}$), then the prolongation axiom is not necessary.

A further generalization of Th. 1.9 for real classes will be given later.

<u>Corollary 1.10</u>: The intersection of a countable amount of compact π-relations is a compact π-relation.

Proof: Proceed analogously to the proof of Th. 1.9.

<u>Corollary 1.11 (P.Vopěnka)</u>: Let R be a reflexive and symmetric π-relation. If $x \subseteq \mathrm{dom}(R)$ is infinite then either there is $y \subseteq x$ such that y is infinite and $(\forall t,u \in y)(\langle t,u \rangle \in R)$ or there is $y \subseteq x$ such that y is infinite and $(\forall t, u \in y)(\langle t,u \rangle \notin R)$.

Proof: Either $R \cap x^2$ is compact, then the first possibility holds by Th . 1.9, or $R \cap x^2$ is not compact and the second possibility holds (there is an infinite $(R \cap x^2)$-net).

The following corollary is quite Ramsey-like.

Corollary 1.12: If X is a π-class having only unordered pairs as its elements, then for every infinite subset $x \subseteq \bigcup X$ there is either an infinite subset $y \subseteq x$ such that $(\forall t,u \in y)$ $(\{t,u\} \notin X)$ or an infinite subset $y \subseteq x$ such that $(\forall t,u \in y)$ $(\{t,u\} \in X)$.

Proof: Put $R = \{\langle t,u\rangle; \{t,u\} \in X\} \cup Id \restriction \bigcup X$ and use Cor. 1.11.

Definition 1.13: Let $\varphi(x,y)$ be a set formula (also a set parameter c is allowed). A set m is said to be homogeneous for φ iff either $(\forall t,u \in m)(t < u \Rightarrow \varphi(t,u))$ or $(\forall t,u \in m)(t < u \Rightarrow \neg \varphi(t,u))$ holds.
(Where $<$ denotes the canonical ordering of V mentioned above.)

Theorem 1.14: For every set formula $\varphi(t,u)$ (also with parameters) the following holds. For every infinite set x there is an infinite subset y homogeneous for φ.

Proof: Put $X = \{\{t,u\}; t < u \Rightarrow \varphi(t,u)\}$ and use Cor. 1.12.

Let us now enumerate all the set formulas of two variables (with parameter c) by $\varphi_i(t,u); i \in FN$.

Definition 1.15 (A. Vencovská): 1) $\langle t,u\rangle \in S_c^n \equiv (\exists m)$ $(\mathrm{card}(m) \geq n \;\&\; m$ is homogeneous for every φ_i such that $i < n) \vee$ $\vee\, t = u$.

2) $S_c = \bigcap\{S_c^n; n \in FN\}$.

Theorem 1.16: For every c, S_c^n are reflexive, symmetric and compact Sd relations and hence S_c is a reflexive, symmetric and compact π-relation.

Proof: It suffices to prove the compactness of S_c^n. If x is infinite then there is an infinite subset $y \subseteq x$ homogeneous for every φ_i such that $i < n$ (Th. 1.14). Any two elements of y are then in S_c^n.

The relation S_c is a very remarkable example of a reflexive, symmetric and compact relation definable with the help of the parameter c. Later we prove that this relation is the finest one among all these relations. Let us now give other examples of reflexive, symmetric and compact π-relations.

Enumerate all set formulas of one free variable (with parameter c) by $\varphi_i(x)$; $i \in FN$.

Definition 1.17: 1) $\langle x,y\rangle \in R_{\varphi_i} \equiv (\varphi_i(x) \equiv \varphi_i(y))$.

2) $\overset{\circ}{=}_{\{c\}} = \bigcap\{R_{\varphi_i}; i \in FN\}$. (See also [V], [ČK1], [V1].)

Theorem 1.18: $\overset{\circ}{=}_{\{c\}}$ is a reflexive, symmetric, transitive and compact π-relation.

Proof: Every R_{φ_i} is an equivalence with two classes of decomposition and thus compact.

$\overset{\circ}{=}_{\{c\}}$ is a compact equivalence definable only with the help of the parameter c. Later we prove that $\overset{\circ}{=}_{\{c\}}$ is the finest of such equivalences.

The following description of $\overset{\circ}{=}$ may be for somebody more acceptable: $x \overset{\circ}{=} y$ iff x, y have the same type.

To the end let us describe an example which is closest to classical topology. Let α be an infinitely large natural number. Let us put $\langle\gamma,\delta\rangle \in R_n \equiv n\cdot|\gamma-\delta| \leq \alpha$ for $\gamma,\delta < \alpha$. R_n are reflexive, symmetric and compact set relations. Hence also $\doteq = \bigcap\{R_n; n \in FN\}$ is a reflexive, symmetric and compact π-relation being moreover an equivalence. This equivalence describes the same topological phenomenon on the bounded interval [0,1] of real numbers as the common topology. Further details can be found in [Č1].

Let us now treat the relations $\overset{\circ}{=}_{\{c\}}$. (More details can be found in [ČK1], [ČK2] and also in [V1].

Definition 1.19: 1) $\mu(x) = (\overset{\circ}{=}_{\{c\}})''\{x\}$. The equivalence classes of $\overset{\circ}{=}_{\{c\}}$ are called monads.

2) $\mathcal{F}ig(X) \equiv X = (\overset{\circ}{=}_{\{c\}})''X$. Such classes are said to be figures.

Theorem 1.20: 1) Classes definable by set formulas with parameter c are figures.

2) The union and the intersection of any system of figures is a figure.

3) The difference of two figures is a figure.

4) $\mathrm{dom}(\mu(\langle x,y\rangle)) = \mu(y)$, $\mathrm{rng}(\mu(\langle x,y\rangle)) = \mu(x)$.

5) The domain and the range of a figure is a figure.

6) The cartesian product of two figures is a figure.

7) If X, Y are figures, then X"Y is a figure.

Proof: 1),2),3) are obvious. 4) If X is $Sd_{\{c\}}$ and $\langle x,y\rangle \in X$, then dom(X) is also $Sd_{\{c\}}$ and $y \in \mathrm{dom}(X)$. Hence $\mathrm{dom}(\mu(\langle x,y\rangle)) \supseteq \mu(y)$. On the other hand: If X is $Sd_{\{c\}}$ and $X \supseteq \mu(y)$, then $V \times X$

is $Sd_{\{c\}}$ and $\langle x,y\rangle \in V \times X$. 5) This is a consequence of 4). 6) Let us prove at first that both $V \times \mu(x)$ and $\mu(x) \times V$ are figures. If $\langle t,u\rangle \in V \times \mu(x)$ and $\langle t_1,u_1\rangle \in \mu(\langle t,u\rangle)$, then $u_1 \in \mu(x)$ (use 4)) and hence $\langle t_1, u_1\rangle \in V \times \mu(x)$. The proof of the second assertion is analogous. If X is a figure, then both $V \times X$ and $X \times V$ are figures and 6) is a consequence of the equality $X \times Y = (X \times V) \cap \cap (V \times Y)$. 7) $X"Y = \mathrm{rng}(X \cap (V \times Y))$.

Theorem 1.21: If $\varphi(x_1,\dots,x_n,Z_1,\dots,Z_k)$ is a normal formula with parameter c and $X_1,\dots,X_k$ are figures, then $X = \{\langle x_1,\dots \dots,x_n\rangle;\ \varphi(x_1,\dots,x_n,X_1,\dots,X_k)\}$ is a figure.

Proof: For the reader who is acquainted with the corresponding Gödel's theorem, it suffices to remind that $\mathrm{Cnv}_2(X)$ and $\mathrm{Cnv}_3(X)$ may be obtained from X as the images by means of suitable Sd functions. We recall that $\{\langle x,y\rangle; x \in y\}$ is Sd and $X \upharpoonright Y = X \cap (V \times Y)$. Let the other readers follow the given instructions. Replace all the subformulas of the form $X_i = X_j$ ($z = X_i$ resp.) by these equivalents: $(\forall t)(t \in X_i \equiv t \in X_j)$ $((\forall t)(t \in X_i \equiv t \in z)$ resp.). Consider the prenex form of the formula. At first we prove the theorem for open formulas (i.e. formulas without quantifiers) in which the above described two types of atomic formulas do not occur. We shall substitute the formula by an equivalent containing only $\neg$, & . Now we proceed by the induction based on the complexity of the formula. For the atomic formula $x_i \in X_j$ let us envelop the class X_j by cartesian products with V in such a way that X_j will be on the i-th coordinate. For $x_i \in x_j$ we have that $Y = \{\langle x_1,\dots \dots,x_n\rangle; x_i \in x_j\}$ is an Sd class and hence a figure. Similarly we proceed in the case $x_i = x_j$. For & we use the operation $Y_1 \cap Y_2$ and for negation the operation $V^n - Y$. To finish the proof it suffices to examine the induction step for quantifiers. For $(\exists x_i)$ we use the operation $\mathrm{dom}_i(Y)$ and for $(\forall x_i)$ the operation $V^n - \mathrm{dom}_i(V^{n+1} - Y)$.

The following theorem is an easy consequence of Th. 1.20.

Theorem 1.22: If a σ-class is an intersection of a countable system of Sd_c classes then X is a figure in $\overset{\circ}{=}_{\{c\}}$. Especially $\overset{\circ}{=}_{\{c\}}$ and S_c are figures in $\overset{\circ}{=}_{\{c\}}$.

Theorem 1.23 (A. Vencovská): If a reflexive, symmetric and compact relation R is a figure in $\overset{\circ}{=}_{\{c\}}$ then $S_c \cap (\mathrm{dom}(R))^2 \subseteq R$.

Proof: We prove at first that $\langle t,u\rangle \in S_c \Rightarrow t \overset{\circ}{=}_{\{c\}} u$. Let e.g. $t<u$ and $t,u \in x$ where x is an infinite set homogeneous for all set formulas with the parameter c (with two free variables). If we put $y=\{\langle v,w\rangle; v<w \;\&\; v,w\in x\}$ then $y \subseteq \mu(\langle t,u\rangle)$ holds and hence $x=\mathrm{dom}(y) \subseteq \mu(u)$. Now we prove the assertion of the theorem. Let $\langle t,u\rangle \in S_c \cap (\mathrm{dom}(R))^2$, let e.g. $t<u$ and let x be an infinite set such that $t,u\in x$ and x is homogeneous. As $x \subset \mu(u)$ and $\mathrm{dom}(R)$ is a figure in $\overset{\circ}{=}_{\{c\}}$ (see Th. 1.20), we have $x \subseteq \mathrm{dom}(R)$. Thus, there are $v,w \in x$ such that $v \neq w \;\&\; \langle v,w\rangle \in R$ (compactness of R). Let e.g. $v<w$. We have $\langle t,u\rangle \overset{\circ}{=}_{\{c\}} \langle v,w\rangle$ (homogeneity of x) and hence $\langle t,u\rangle \in R$ (R is a figure).

Now we want to prove the equality $S_c \circ S_c = \overset{\circ}{=}_{\{c\}}$. The following theorem helps us to prove it.

Theorem 1.24: If R is a reflexive, symmetric and compact $Sd_{\{c\}}$ relation then there is a maximal R-net y having only elements definable by set formulas with parameter c.

Proof: Let us put $Y=\{x;\ x$ is a maximal R-net$\}$. Y is definable by a normal formula with parameter R and hence Y is $Sd_{\{c\}}$. Let x be the smallest (in $<$) element of Y. x is definable by a set formula with the parameter c and as x is finite we have that every element of x is definable by a set formula with the parameter c, too.

Theorem 1.25 (A. Vencovská): $S_c \circ S_c = \overset{\circ}{=}_{\{c\}}$.

Proof: Let $x \overset{\circ}{=}_{\{c\}} y$. For every S_c^n there is a maximal S_c^nnet consisting of definable elements. Let z_n be the smallest element of this net such that $\langle x,z_n\rangle \in S_c^n$. Due to the definability of z_n and the validity of $x \overset{\circ}{=}_{\{c\}} y$ we obtain $\langle x,z_n\rangle \overset{\circ}{=}_{\{c\}} \langle y,z_n\rangle$ and hence $\langle y,z_n\rangle \in S_c^n$ (S_c^n is a figure). Using the axiom of prolongation and overspill we obtain z_α such that $\langle x,z_\alpha\rangle \in S_c$ and $\langle y,z_\alpha\rangle \in S_c$.

Theorem 1.26 (A. Vencovská): If R is a compact equivalence relation being a figure in $\overset{\circ}{=}_{\{c\}}$ then $\overset{\circ}{=}_{\{c\}} \cap (\mathrm{dom}(R))^2 \subseteq R$.

Proof: By Th. 1.23 we have $S_c \cap (\mathrm{dom}(R))^2 \subseteq R$ hence $(S_c \cap (\mathrm{dom}(R))^2) \circ (S_c \cap (\mathrm{dom}(R))^2) = \overset{\circ}{=}_{\{c\}} \cap (\mathrm{dom}(R))^2 \subseteq R \circ R = R$.

§ 2. Examples of applications of theorems and some other examples

Let us now give some interesting consequences of the previous theorems.

Definition 2.1: Let $\varphi(x_1,\ldots,x_n)$ be a set formula. A class X is said to be homogeneous for φ iff for every n-tuples $\langle t_1,\ldots,t_n\rangle, \langle u_1,\ldots,u_n\rangle$ of elements of X we have $t_1 < t_2 < \ldots$ $\ldots < t_n \,\&\, u_1 < u_2 < \ldots < u_n \Rightarrow \varphi(t_1,\ldots;t_n) \equiv \varphi(u_1,\ldots,u_n)$.

The reader who is acquainted with Ramsey theorem proves easily that in every infinite set there is an infinite subset homogeneous for φ . Some hints for proving Ramsey theorem (and especially the given assertion) - as a byproduct of our considerations - may be found in Section 3.

Definition 2.2: A class X is said to be the class of indiscernibles iff X is homogeneous for every set formula.

The classes of indiscernibles are often used in model theory. With respect to this fact the following consequences of above theorems are of some interest. The first of them was noticed by J. Mlček.

Theorem 2.3: $\langle x,y\rangle \in S$ iff $x=y$ or there is an infinite set of indiscernibles i such that $x,y \in i$. (S without subscript denotes that no parameter is used.)

Proof: Let us put $\langle x,y\rangle \in \bar{S} \equiv x=y \vee$ there is an infinite set of indiscernibles i such that $x,y \in i$. $\bar{S}$ is a reflexive and symmetric relation being a π-class. The fact $\bar{S} \subseteq S$ is an immediate consequence of definitions. For the proof of $S \subseteq \bar{S}$ it suffices to justify the compactness of $\bar{S}$. To prove this fact it is sufficient to realize that in every infinite set there is an infinite set of indiscernibles (a set homogeneous for all set formulas of an arbitrary number of free variables). Now we use the above mentioned fact that for any set formula and any infinite set there is an infinite subset homogeneous for this formula, the axiom of prolongation and overspill.

Remark: If we consider all set formulas of the language $FL_{\{c\}}$ in the definition of indiscernibles then we obtain the analogue of the theorem for the relation $S_{\{c\}}$.

Theorem 2.4: If $x \hat{=} y$ and $x \neq y$ then there are two infinite sets of indiscernibles i_x, i_y such that $x \in i_x \,\&\, y \in i_y \,\&\, i_x \cap i_y \neq 0$.

Proof: Use the fact $S \circ S = \hat{=}$ (Th. 1.25).

The last theorem acquires a model theoretical form if we change "$x \hat{=} y$" by "x, y have the same type". It is obvious that the last theorem can be reformulated into a version with parameters.

Now a question arises if the sets i_x, i_y must be different, i.e. if the equality $S_c = \overset{\circ}{=}_c$ holds. In the paper [SVe] the existence of a proper class of indiscernibles is proved. Moreover, this class is a π-class (without parameters) and hence a figure in $\overset{\circ}{=}$ and thus a monad in $\overset{\circ}{=}$ (indiscernibles). If we denote by In one of such classes then we have obviously that for any $x,y \in$ In we can find an infinite set of indiscernibles i such that both x and y are elements of i. On the other hand the following example proves that for every $\overset{\circ}{=}_{\{c\}}$ there are x, y such that $x \overset{\circ}{=}_{\{c\}} y$ and that it is not possible to find such z that the set $\{x,y,z\}$ is homogeneous for any set formula of the language $FL_{\{c\}}$ with two variables.

<u>Example 2.5</u>: Let $t \overset{\circ}{=}_{\{c\}} u \,\&\, t \neq u$. As $\mathrm{rng}(\mu(\langle t,u\rangle)) = \mu(t)$ (Th. 1.20) there is a v such that $v \overset{\circ}{=}_{\{c\}} u$ and $\langle t,u\rangle \overset{\circ}{=}_{\{c\}} \langle u,v\rangle$. We prove $\langle\langle t,u\rangle,\langle u,v\rangle\rangle \notin S_c$. The assumption that there are w, z such that $\{\langle t,u\rangle,\langle u,v\rangle,\langle w,z\rangle\}$ is homogeneous implies namely a contradiction with $t \neq u$. If we suppose e.g. $\langle t,u\rangle < \langle u,v\rangle < \langle w,z\rangle$ then the first pair (of these ordered pairs) has the property: "the second component of the less element is equal to the first component of the larger one", hence (due to homogeneity) all pairs have this property and thus we obtain w=u, w=v. Similarly we proceed further and prove t=u=v=w in contradiction to $t \neq u$. In a different ordering of pairs we proceed analogously.

Now we give some consequences of our theorems to topology in AST.

<u>Theorem 2.6</u>: Let R, T be two reflexive, symmetric and compact π-relations. If $X = \mathrm{dom}(R) \cap \mathrm{dom}(T)$ then there is a c such that $\overset{\circ}{=}_{\{c\}} \cap X^2 \subseteq R \circ T$.

Proof: Let $\{c_i; i \in FN\}$ be the sequence of all the parameters occurring in set formulas defining Sd relations such that R, T are intersections. Let us prolong this sequence to c (thus $c(i) = c_i$). We have $S_c \cap X^2 \subseteq R \,\&\, S_c \cap X^2 \subseteq T$ (Th. 1.23). Now it suffices to use $(S_c \cap X^2) \circ (S_c \cap X^2) = \overset{\circ}{=}_{\{c\}} \cap X^2$.

Especially, if we put $R = S = \overset{\ast}{=}$, where $\overset{\ast}{=}$ is an indiscernibility equivalence (a compact π-equivalence defined on V) we obtain the following theorem proved also in [V1].

<u>Theorem 2.7 (P. Vopěnka)</u>: For every indiscernibility equivalence $\stackrel{*}{=}$ there is c such that $\stackrel{\circ}{=}_{\{c\}} \subseteq \stackrel{*}{=}$.

<u>Definition 2.8</u>: A class X is called real iff there is c such that X is a figure in $\stackrel{\circ}{=}_{\{c\}}$ (see also [ČV]).

Note that if $X_1,\dots,X_n$ are real classes then there is c such that they are figures in $\stackrel{\circ}{=}_{\{c\}}$. If, namely, X_i is a figure in $\stackrel{\circ}{=}_{\{c_i\}}$ it is also a figure in $\stackrel{\circ}{=}_{\{\langle c_1,\dots,c_n\rangle\}}$.

From the theorem 1.21 we know that real classes are closed on the definitions by normal formulas. In the paper [ČV] it is proved that real classes are closed also on definition by non-normal formulas. We shall not need this fact in this paper.

<u>Theorem 2.9</u>: The following properties are equivalent:

1) X is a real class.

2) X is a figure in an indiscernibility equivalence.

Proof: A consequence of Th. 2.7.

Now we prove that in some theorems of § 1 it is possible to replace the assumption X is a π-class by X is a real class.

<u>Theorem 2.10:</u> If R is a real, reflexive and symmetric relation then the following properties are equivalent:

1) R is compact (i.e. in every infinite set $x \subseteq \mathrm{dom}(R)$ there are $t,u \in x$ such that $t \neq u \;\&\; \langle t,u\rangle \in R$).

2) Every R-net is finite.

3) In every infinite set $x \subseteq \mathrm{dom}(R)$ there is an infinite subset $y \subseteq x$ such that $(\forall t,u \in y)(\langle t,u\rangle \in R)$.

Proof: 3) $\Rightarrow$ 1) obvious, $\neg$2) $\equiv$ $\neg$1) obvious, 1) $\Rightarrow$ 3). If R is a figure in $\stackrel{\circ}{=}_{\{c\}}$ then $S_c \cap (\mathrm{dom}(R))^2 \subseteq R$ and there is $y \subseteq x$ such that y is infinite and $(\forall t,u \in y)(\langle t,u\rangle \in S_c)$(Th. 1.7) and hence also $(\forall t,u \in y)(\langle t,u\rangle \in R)$.

<u>Theorem 2.11</u>: If R is a real equivalence then the following properties are equivalent: 1) R is compact.

2) $(\forall \gamma \in N\text{-}FN)(\exists x)(\mathrm{card}(x) \leq \gamma \;\&\; R''x = \mathrm{dom}(R))$.

Proof: We prove at first that $\stackrel{\circ}{=}_{\{c\}}$ has the property 2). If x is a set containing all the elements definable with the parameter c then $V = (\stackrel{\circ}{=}_{\{c\}})''x$. To prove the given property it suffices to show that the intersection of x and every monad in $\stackrel{\circ}{=}_{\{c\}}$ is non-empty. For this it suffices to prove that the intersection of x and any class X definable by a set formula with the parameter c is nonempty and use the prolongation and overspill. If X is an $Sd_{\{c\}}$-class then e.g. the least element of X is definable by a set formula with the parameter c and hence it is an element of x. We have proved $V=(\overset{\circ}{=}_{\{c\}})''x$. As there are only countably many elements definable with the parameter c, they may be included (using the prolongation) into a set with an arbitrary (small) infinite cardinality. We have proved that $\overset{\circ}{=}_{\{c\}}$ has the property 2). Now we prove 1) $\Rightarrow$ 2). Let R be a figure in $\overset{\circ}{=}_{\{c\}}$. Hence $\overset{\circ}{=}_{\{c\}} \cap (\mathrm{dom}(R))^2 \subseteq$ $\subseteq R$ and if x is such that $V=(\overset{\circ}{=}_{\{c\}})''x$ then the intersection of every monad and x is nonempty. As dom(R) is a figure, the intersection of x and every monad of this figure is nonempty, too. Thus we have even $\mathrm{dom}(R)=(\overset{\circ}{=}_{\{c\}}(\mathrm{dom}(R)^2)''x$. Now we prove 2) $\Rightarrow$ 1). Let $y \subseteq \mathrm{dom}(R)$ be infinite. From the assumption $(\forall t,u \in y)(\langle t,u\rangle \notin R)$ we shall deduce a contradiction. Let x be a set having the property $(\mathrm{card}(x))^2 \leq \mathrm{card}(y)\ \&\ \mathrm{dom}(R)=R''x$. Let us put $F=R \cap (y \times x)$. From our assumption and from the transitivity we obtain that F is a function. We also have $\mathrm{rng}(F)=y$ as $R''x \supseteq y$. F is also a real class and this fact - as will be proved - leads to the contradiction (cf. also [ČV]). If we put $d=\langle c,x,y\rangle$ we obtain that R,y,x are figures in $\overset{\circ}{=}_{\{d\}}$ and hence F is a figure in $\overset{\circ}{=}_{\{d\}}$, too (Th. 1.20). Every monad $\mu \subseteq F$ must be also a function. Using the prolongation and overspill we obtain that μ is a subclass of a set function definable with the parameter d having its domain included in x. Let $\{f_i; i \in FN\}$ denote an enumeration of these functions. We have $F \subseteq \bigcup\{f_i; i \in FN\}$. Let us prolong this sequence. If $\alpha \in N-FN$ is such that $(\forall \beta < \alpha)(f_\beta$ is a function $\&\ \mathrm{dom}(f_\beta) \subseteq$ $\subseteq x)\ \&\ \alpha < \mathrm{card}(x)$ then $y \subseteq \mathrm{rng}(\bigcup\{f_\beta; \beta < \alpha\})$. But we have $\mathrm{card}(y) <$ $< \mathrm{card}(\bigcup\{f_\beta; \beta < \alpha\}) \leq \alpha \cdot \mathrm{card}(x)$ - a contradiction.

<u>Definition 2.12</u>: Let R be a reflexive and symmetric relation. We denote by $R^{\mathcal{P}}$ the power relation to R and define as follows: $\langle x,y\rangle \in R^{\mathcal{P}} \equiv x \subseteq R''y\ \&\ y \subseteq R''x$.

The proofs of the following easy assertions are left to the reader.

<u>Theorem 2.13</u>: 1) $R^{\mathcal{P}}$ is reflexive snd symmetric.

2) $\mathrm{dom}(R^{\mathcal{P}}) = \mathcal{P}(\mathrm{dom}(R))$.

3) $R_1 \subset R_2 \Rightarrow R_1^{\mathcal{P}} \subset R_2^{\mathcal{P}}$.

4) R is a π-class $\Rightarrow R^{\mathcal{P}}$ is a π-class.

5) R is transitive $\Rightarrow R^{\mathcal{P}}$ is transitive and $\langle x,y\rangle \in R^{\mathcal{P}} \equiv R''x = R''y$.

Using the property 2) from Th. 2.11, the following theorem may be proved (see also [V]).

<u>Theorem 2.14</u>: If R is a real compact equivalence then $R^{\mathcal{P}}$ is also a real compact equivalence.

Proof: Essential is only the proof of the compactness and hence it is sufficient to prove the theorem only for $\overset{\circ}{=}_{\{c\}}$. Here we use the fact that $V=(\overset{\circ}{=}_{\{c\}})''x \Rightarrow \mathcal{P}(V)=(\overset{\circ}{=}_{\{c\}}^{\mathcal{P}})''\,\mathcal{P}(x)$ and Th. 2.11.

The essentiality of the requirement of the transitivity of R points out the following theorem.

<u>Theorem 2.15</u>: For any c the relation $S_c^{\mathcal{P}}$ is not compact.

Proof: The compactness of $S_c^{\mathcal{P}}$ implies $S_c \subseteq S_c^{\mathcal{P}}$ as $S_c^{\mathcal{P}}$ is definable by a normal formula from S_c and hence it is a figure in $\overset{\circ}{=}_{\{c\}}$. From this inclusion we deduce a contradiction. Let $\alpha \in$ N-FN. Let us put $x=\{\beta\times\{\beta\};\ \beta\in\alpha\}$. x is infinite and due to the compactness of S_c there must be $\beta,\gamma\in\alpha$ such that $\beta\neq\gamma$ & $\langle\beta\times\{\beta\},\ \gamma\times\{\gamma\}\rangle\in S_c\subseteq S_c^{\mathcal{P}}$. Let e.g. $\beta\in\gamma$. As $\gamma\times\{\gamma\}\subseteq S_c''(\beta\times\{\beta\})$ there is $\delta\in\beta$ such that $\langle\langle\delta,\beta\rangle,\langle\beta,\gamma\rangle\rangle\in S_c$ which contradicts the example 2.5.

Let us now prove some consequences of our results to the graph theory. Let us limit ourselves only on finite graphs. Remember that an undirected graph is a set with a symmetric relation R such that dom(R) $\subseteq$ x. Elements of x are called vertices, pairs $\{t,u\}$ such that $\langle t,u\rangle\in R$ are called edges and if $\langle t,t\rangle\in R$, it is said that the graph has a loop in t. The largest number k such that there are k vertices such that no pair of them forms an edge, is called the independence of the graph (cf. with the definition of an R-net). To simplify our considerations and to be consistent with our previous investigations let us add to all the graphs, we shall work with, all their loops. (The reflexivity of the corresponding relation.) Remember that a graph is called complete iff every tuple of vertices forms an edge and connected ff any two vertices may be connected by a path. Maximal connected parts of graphs are called components. If the corresponding relation is transitive then the graph consists of components (classes of the equivalence) which are complete graphs.

Theorem 2.16: For every k there is m such that if R_1, R_2 are undirected graphs with the same vertices having the independence less or equal to k, then the composition of these graphs (this graph may be mixed) has a subgraph with the same vertices (only some arrows and edges can be cancelled) consisting of maximally m components which are complete graphs.

Proof: (By contradiction.) Let for a fixed k we have that for every n there are reflexive, symmetric relations R_1^n, R_2^n such that $dom(R_1^n)=dom(R_2^n)$ and that there is no equivalence on $dom(R_1^n)$ which is a subclass of $R_1^n \circ R_2^n$ and having maximally n equivalence classes. Let us form two sequences $\{R_1^n; n \in \omega\}$ and $\{R_2^n; n \in \omega\}$ such that the n-th elements have the mentioned property (we restrict ourselves on finite graphs). Let us prolong these sequences in such a way that the mentioned set properties hold also for infinite superscripts. Let R_1 and R_2 denote the prolonged sequences. If we put $c=\langle R_1, R_2, \alpha\rangle$ for an infinite α then R_1^α, R_2^α are Sd_c-classes and they are compact (the corresponding graphs have the independence at most k). If we put $d=dom(R_1^\alpha)=$ $=dom(R_2^\alpha)$ we obtain $S_c \cap d^2 \subseteq R_1^\alpha$ and $S_c \cap d^2 \subseteq R_2^\alpha$ and hence $\overset{\circ}{\{c\}} \cap d^2 \subseteq R_1^\alpha \circ R_2^\alpha$. As $\overset{\circ}{\{c\}} \cap d^2$ is a compact equivalence being an intersection of a countable decreasing (in c) sequence of Sd equivalences on d (let us denote this sequence by $\{e_n; n \in \omega\}$), there is $n \in \omega$ such that $e_n \subseteq R_1^\alpha \circ R_2^\alpha$. But e_n has only a finite number of equivalence classes and thus the corresponding graph consists of a finite number of complete subgraphs - a contradiction.

From the example 2.5* we can obtain the following assertion.

Theorem 2.17: For every n there is an undirected graph with the independence maximally 6 such that there is no subgraph (with the same vertices) consisting of at most n complete components.

Proof: Note that $6=R(2,3,2)$, hence for any partition of pairs of a set x, such that $card(x) \geq 6$, on two subsets there is a homogeneous set y such that $card(y)=3$. Let r_k be a reflexive and symmetric relation defined on unordered pairs of natural numbers less than k as follows: For a an unordered pair of natural numbers a_1, a_2 denotes the smaller, larger element of a, respectively. Now we put $\langle a,b\rangle \in r_k \equiv a=b \vee (\exists u)(card(u) \geq 3 \,\&\, a, b \in u \,\&\, u$ is homogeneous for the formula $\varphi(x,y) \equiv x_2=y_1)$. We deduce a contradiction from the assumption that there is m such that $(\forall k)(\exists e_k \subseteq r_k)(e_k$ is an equivalence having m equivalence classes & $\mathrm{dom}(e_k)=\mathrm{dom}(r_k))$. Let us prolong the sequence $\{e_n; n \in \omega\}$ and denote it e. For $\alpha \in$ N-FN we put $c = \langle e, \alpha \rangle$. Now e_α is a compact Sd equivalence being a figure in $\stackrel{\circ}{=}_{\{c\}}$. Hence $\stackrel{\circ}{=}_{\{c\}} \cap$ $\cap\, (\mathrm{dom}(e_\alpha))^2 \subseteq e_\alpha$. Now we can find numbers β, γ, δ such that $\beta < \gamma < \delta < \alpha \ \& \ \beta \stackrel{\circ}{=}_{\{c\}} \gamma \stackrel{\circ}{=}_{\{c\}} \delta \ \& \ \{\beta, \gamma\} \stackrel{\circ}{=}_{\{c\}} \{\gamma, \delta\}$ and deduce a contradiction similarly as in the example 2.5.

By a quite analogous manner we obtain from Th. 2.13 and Th. 2.14 the assertion of the following theorem 2.19. Let us give a definition before.

Definition 2.18: For an undirected graph G (with all loops) we define on the powerset of the set of vertices the graph $G^{\mathcal{P}}$ by the following way: sets u, v are connected by an edge iff the neighbourhood of the first one covers the second one and vice versa. (If r is the corresponding relation to G then $r^{\mathcal{P}}$ is the corresponding one to $G^{\mathcal{P}}$.) We define analogously the graph $G_2^{\mathcal{P}}$ using the two-steps neighbourhood. (If r is the corresponding relation to G then $(r \circ r)^{\mathcal{P}}$ is the corresponding one to $G_2^{\mathcal{P}}$.)

Theorem 2.19: 1) For every k there is m such that if the independence of G is less than or equal to k then the independence of $G_2^{\mathcal{P}}$ is less than or equal to m.

2) For every n there is a graph with the independence ≤ 6 such that the independence of $G^{\mathcal{P}}$ is larger than n.

§ 3. The possibilities of the generalization of the ideas in the paper

The author suggests five directions for a generalization of the ideas contained here.

1) The investigation in higher dimensions - i.e. ternary and more-ary relations.

2) The investigation for other basic systems of classes (different from the system of Sd classes).

3) The usage in poorer models than these of AST.

4) The usage of other forms of "finiteness".

5) Using "compactness" (e.g. the compactness theorem from mathematical logic) to extend our results for infinite relations in the classical set theory.

As the author does not want to burden the shelves of libraries with another unread monography, as he is not able to estimate the importance of these generalizations and as he moreover means that for an open-minded man (or woman) a hint at an ideá is much better than hiding of ideas in formal details and also due to the author´s "spring fever", the following text is much more shortened than the previous one.

1) For higher dimensions the investigation is much more "Ramsey like". The assumption of the symmetry of a relation may be exchanged by the investigation of unordered pairs. The assumption of the reflexivity of relations may be exchanged by specifying their supports (dom(R) in the initial point of view). If we denote by $\mathcal{P}_k(C)$ the class of subsets of C having cardinality k ($\mathcal{P}_k(C)=\{x; x\subseteq C \,\&\, \mathrm{card}(x)=k)$ then for $R\subseteq\mathcal{P}_k$ (C) we call the class X to be an R-net iff $(\forall t\in\mathcal{P}_k(X))(t\notin R)$. $R\subseteq\mathcal{P}_k(V)$ is called compact on C iff every subset $x\subseteq C$ being an R-net is finite. The compactness is a hereditary property in the following sense: If R is compact on C and $D\subset C$ then R is compact on D. The technical lemma 1.8.2) obtains the following form : Let $R\subseteq\mathcal{P}_{k+1}(V)$ be set-theoretically definable. If x is an infinite set such that R is compact on x then there is $t\in x$ and an infinite subset $y\subseteq x$ such that if we put $\overline{R}=\{u\in\mathcal{P}_k(V); u\cup\{t\}\in R\}$ then $\overline{R}$ is compact (in the dimension k) on y. Theorem 1.7 obtains the form: Let $R\subseteq\mathcal{P}_k(V)$ be Sd. If x is infinite and R is compact on x then there is an infinite set $y\subseteq x$ such that $(\forall u\in\mathcal{P}_k(y))$ $(u\in R)$. The proof may be done by an iteration analogous to that one in Theorem 1.7 for the case k=2 when using the induction hypothesis and the given adaption of L. 1.8.2). The given procedure may be also compared with the proof of Ramsey theorem given in [G]. Now the reader is able to prove the assertion from the previous text and namely that for every infinite set x and every set formula φ (also with more than two variables) there is an infinite subset $y\subseteq x$ homogeneous for φ . The definition 1.15 may be adapted to the form: a) $u\in{}^{k}S^{n}_{c}\equiv u\in\mathcal{P}_k(V)\,\&$ $\&\,(\exists v)(\mathrm{card}(v)\geq n \,\&\, v$ is homogeneous for all φ_i, $i\leq n)$.

b) ${}^{k}S_{c}=\bigcap\{{}^{k}S^{n}_{c}; n\in FN\}$. Theorem 1.23 obtains the following form: Let R, Y be figures in $\overset{\circ}{=}_{\{C\}}$. If $R\subseteq\mathcal{P}_k(V)$ is compact on Y then $(\mathcal{P}_k(Y)\cap{}^{k}S_{c})\subseteq R$. Theorem 1.25 obtains the form:
$(\forall k>0)(x\neq y \,\&\, x\overset{\circ}{=}y\equiv(\exists u\in\mathcal{P}_k(V))(u\cup\{x\}\in{}^{k+1}S_{c}\,\&\,u\cup\{y\}\in{}^{k+1}S_{c}))$.

Theorem 2.3 may be adapted to: $u \in {}^kS$ = there is an infinite set of indiscernibles i such that $u \subseteq i$. Theorem 2.4 may be strengthened to: $x \overset{\circ}{=} y \;\&\; x \neq y \equiv$ there is an infinite set of indiscernibles i such that both $i \cup \{x\}$ and $i \cup \{y\}$ are sets of indiscernibles. If we suppose moreover that x (and hence also y) is larger than all definable elements then we may ask that all the elements of i are less than x and y. Concerning the relation of S_c^k and S_c^m for different k, m, we have that if $k < m$ then $u \in {}^kS_c \equiv$ $\equiv (\exists v \supset u)(v \in S_c^m)$. The proof is an easy consequence of the characterization of S^m by infinite sets of indiscernibles.

2) The fact that we start in our considerations from Sd classes is not substantial. We may start from a system of classes having "similar properties". Substantial properties seemed to be that the system is closed on definitions by normal formulas and that classes have set intersections with sets (hence classes are fully revealed). As an example we remember the system Sd_V^* (see [SV]) and papers [ČVj] and [Ve].

3) The paper is written in the framework of AST and technical means of this theory are used. But the author has intentionally used as few specific axioms of AST as possible (actually only the axiom of prolongation). Moreover, we have tried to specify countable semisets as much as possible so that we could argue also by another way for the existence of their prolongation. If e.g. a countable semiset is in the standard system of a model of PA (Peano's arithmetic) then the prolongation may be proved only using the overspill. It is possible to use also the property that the model is recursively saturated or the existence of the universal relation for relations of a given type from arithmetical hierarchy (see Th. Cleene § 7.5 [Sh]). More information about the connection of models of AST and PA can be found in [PS].

4) The finiteness can be understood more generally, too. The usage of SD_V^* classes and $card(x) \in FN^*$ (instead of the finiteness) is almost evident. It is possible to use also another type of cuts than FN. The usage of such a type of "finiteness" for a construction of an alternative of real numbers can be found in [Č1]. Also in the paper [PS] some models of AST in which the interpretation of FN must differ from ω, are pointed out.

5) To this point let us give an illustrating example. From Th. 2.15 the following assertion may be obtained: For every k

there is m with the following property: If R is a reflexive symmetric relation such that dom(R) is infinite and in every k-element subset of dom(R) there are x, y such that $x \neq y$ and $\langle x,y\rangle \in R$ then dom(R) can be divided on m disjoint parts such that if x, y are in a part then $\langle x,y\rangle \in R \circ R$. We prove this using nonstandard analysis: Let $d \subset {}^*(\mathcal{P}_{fin}(dom(R)))$ be such that $dom(R) \subset d$ (i.e. every standard element of dom(R) is an element of d). Let us put $r = {}^*R \cap d^2$. We have $R \subseteq r$ and r is *finite. Hence we may apply Th. 2.15 on r and we obtain a partition of d on maximally m parts. This partition generates a partition of dom(R) on maximally m parts. The proof of the existence of a reflexive symmetric relation such that in every unordered 6-tuple of its domain there are two elements being in this relation and there is no finite partition of the domain finer than the relation is left to the reader.

The sources of ideas in this paper: As the main source have served the considerations of A. Vencovská obtained in theorems 1.23, 1.25, 1.26. These considerations have been partially motivated by the attempt of the author to generalize the alternative view on topology (see [Č2]). The theorems being consequences of the mentioned theorems and examples are due to the author. They have arisen when working with one of the author's student (L. Paroha) on the first attempt of the comprehensive elaboration of the matter. The arrangement of theorems up to Corollary 1.11 is taken from the new P. Vopěnka's book; it allows to avoid the quotation on Ramsey theorem (this one appears to be a byproduct). These theorems (except Corollary 1.11) are obvious consequences of Theorem 1.23 proved before.

When discussing the matter with J. Mlček he noticed that the generalization to higher dimensions is quite interesting. The special way of this generalization described here should be compared with the proof of Ramsey theorem given in [G]

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