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PERIODIC SOLUTIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS OF LIENARD TYPE WITH JUMPING NONLINEARITIES

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Abstract: An existence theorem of periodic solutions for the first order differential system $x' = y - F(x) + E_1(t)$, $y' = -g(x) + E_2(t)$ is proved and applications are given to the scalar Liénard equation $x'' + f(x)x' + g(x) = e(t)$, under suitable assumptions on the limits $\lim_{x\to\pm\infty} F(x)/x$ and $\lim_{x\to\pm\infty} g(x)/x$. In this way, a theorem of S. Fučík, concerning the second order equation $x'' + g(x) = e(t)$, is extended to the Liénard equation.

Key words: Periodic solutions, Liénard equation, jumping nonlinearities, homogeneous equations, conditions for the existence of a global center.

Classification: 34 C 25, 34 B 15

Introduction. In [8], S. Fučík considered the problem of the existence of periodic solutions to the second order scalar differential equation

$$x'' + g(x) = e(t),$$

assuming that the limits

$$\lim_{x\to-\infty} g(x)/x = g_1 , \qquad \lim_{x\to+\infty} g(x)/x = g_2$$

exist finite. More precisely, he found a critical set K in the (g_1,g_2)-plane such that the existence of periodic solutions (having the same period of $e(\cdot)$) is ensured provided that $(g_1,g_2) \notin K$. For related results, see also [2]. Fučík's theorem was further generalized by several authors (see, e.g. [3], [6], [9], [12], [13]) either considering conditions on the lim inf and lim sup of $g(x)/x$, or dealing with more general equations. We refer to [9] and [5] for more details and a more complete bibliography.

In the present paper, we extend Fučik's result to the differential equation of Liénard type

$$x'' + f(x)x' + g(x) = e(t)$$

allowing suitable "jumping" behaviour not only on g but also on f. To this end we prove an existence theorem for a more general first order differential system in $\mathbb{R}^2$. Different results in this direction were obtained by E.N. Dancer in [4] who also studied the Rayleigh equation $x'' + F(x') + g(x) = e(t)$ and, more recently, by P. Drábek and S. Invernizzi in [6] who considered the Duffing equation $x'' + cx' + g(t,x) = e(t)$. Our main argument, which combines topological degree tools with phase plane analysis of homogeneous equations, provides a new proof of Fučik's theorem in the case $f \equiv 0$ as well.

For the significance of the periodic problem to the Liénard equation and for an exhaustive collection of the most classical results, see, e.g. [17],[18].

<u>Main results</u>. In this paper we deal with the existence of T-periodic solutions to the following differential system:

$$(1) \qquad x' = y - F(x) + E_1(t) \quad , \quad y' = - g(x) + E_2(t)$$

where $F, g : \mathbb{R} \to \mathbb{R}$ are <u>continuous</u> functions and E_1 , E_2 : $\mathbb{R} \to \mathbb{R}$ are continuous and T-periodic.

System (1) is usually considered in the study of some second order scalar differential equations like, for instance, Liénard or Rayleigh equations.

We assume that the limits

$$F_1 := \lim_{x \to -\infty} - F(x)/x \quad , \quad F_2 := \lim_{x \to +\infty} F(x)/x$$

$$g_1 := \lim_{x \to -\infty} g(x)/x \quad , \quad g_2 := \lim_{x \to +\infty} g(x)/x$$

exist and are finite.

Throughout the paper we also suppose that g_1 and g_2 are both positive. The case in which g_1 and g_2 are both negative was already examined in [16] where it was proved that existence of T-periodic solutions to (1) is achieved without any other assumption on F. The case in which g_1 and g_2 have opposite sign is contained in [7]. In this case, for any choice of F, there always exist forcing terms (E_1, E_2) such that (1) has no periodic solutions. Finally, if $g_i = 0$ for some $i = 1, 2$, further informations on the behaviour of the function g are needed in order to get results of existence of periodic solutions. Accordingly, this situation is left out in the present discussion.

Our main theorem is a generalization of some results obtained by Fučik et al. (see e.g. [9]) in the case $F \equiv 0$.

In order to prove our result we need to study the existence of non-trivial T-periodic solutions to the homogeneous system

$$(2) \qquad x' = y - F_1 x^- - F_2 x^+ \quad , \quad y' = g_1 x^- - g_2 x^+$$

where we have set $x^- := \max\{0,-x\}$, $x^+ := \max\{0,x\}$, $x = x^+ - x^-$.

Lemma 1. System (2) has nontrivial periodic solutions if and only if the following conditions hold:

$$(k1) \qquad A_i := g_i - (F_i/2)^2 > 0 \qquad (i = 1, 2)$$

$$(k2) \qquad F_1/\sqrt{g_1} = F_2/\sqrt{g_2} .$$

Moreover, system (2) has nontrivial solutions of period T if and only if $(k1)$, $(k2)$ and

$$(k3) \qquad 2\sqrt{A_1}\sqrt{A_2}/\omega(\sqrt{A_1} + \sqrt{A_2}) \in \mathbb{N}$$

hold, with $\omega := 2\pi/T$.

Proof. First we observe that a nontrivial periodic solution to system (2) exists if and only if the origin is a global center. This is due to the fact that the nonlinearities in the vector field of (2) are positively homogeneous functions. Accordingly, we can use some standard facts about the theory of plane dynamical systems (see [18], [11]).
Assume $(k1)$ and $(k2)$; then, Theorem 4.2 in [11] assures that (2) has a global center at the origin. If $(k2)$ holds but $(k1)$ does not, Corollary 4.1 in the same paper guarantees the nonexistence of nontrivial periodic orbits. Finally, let us consider the case when $(k2)$ does not hold. By Lemma 3.1 in [11], there is a homeomorphism between the orbits of (2) and those of

$$(3) \qquad u' = v - (F_1/\sqrt{g_1})u^- - (F_2/\sqrt{g_2})u^+ \quad , \quad v' = - u$$

and it is easily seen that if $F_1/\sqrt{g_1} \neq F_2/\sqrt{g_2}$, (3) does not have any nontrivial closed orbit.

The minimal positive period T^* of a nontrivial periodic orbit of (2) can be computed by examining separately the cases $x(t) \leq 0$ and $x(t) \geq 0$ (observe that all the nontrivial periodic solutions of (2) have the same minimal period).

More precisely, given a closed orbit which intersects the y-axis at the points $P_1 = (0,y_1)$ and $P_2 = (0,y_2)$ with $y_1 > 0 > y_2$, by simple calculations we find that the minimal time T_1 (resp. T_2) needed to reach P_1 (resp. P_2) from P_2 (resp. P_1) is $\pi/\sqrt{A_1}$ (resp. $\pi/\sqrt{A_2}$). Hence we have

$$T^* = T_1 + T_2 = \pi(\sqrt{A_1} + \sqrt{A_2})/\sqrt{A_1}\sqrt{A_2}$$

and $(k3)$ easily follows.

The set of all 4-tuples (F_1,F_2,g_1,g_2) such that $(k1)$, $(k2)$ and $(k3)$ hold true will be denoted by $K(T)$.

The proof of our main theorem is based on the following abstract existence result (cf. [10], [14] for the terminology).

<u>Lemma 2</u>. Let X, Z be real Banach spaces. We consider the operators:

$L : \mathrm{dom}(L) \subset X \to Z$ is a linear (not necessarily continuous) Fredholm mapping of index zero, with a continuous generalized inverse $K_{P,Q}$;

$T : X \to Z$ is an L-completely continuous operator such that

$(t1)$ $\quad \lim_{\|x\| \to \infty} \|Tx\|/\|x\| = 0;$

$S^* : [0,1] \times X \to Z$ is an L-completely continuous operator such that

$(s1)$ $\quad S^*(0,\cdot)$ is linear,

$(s2)$ $\quad S^*(\lambda,\mu x) = \mu S^*(\lambda,x)$ for every $\lambda \in [0,1]$, $x \in X$, $\mu > 0$,

$(s3)$ $\quad$ for each $\lambda \in [0,1]$, $Lx = S^*(\lambda,x)$ if and only if $x = 0$.

Set $S = S^*(1,\cdot)$. Then, for each $z \in Z$, the operator equation

(4) $\qquad Lx = Sx + Tx + z$

has at least one solution in $\mathrm{dom}(L)$.

<u>Proof</u>. It is sufficient to find an a-priori bound for the solutions of eq.

(4_λ) $$Lx = S^*(\lambda,x) + \lambda Tx + \lambda z \qquad (\lambda \in [0,1]).$$

In fact, if this is the case, the coincidence degree of the homotopy

$$H(\lambda,x) = Lx - S^*(\lambda,x) - \lambda Tx - \lambda z \qquad (\lambda \in [0,1])$$

is well defined and nonzero - by $(\delta 1)$, $(\delta 3)$ - on balls with sufficiently large radius (see [14, Prop. II.16, Th. IV.1]).

Suppose, by contradiction, that no a-priori bound exists. Then for each $n \in \mathbb{N}$, there are $\lambda_n \in [0,1]$ and $x_n \in \mathrm{dom}(L)$ such that $\|x_n\| \to \infty$ and x_n is a solution to (4_{λ_n}). Then, dividing (4_{λ_n}) by $\|x_n\|$, we have, by $(\delta 2)$,

(5_{λ_n}) $$L(x_n/\|x_n\|) = S^*(\lambda_n,x_n/\|x_n\|) + \lambda_n(Tx_n)/\|x_n\| + \lambda_n z/\|x_n\| := y_n .$$

Equation (5_{λ_n}) is equivalent, (cf. [14]), to

(6_{λ_n}) $$x_n/\|x_n\| = Px_n/\|x_n\| + JQy_n + K_{P,Q}y_n ,$$

where $P: X \to \ker(L)$, $Q: Z \to \mathrm{coker}(L)$ are continuous projectors, $J: \mathrm{Im}(Q) \to \ker(L)$ is a linear isomorphism and $K_{P,Q}$ is the (continuous) generalized inverse of L associated to the pair (P,Q). Assumption $(t1)$ and the L-complete continuity of the operators imply, by standard arguments, that the set $\{x_n/\|x_n\|,\ n \in \mathbb{N}\}$ is relatively compact in X. Therefore, passing if necessary to subsequences, we can assume $\lambda_n \to \lambda^* \in [0,1]$, $x_n/\|x_n\| \to x^* \in X$ and $\|x^*\| = 1$. Taking the limits in (5_{λ_n}) - (6_{λ_n}) as $n \to \infty$ and taking into account $(t1)$, we obtain $x^* = Px^* + JQS^*(\lambda^*,x^*) + K_{P,Q}S^*(\lambda^*,x^*)$, that is $x^* \in \mathrm{dom}(L)$ and $Lx^* = S^*(\lambda^*,x^*)$. By $(\delta 3)$, $x^* = 0$ and the contradiction is achieved.

<u>Theorem 1.</u> System (1) has at least one T-periodic solution provided that

$(h1)$ $$(F_1,F_2,g_1,g_2) \notin K(T)$$

holds.

<u>Proof.</u> In order to use Lemma 2 we set

$$X = Z = \{h \in C^0(\mathbb{R},\mathbb{R}^2): h \text{ is } T\text{-periodic}\},$$

equipped with the sup norm $\|h\| = |h|_\infty = \sup\{|h(t)|: 0 \le t \le T\}$, where $|\cdot|$ is the euclidean norm in $\mathbb{R}^2$.

$$\mathrm{dom}(L) = X \cap C^1(\mathbb{R}, \mathbb{R}^2).$$

$$L : (x(\cdot),y(\cdot)) \longmapsto (x'(\cdot),y'(\cdot)),$$

$$T : (x(\cdot),y(\cdot)) \longmapsto (F_1x(\cdot)^- + F_2x(\cdot)^+ - F(x(\cdot)),\ - g_1x(\cdot)^- + g_2x(\cdot)^+ - g(x(\cdot)))$$

$$S^*(\lambda,\cdot) : (x(\cdot),y(\cdot)) \longmapsto (y(\cdot) - F_1(\lambda)x(\cdot)^- - F_2(\lambda)x(\cdot)^+,\ g_1(\lambda)x(\cdot)^- - g_2(\lambda)x(\cdot)^+)$$

$$S^*(1,\cdot) = S : (x(\cdot),y(\cdot)) \longmapsto (y(\cdot) - F_1x(\cdot)^- - F_2x(\cdot)^+,\ g_1x(\cdot)^- - g_2x(\cdot)^+)$$

$$z := (E_1(\cdot),E_2(\cdot)),$$

where

$$g_1(\lambda) = \begin{cases} 2\lambda g_1 + (1 - 2\lambda)a & \text{if } 0 \le \lambda \le 1/2 \\ g_1 & \text{if } 1/2 < \lambda \le 1 \end{cases}$$

$$g_2(\lambda) = \begin{cases} 2\lambda g_2 + (1 - 2\lambda)a & \text{if } 0 \le \lambda \le 1/2 \\ g_2 & \text{if } 1/2 < \lambda \le 1 \end{cases}$$

$$F_1(\lambda) = \begin{cases} vc & \text{if } 0 \le \lambda \le 1/2 \\ (2\lambda - 1)F_1 + 2(1 - \lambda)vc & \text{if } 1/2 < \lambda \le 1 \end{cases}$$

$$F_2(\lambda) = \begin{cases} -vc & \text{if } 0 \le \lambda \le 1/2 \\ (2\lambda - 1)F_2 - 2(1 - \lambda)vc & \text{if } 1/2 < \lambda \le 1 \end{cases}$$

with $a > 0$, $c > 0$ and $v \in \{-1, 1\}$ to be chosen in the following way:

$$v = \begin{cases} 1 & \text{if } F_1/\sqrt{g_1} \ge F_2/\sqrt{g_2} \\ -1 & \text{otherwise .} \end{cases}$$

It is clear that $w(t) = (x(t),y(t))$ is a T-periodic solution of system (1) if and only if $w \in \mathrm{dom}(L)$ fulfils the operator equation $Lw = Sw + Tw + z$.

It can be easily seen that, by the choice of a, c and v,

$$F_1(\lambda)/\sqrt{g_1(\lambda)} \ne F_2(\lambda)/\sqrt{g_2(\lambda)},$$

for every $\lambda \in [0,1[$, so that, by definition of $K(T)$ and using hypothesis $(h1)$, we have that, for each $\lambda \in [0,1]$, $(F_1(\lambda),F_2(\lambda),g_1(\lambda),g_2(\lambda)) \notin K(T)$. Hence, by Lemma 1, $(\delta 3)$ of Lemma 2 holds.

All the other assumptions of Lemma 2 are easily verified and the result follows.

<u>Remark 1.</u> Theorem 1 generalizes a previous result of S. Fučík in [8], since in the case $F_1 = F_2 = 0$ we have $A_i = g_i$. However the proof is carried

out in a different way even in this case.

Corollary 1. Consider the equation of Liénard type

(7) $$x'' + f(x)x' + g(x) = e(t)$$

with $f, g, e : \mathbb{R} \to \mathbb{R}$ continuous and $e(\cdot)$, T-periodic.

Suppose the limits

$$f_1 := \lim_{x \to -\infty} -f(x) \quad , \quad f_2 := \lim_{x \to +\infty} f(x)$$

$$g_1 := \lim_{x \to -\infty} g(x)/x \quad , \quad g_2 := \lim_{x \to +\infty} g(x)/x$$

exist and are finite, with g_1 and g_2 positive.
Then equation (7) has a T-periodic solution if $(f_1, f_2, g_1, g_2) \notin K(T)$.

Proof. For the proof it is sufficient to observe that equation (7) is equivalent to the system

$$x' = y - F(x) + \tilde{E}(t) \quad , \quad y' = -g(x) + \bar{e} ,$$

where $F(x) = \int_0^x f(u)du$, $\bar{e} = (1/T)\int_0^T e(s)ds$, $\tilde{E}(t) = \int_0^t (e(s) - \bar{e})ds$, and then apply Theorem 1 observing that $F_i = f_i$ $(i = 1, 2)$.

As already remarked, the case in which g_1 and g_2 are both negative is contained in [16] where the existence of T-periodic solutions to (7) is proved for f arbitrary.

Remark 2. If, in the situation of Corollary 1, we have $f_1 \cdot f_2 < 0$, with g_1 and g_2 positive, then the existence of a T-periodic solution to (7) is ensured, since $(k2)$ fails and so, $(f_1, f_2, g_1, g_2) \notin K(T)$. This fact, which could also be obtained as a consequence of more classical results (like, e.g. [15]) was also covered by [6, Th. 3.2] in the case of the Duffing equation, that is, with $f_2 = - f_1 = c \neq 0$ (see also [1]).

Remark 3. Theorem 1 also applies to equations of the form

(8) $$x'' + b|x'| + ax = e(t) = e(t+T) \qquad (a > 0).$$

We have: (8) has a T - periodic solution whenever

$$a - (b^2/4) \neq k^2\omega^2 \qquad (k = 0, 1, \dots).$$

This result shows that our theorem is not contained in those by E.N. Dancer in [4].

Remark 4. The regularity assumptions on the functions E_1 and E_2 could be weakened by standard arguments provided that solutions were considered in a suitable generalized sense. In particular, Corollary 1 can be extended to

$$x'' + f(x)x' + g(x) = e(t), \qquad e \in L^1(0,T; \mathbb{R}),$$

with boundary conditions

$$x(T) - x(0) = x'(T) - x'(0) = 0.$$

In this situation, however, it is convenient to produce the proof directly on equation (7) and use the homotopy on system (1) only for getting the needed a-priori bounds.

In the case of 2 - points BVP we are able to obtain similar results, which will be discussed elsewhere.

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