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NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES

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Abstract: The existence of a weak solution of a nonlinear parabolic variational inequality (with quadratic growth in the spatial gradient) is studied using a Hölder continuity result: a Meyers estimate and a local uniqueness result are also obtained in the case of continuous weak solutions.

Key words: Nonlinear variational inequalities, nonlinear parabolic equations and systems.

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§ 1. Notations

Ω is a bounded open set in R^N with smooth boundary $\partial\Omega = \Gamma$, $N \geq 3$.

$Q = (0,T) \times \Omega$

$B(R;x_0) = B_R(x_0) = \{x \in \Omega \mid |x-x_0| < R\}$

$Q(R;z_0) = Q_R(z_0) = \{(t,x) \in Q \mid |x-x_0| < R,\ |t-t_0| < R^2\}\ z_0 = (t_0,x_0)$

$Q^-(R;z_0) = Q_R^-(z_0) = \{(t,x) \in Q \mid |x-x_0| < R,\ t_0-R^2 < t < t_0\}$

$Q_\theta^-(R;z_0) = \{(t,x) \in Q \mid |x-x_0| < R,\ t_0-R^2 < t < t_0 - 6\theta R^2\}$, $\theta \in (0,1)$

$\Psi : Q \longrightarrow R \cup \{-\infty\}$ is a Borel function everywhere defined in Q

Let now ε be a positive real number

$E(\varepsilon, z_0, \Psi, r) = \{z = (t,x) \in Q_\theta^-(r,z_0),\ \Psi(t,x) \geq$

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$\geq \operatorname{Sup}_{(t,t_o+r^2/4)\times B(r/2;x_o)} \Psi - \varepsilon \}$

$\Delta_\theta(\varepsilon, z_o, \Psi, r) = \Delta_\theta(\varepsilon, r) = \operatorname{cap}_{Q(2r;z_o)} E(\varepsilon, z_o, \Psi, r)$,
where the definition of the capacity used in the paper is given in § 2.

$\delta_\theta(\varepsilon, z_o, \Psi, r) = \delta_\theta(\varepsilon, r) = \Delta_\theta(\varepsilon, r)\,\sigma_N^{-1}\, r^{-N}$, where σ_N is the capacity of the parabolic cylinder with $r=1$ in R^{N+1}.

For the Sobolev spaces on Ω or Q we assume the usual notations

Let $a_{ik}(t,x)$ be bounded measurable functions on Q, $i,j=1,2,\dots,N$, such that

$$\sum_{i,j=1}^{N} a_{ij}(t,x)\,\xi_i\,\xi_j \geq \nu\,|\xi|^2 \qquad \nu>0$$

$A: L^2(0,T;H_0^1(\Omega)) \longrightarrow L^2(0,T;H^{-1}(\Omega))$ is the operator defined by

$$\langle Au,v\rangle = \int_Q \sum_{i,j=1}^{N} a_{ij}(t,x) D_{x_j} u\, D_{x_i} v \,dxdt$$

G^z is the Green function relative to A (or its extension by $-\Delta$ to $L^2(0,T;H^1(R^N))$ in the case of boundary points z) with singularity in z

G^z_ρ is the regularized Green function defined by the problem

$$-\int v_t\, G^z_\rho \,dxdt + \langle AG^z_\rho, v\rangle = \fint_{Q(\rho;z)} v\,dxdt$$

$$G^z_\rho \in L^2(0,T;H^1(R^N)),\ G^z_\rho(t-2,\cdot) = 0 \qquad v\in D(R^{N+1})$$

where we indicate again by A its extension to R^{N+1} by $-\Delta$ and $\fint_{Q(\rho;z)} v\,dxdt$ denotes the average of v on $Q(\rho;z)$.

§ 2. Introduction and results. Recently some attention has been paid to the parabolic variational inequalities with a non-

linear term, which is quadratic in the spatial gradient, in connection with some problems of optimal stochastic control [2].
In the present paper we will study for these variational inequalities the existence, the uniqueness (global or local) and the regularity of a solution. In the case of equations, a general result of existence of a solution has been obtained by L. Boccardo, F. Murat [8], for variational inequalities some partial results, depending essentially on a Hölder continuity result for bounded solutions, has been given by M. Biroli [4], J. Naumann and M.A. Vivaldi have solved the problem of the quasi-variational inequality of the stochastic impulse control.
For nonlinear elliptic variational inequalities with irregular obstacles a general result on the Hölder continuity of the solutions has been proved by J. Frehse, U. Mosco [11], and U. Mosco [19],[20]; using the methods of these papers, M. Struwe, M. A. Vivaldi prove the Hölder continuity of a bounded solution of a nonlinear parabolic obstacle problem with an obstacle, which is Hölder continuous in time and one sided Hölder continuous in space variables, and M. Biroli, U. Mosco prove a general result in the linear case [22],[7].
Here, using some tricks, given in [22], we extend the result of [7] to the nonlinear case and we use this new result to prove the existence of a solution of our variational inequality.
The uniqueness of the solution in the linear case and the local uniqueness in the nonlinear case are investigated in § 4, giving a result extending the one proved for elliptic equations in [14].
Further regularity of the solution is studied in § 5 proving the dual inequalities for our problem, this result appears in

[16] and the proof given here is the same as in [16].
We state now the results precisely.
Let E be a compact set, $E \subset P$, where $P=(t_1,t_2)\times B$ and

$$\mathrm{cap}_P(E) = \mathrm{Inf}\,\{ \int_P |D_x w|^2 ;\ w\in D(P)\quad w=1 \text{ in a neighbourhood of } E\}$$

We have so defined a Choquet capacity [9], and we can prove that if a set E is capacitable, then

$$\mathrm{cap}_P\,(E) = \int_{t_1}^{t_2} \mathrm{cap}_N\,(E_t)\,dt,$$

where cap_N is the usual Newtonian capacity and E_t is the section at time t of E.

Let $H(t,x,u,p)$ be a function measurable in $(t,x)\in Q$ and continuous for $(u,p)\in R\times R^N$ such that

$$(2.1)\qquad |H(t,x,u,p)| \le K_1 + K_2\,|p|^2$$

$\forall (t,x)\in Q$, $|u|\le C$, $p\in R^N$, where K_1, K_2 depend on C.

A function $u\in L^2(0,T;H^1(\Omega))\cap L^\infty(\Omega)$ is a <u>local</u> solution of the parabolic obstacle problem relative to A,H,Ψ if

(a) $u\ge \Psi$ q.e. in Q for the above defined capacity

(b) $$\int_0^t \int_\Omega \{ v_t\,\varphi(v-u) + \sum_{i,j=1}^{N} a_{ij}(t,x)\, D_{x_j}u\, D_{x_i}(\varphi(v-u)) +$$
$$+ H(\cdot,\cdot,u,D_x u)\,\varphi(v-u) + 1/2\,\varphi_t(v-u)^2\}\,dxdt \ge$$
$$\ge 1/2\,\|\varphi^{1/2}(v-u)\|^2_{L^2(\Omega)}(t),$$

$\forall v\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1_0(\Omega))\cap L^\infty(Q)$, $\quad v\ge\Psi$
and where $\varphi\in D(\bar{Q})$ with $\varphi=0$ in $(0,T)\times\partial\Omega$ and $\varphi(0,\cdot)=0$

(c) for every constant $d\ge\Psi$ in $\mathrm{supp}(\varphi)\cap(0,t)\times\Omega$

$$1/2\,\|\varphi^{1/2}(u-d)^+\|^2_{L^2(\Omega)}(t) \le C\int_0^t [D_x u\, D_x\varphi u +$$
$$+ |D_x u|^2\varphi + \varphi_t(u-d)^2]\,dxdt$$

A function u is a solution of the parabolic obstacle problem relative to A,H, Ψ if (a),(b) and (c) hold for $\varphi \in D(\bar{Q})$, while we consider a null initial value. The <u>Wiener modulus</u> of Ψ is defined by

$$\omega_\theta(r,R) = \operatorname{Inf}\{\omega \geq 0;\ \int_r^R \delta_\theta(\omega,\rho)\, d\rho/\rho \geq 1\}$$

We prove the following result:

<u>Theorem 1.</u> <u>Let</u> u <u>be a local solution of our problem and</u> $z_0 \in Q$; <u>there exists</u> Θ_0 <u>such that for</u> $\theta \in (0, \theta_0)$ <u>we have</u>

$$\operatorname{osc}_{Q(r;z_0)} u \leq K\{M(R)\,\omega_\theta(r,R)^{\beta_\theta} + \omega_\theta(r,R) \wedge \operatorname{osc}_{Q(R;z_0)} \Psi\},$$

<u>where</u> $0 \leq r < \theta^{1/2}R < R < \theta^{1/2}R_0$ (R_0 suitable) <u>and</u>

$$M(r) = \left(\int_{Q^-(r,z_0)} |D_x u|^2\, dxdt\right)^{1/2} + \operatorname{osc}_{Q(r;z_0)} u.$$

<u>Moreover if there exists</u> $\bar{u} \in H^{1,p}(Q)$, $p > N+1$, $\bar{u} = 0$ <u>in</u> $(0,T) \times \partial\Omega \cup (0) \times \Omega$ $\bar{u} \geq \Psi$ q.e. in Q <u>and</u> $z_0 \in (0,T) \times \partial\Omega \cup (0) \times \Omega$ <u>and</u> u <u>is a solution,</u>

$$\operatorname{osc}_{Q(r;z_0)} u \leq K R^\beta \quad \beta \in (0,1),\ r \leq R_0,\ R_0 \text{ \underline{suitable.}}$$

<u>Corollary 1.</u> <u>Let</u> u <u>be a local solution of our problem and let the assumptions of Th. 1 hold, then</u>

$$\operatorname{osc}_{Q(r;z_0)} u \leq K(R^{\gamma_\theta} + \operatorname{osc}_{Q(R;z_0)} \Psi)\,\omega(r,R)^{\beta_\theta} +$$
$$+\ \omega_\theta(r,R) \wedge \operatorname{osc}_{Q(R;z_0)} \Psi$$

A point $z_0 \in Q$ such that there exists a $\theta \in (0,1)$ with

$$\lim_{r \to 0} \omega_\theta(r,R) = 0 \qquad R \leq R_0$$

is a <u>Wiener point</u>, if

$$\omega_\theta(r,R) \leq K(r/R)^\alpha \quad \alpha \in (0,1) \qquad R \leq R_0$$

z_0 is a <u>Hölder Wiener point</u>.

Corollary 2. Let u be a local solution of our problem; if z_0 is a (Hölder) Wiener point, then u is (Hölder) continuous at z_0.

Corollary 3. Let u be a local solution of our problem; if u is one sided (Hölder) continuous at z_0, then u is (Hölder) continuous at z_0.

Remark 1. The result of Th. 1 at time t=0 holds also if we have as initial data $u_0 \in H^{1,q}(\Omega)$, $q > N$.

We consider now the problem of the existence of a solution to our problem. We suppose

(a) every point $z_0 \in Q$ is a Wiener point or Ψ is one sided continuous at z_0

(b) there exists a function $\bar{u}$ as in Th. 1 and

$$H(t,x,u,p)(u-\bar{u}) \geq -c\,|p|^2 - K(|u|^2 + 1) \qquad c < \nu .$$

Theorem 2. Suppose that Ψ is quasi continuous on the set $Y = \{\Psi > -\infty\}$ and that there is a measure m on Y "weaker" than the capacity. Let Ψ be bounded from above and (a) and (b) hold, then there exists a continuous solution of our problem.

Remark 2. The result of Th. 2 can be extended to the case of general initial data and Ψ quasi l.s.c. on Y, if in (b) $v_0(0) = u_0$.

Theorem 2'. Let $u \in C(Q)$ be a local solution, $D_t \Psi \in L^q(0,T;H^{-1,q}(\Omega))$, $D_x \Psi \in L^q(Q)$, $q > 2$, then $D_x u \in L^p(Q)$, $p > 2$.

For the problem of the uniqueness or of the local uniqueness of the solution of our problem we obtain the following result:

Theorem 3. In the linear case ($H = f(t,x)$) the solution $u \in C(\bar{Q})$ (if there exists) of our variational inequality is unique.

Let H *be differentiable in* (u,p) *and such that*

$$|H_{p_i}(t,x,u,p)| \le K\,(1+|p|) \qquad (t,x)\in Q,\quad |u|\le C.$$
$$|H_u(t,x,u,p)| \le K\,(1+|p|^2).$$

Consider two local solutions $u_1,\ u_2 \in C(Q)$ *of our variational inequality and suppose*

$u_1 = u_2$ *in* $(t_0-R^2, t_0+R^2)\times \partial B(R;x_0) \cup \{t_0-R^2\}\times B(R;x_0) \subset Q$

then, if $R<R_0$, R_0 *suitable*, $u_1=u_2$ *in* $Q(R;z_0)$ $(z_0=(t_0,x_0))$.

Remark 3. The result of Th.3 holds also in the case of general initial data. Consider now the following two conditions:

(c) $\Psi \in H^{1,\infty}(Q)$ and there exists $v_0 \in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)\cap$ $\cap\, H^1(0,T;H^{-1}(\Omega))$ with $v_0 \ge \Psi$ q.e. in Q.

(d) $\Psi_t + A\Psi + H(\cdot,\cdot,\Psi,D_x\Psi) \le k$, $k>0$, in the sense of measures.

Theorem 4. Let the assumptions (c) *and* (d) *hold; then, if* u *is a solution of our variational inequality, we have*

$$0 \le u_t + Au + H(\cdot,\cdot,u,D_xu) \le (\Psi_t + A\Psi + H(\cdot,\cdot,\Psi,D_x\Psi)) \quad \forall\, 0\le k$$

in the sense of distributions on Q, *hence, if* $a_{ij}\in H^{1,\infty}(Q)$, u *belongs to* $H^{2,1,q}(Q)$, $1<q<+\infty$.

Remark 4. The result of Th. 4 holds also for general initial data, of course for the last part of the result a regularity assumption on the initial data is necessary.

§ 3. *Sketch of the proof of Theorem 1.* The main tool in the proof of Th. 1 is a Poincaré's type inequality involving only the spatial gradient, which is given for local solutions of our variational inequalities.

Lemma 1. There exists a constant $\hat{d}$ *such that*

$$\hat{d} \ge \Psi(t,x) - \varepsilon \quad \text{in } (t_0-6\theta R^2, t_0+R^2)\times B(3/8\,R;x_0)$$

and

$$\int_{t_0-6\theta R^2}^{t_0-\theta R^2} \int_{B(3/8 R; x_0)} (u-\hat{d})^2 \, dxdt \leq$$

$$\leq C(R^2 \sigma(\varepsilon, R)^{-1} \int_{t_0-6\theta R^2}^{t_0-\theta R^2} \int_{B(R;x_0)} |D_x u|^2 \, dxdt + \varepsilon^2).$$

We observe at first that we consider here bounded solutions of our variational inequality.

We consider at first the case of interior points.

Let $\bar{z} = (\bar{x}, \bar{t}) \in Q(R/4; z_0)$ and consider $\eta = \eta(x)$ such that

$$\eta \in D(R^N), \quad \eta = 1 \text{ in } B(R/8; \bar{x}), \quad \eta = 0 \text{ for } x \notin B(R/4; \bar{x})$$

$$0 \leq \eta \leq 1 \quad \text{in } B(R/4; \bar{x})$$

$$|D_x \eta| \leq CR^{-1}$$

and $\tau = \tau(t)$ such that

$$\tau \in D(R), \quad \tau = 1 \text{ for } t \geq \bar{t} - 3\theta R^2, \quad \tau = 0 \text{ for } t \leq \bar{t} - 5\theta R^2,$$

$$0 \leq \tau \leq 1 \quad \text{in } (\bar{t} - 5\theta R^2, \bar{t} - 3\theta R^2),$$

$$|D_t \tau| \leq C(\theta R^2)^{-1}.$$

Choosing in the variational inequality $v = d$, where $d \geq \Psi$ in $(\bar{t} - 6\theta R^2, \bar{t}) \times B(R/4; \bar{x})$ and $\varphi = \tau^2 \eta^2 G_\rho^{\bar{z}} \operatorname{sink} ((u-d)^2)_\varepsilon$,

$$(((u-d)^2)_\varepsilon)_t + ((u-d)^2)_\varepsilon = (u-d)^2,$$

$$((u-d)^2)_\varepsilon (\bar{t} - R^2) = (u-d)^2 (\bar{t} - R^2),$$

we obtain, after some computations,

$$\int_{\bar{t}-\theta R^2}^{\bar{t}} \int_{B(\theta^{1/2} R; \bar{x})} |D_x u|^2 G^{\bar{z}} \, dxdt + |u-d|^2(\bar{z}) \leq$$

$$\leq C_1 \exp(-C_2 \theta^{-1}) \; \theta^{-3N/4} \sup_{(\bar{t}-3\theta R^2, \bar{t}) \times B(R/4; \bar{x})} |u-d|^2 +$$

$$+ C_3 \theta^{-(1+3N/4)} R^{-(N+2)} \int_{\bar{t}-5\theta R^2}^{\bar{t}-3\theta R^2} \int_{B(R/4; \bar{x})} |u-d|^2 \, dxdt .$$

Taking now the supremum for $\bar{z} \in Q(\theta^{1/2} R; z_0)$ we obtain:

<u>Lemma 2</u>. <u>Let</u> $d \geq \Psi$ <u>in</u> $(t_0 - 6\theta R^2, t_0 + R^2/4) \times B(R/2; x_0)$,

$\theta \in (0,1/64)$ <u>the following relation holds</u>

$$\int_{t_0-\theta R^2}^{t_0} \int_{B(\theta^{1/2}R;x_0)} G^{z_0} |D_x u|^2 \, dxdt + \operatorname{Sup}_{Q(\theta^{1/2}R;z_0)} |u-d|^2 \leq$$

$$\leq K_1 \exp(-K_2\theta^{-1}) \ \theta^{-3N/4} \operatorname{Sup}_{Q(R;z_0)} |u-d|^2 +$$

$$+ K_3 \theta^{-(1+3N/4)} R^{-(N+2)} \int_{t_0-6\theta R^2}^{t_0-2\theta R^2} \int_{B(3/8\,R;x_0)} |u-d|^2 \, dxdt \, .$$

Choosing now $d = \hat{d} + \varepsilon$, $\hat{d}$ as in the lemma 1, using the lemma 1 and taking into account the estimates on the Green function [1], we obtain the following relation

$$\int_{Q^-(\theta^{1/2}R;z_0)} |D_x u|^2 \, G^{z_0} \, dxdt + (\operatorname{osc}_{Q(\theta^{1/2}R;z_0)} u)^2$$

$$K_4 K_5(\theta) \ (\operatorname{osc}_{Q(R;z_0)} u)^2 + (K_6(\theta)\,\sigma(\varepsilon,R))^{-1} \, .$$

$$\int_{t_0-R^2}^{t_0-\theta R^2} \int_{B(R;x_0)} |D_x u|^2 G^{z_0} \, dxdt + K_7(\theta)\,\varepsilon^2,$$

where

$$K_5(\theta) = \exp(-K_2\theta^{-1}) \ \theta^{-3N/4},$$

$$K_6(\theta) = K_8 \exp(-K_9\theta^{-1}) \ \theta^{-(1+N/4)},$$

$$K_7(\theta) = K_{10} \ \theta^{-3N/4},$$

Taking into account that $K_5(\theta)$, $K_6(\theta) \longrightarrow 0$ as $\theta \longrightarrow 0$ we obtain the following relation

$$\int_{Q^-(\theta^{1/2};x_0)} |D_x u|^2 \, G^{z_0} \, dxdt + (\operatorname{osc}_{Q(\theta^{1/2};z_0)} u)^2$$

$$(1 + K_{11}(\theta)\,\sigma(\varepsilon,R))^{-1} \ (\int_{Q^-(R;x_0)} |D_x u|^2 \, G^{z_0} \, dxdt +$$

$$(\operatorname{osc}_{Q(R;z_0)} u)^2) + K_{12}(\theta)\,\varepsilon^2,$$

where $\theta \leq \theta_0$, θ_0 suitable.

Using the same methods in the elliptic case [11], we obtain

$$\operatorname{osc}_{Q(r;z_0)} u \leq K(M(R) \exp(-\beta \int_r^R \sigma(\varepsilon,s)\, s^{-1} ds) + \varepsilon)$$

where $r \leq \theta^{1/2}R$, K depends on θ and

$$M(R) = \left(\int_{Q_1^-(R;z_0)} |D_x u|^2 \, dxdt\right)^{1/2} + \operatorname{osc}_{Q(R;z_0)} u.$$

Choosing now $\varepsilon = \omega(r,R)$, we have the result of Th. 1.
The result of Coroll. 1 follows by an iteration method taking into account the result of Lemma 2 [19],[20].
Coroll. 2.3 are easy consequences of Coroll. 1.
The proof of the Hölder continuity at boundary points can be given by the same methods if we replace in the test function d by $\bar{u}$.

§ 4. Existence result. The proof of the existence result is divided into several steps.

(1) We consider at first the linear problem and we prove the existence of a solution by penalization using the same methods as in [17].

We observe that in this case the sequence of the solutions of the penalized problems converges in $L^2(Q)$; then only one solution is characterized as limit of the sequence of the solutions of the penalized problems.

In the following we consider always such a solution in the linear case. Consider now two solutions of the linear problem; using the penalization we have easily

$$1/2 \, \|u_1(t)-u_2(t)\|^2_{L^2(\Omega)} + \int_0^t \|D_x(u_1(s)-u_2(s))\|^2_{L^2(\Omega)} ds \leq$$

$$\leq 1/2 \, \|u_1(0)-u_2(0)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega (f_1-f_2)(u_1-u_2) \, dxd\tau$$

(2) Consider now the case in which the nonlinear term H(t,x,u,p) is bounded; in such a case the existence of a solution is proved by Schauder's fixed point theorem, using the Hölder continuity result proved in § 3.

(3) In the general case we denote

$$H_n(t,x,u,p) = H(t,x,u,p)\ (1 + n^{-1}H(t,x,u,p))^{-1}$$

We observe that $H_n(t,x,u,p)$ is bounded and we indicate by u_n the solution given in (2).

We prove as in [15] that the sequence u_n is uniformly bounded; then, from the result on the Hölder continuity of the solutions proved in § 3, the sequence u_n is also bounded in $C^{\alpha}(\bar{Q})$, $\alpha \in$ $\in (0,1)$.

From the above we can suppose that u_n converges to u in $C(\bar{Q})$.

We have

$$1/2\ \|u_n(t)-u_m(t)\|_{L^2(\Omega)} + \int_0^t \int_\Omega |D_x(u_n-u_m)|^2\ dxds \leq$$

$$\leq \int_0^t \int_\Omega (H_n(s,x,u_n,D_xu_n)-H_m(s,x,u_m,D_xu_m))(u_n-u_m)\ dxds.$$

We observe that the sequence $H_n(\cdot,\cdot,u_n,D_xu_n)$ is bounded in $L^1(Q)$ (u_n being bounded in $L^2(0,T;H^1_0(\Omega))$) then u_n converges in $L^2(0,T;H^1_0(\Omega))$ to u. Consider now the sequence $H_n(\cdot,\cdot,u_n,D_xu_n)$; this sequence is equi-integrable and converges pointwise to $H(\cdot,\cdot,u,D_xu)$. Then it converges in $L^1(Q)$ to $H(\cdot,\cdot,u,D_xu)$.

Summing up, we have

u_n converges to u in $C(\bar{Q})$ and in $L^2(0,T;H^1_0(\Omega))$

$H_n(\cdot,\cdot,u_n,D_xu_n)$ converges to $H(\cdot,\cdot,u,Du)$ in $L^1(Q)$.

Then we can easily prove that u is a solution of our variational inequality.

§ 5. A Meyers type result (Th. 2'). The proof can be obtained by standard methods ([12] for the elliptic case, [21] for parabolic case with small nonlinearities) using the variational inequality with $v=u_R$ (u_R is the average of u in the parabolic cylinder Q_R) and φ as a cut off function relative to Q_R.

§ 6. Uniqueness and local uniqueness results. Consider at first the linear case (H=f does not depend on u, p). We observe that if u is a solution of our variational inequality

$$u_t + Au - f \in M^+(Q)$$

($M^+(Q)$ is the space of positive measures on Q), then we have

$$(6.1) \qquad \langle u_t + Au, \varphi(v-u)\rangle_{M(Q),C^o(Q)} \geq \int_Q f\ (v-u)\ dxdt$$

($C^o(Q)$ is the space of the functions in C(Q) with compact support in Q) where $\varphi \in C^o(Q)$ and $v \in C(Q)$, $v \leq \Psi$.

Now let u_1 and u_2 be solutions of our variational inequality in $C(\bar{Q})$ and denote $w=u_1-u_2$.

Let $\varphi_n \in C^o(Q)$ such that

$$\varphi_n = 1 \text{ in } Q_{2n}\ (Q_n = \{z \in Q,\ \text{dist}(z, \partial Q) > n^{-1}\}),$$

$$\varphi_n = 0 \text{ in } Q-Q_n,$$

$$|D_x \varphi_n|,\ |D_t \varphi_n| \leq K_1\ n^{-1}.$$

Using (6.1) with $v = 2^{-1}(u_1+u_2)$, $\varphi = \varphi_n$ and passing to the limit as $n \to +\infty$, we have

$$1/2\ \|u_1(T)-u_2(T)\|^2_{L^2(\Omega)} - 1/2\ \|u_1(0)-u_2(0)\|^2_{L^2(\Omega)} + \\ + \int_Q |D_x w|^2\ dxdt \leq 0,$$

from where $u_1 = u_2$.

We consider now the nonlinear case; our aim is to prove a result on local uniqueness analogous to the one given in [14] for elliptic equations. Let u_1 and u_2 be solutions of our variational inequality, which are continuous in $\bar{Q}$ and $w=u_1-u_2$.

It is easily seen from the variational inequality that

$$(u_i)_t + Au_i \in M(Q) \qquad i=1,2.$$

Let $Q_R(z_o)$ be a parabolic cylinder such that $w=0$ in $(t_o-R^2, t_o+R^2)\times \times B_R(x_o)$ and in $\{t_o-R^2\}\times B_R(x_o)$.

Denote by $o(R)$ the supremum between the oscillations of u_1 and u_2 in $Q_R(z_o)$ and by i_R the characteristic function of $Q_R(z_o)$; by the same methods used in [14] p. 234 for the elliptic equation we have

$$(6.2) \quad \int_{Q_R(z_o)} |D_x w|^2 \, dxdt \leq K_1 \int_{Q_R(z_o)} (1+ |D_x u_1|^2) + |D_x u_2|^2) \, w^2 \, dxdt$$

Using now the same methods of the lemma 1.3 [14] p. 231 we obtain

$$(6.3) \qquad \int_{Q_R(z_o)} (1+ |D_x u_i|)^2 \, w^2 \, dxdt \leq K_2 o(R) \int_{Q_R(z_o)} |D_x w|^2 \, dxdt + \\ + \langle w_t, (u_i-u_i(z_o)-o(R))^2 \, w \rangle$$

where the duality in the last term is between $M_b(Q_R(z_o)) + L^2(t_o-R^2, t_o+R^2; H^{-1}(B_R(x_o)))$ and $C(\overline{Q_R(z_o)}) \cup L^2(t_o-R^2, t_o+R^2; H_o^1(B_R(x_o)))$.

Consider the last term in (6.3), using in the variational inequality relative to u_1 the test function $((1-(u_i-u_i(z_o)-o(R)))^2 u_1 + +(u_i-u_i(z_o)-o(R))^2 u_2) i_R + (1-i_R)u_1$ and in the variational inequality relative to u_2 test function $((1-(u_i-u_i(z_o)-o(R))^2)u_2 + + (u_i-u_i(z_o)-o(R))^2 u_1) i_R + (1-i_R)u_2$ we obtain

$$(6.4) \qquad \langle w_t, (u_i-u_i(z_o)-o(R))w \rangle \leq 1/2 \int_{Q_R(z_o)} (1+ |D_x u_i|^2) w^2 \, dxdt + \\ + K_3 o(R) \int_{Q_R(z_o)} |D_x w|^2 \, dxdt .$$

Then from (6.3),(6.4) we have

$$(6.5) \quad \int_{Q_R(z_o)} w^2 (1+ |D_x u_i|^2) \, dxdt \leq K_4 o(R) \int_{Q_R(z_o)} |D_x w|^2 \, dxdt$$

From (6.2),(6.5) we have

(6.6) $\int_{Q_R(z_0)} |D_x w|^2 \, dxdt \le K_5 o(R) \int_{Q_R(z_0)} |D_x w|^2 \, dxdt$.

We recall that u_1 and u_2 are supposed to be continuous; then there exists R_0 such that for $R \le R_0$ we have $o(R) < K_5$ and in such a case we have from (6.6) $w=0$.

§ 7. Dual inequalities. The proof of the dual inequalities uses a method which is an adaptation of the one used for the elliptic case in [10] (regularization of the nonlinear term H). Let $H_m(t,x,u,p)$ be such that

(7.1) $H_m(t,x,u,p) \xrightarrow{m\to+\infty} H(t,x,u,p)$

a.e. in (t,x), $\forall r \in R$, $\forall p \in R^N$,

(7.2) $|H_m(t,x,u,p)| \le c_m \le K_1 + K_2 |p|^2$

a.e. in (t,x), $|u| \le C$, $\forall p \in R^N$,

(7.3) $|H_m(t,x,u,p) - H_m(x,t,u',p')| \le K_m |u-u'| + K_m |p-p'|$

a.e. in (t,x), $|u|$, $|u'| \le C$, $p,p' \in R^N$.

We observe that u is also a solution of the variational inequality

(7.4) $\langle v_t, v-u\rangle + a_m(u,v-u) - 1/2 \, \|v(0)\|^2_{L^2(\Omega)} \ge \langle f_m, v-u\rangle$

$$\forall v \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \cap L^\infty(\Omega), \; v \ge \Psi$$

$$u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega), \; u \ge \Psi ,$$

and the solution of the variational inequality (7.4) is unique [3],[13],[10], ($a_m(u,v) = \langle Au,v\rangle + \int_Q (H_m(\cdot,\cdot,u,D_x u) + \lambda_m u) v \, dxdt$), where λ_m is large enough for the strict monotonicity of a_m), $f_m = H_m(\cdot,\cdot,\varkappa,D_x u) - H(\cdot,\cdot,u,D_x u) - \lambda_m u$.

Let now

$$T_m = \Psi_t + A\Psi + H_m(\cdot,\cdot,\Psi,D_x\Psi).$$

We consider the auxiliary variational inequality

(7.5) $\langle v_t, v-z\rangle + a_m(z,v-z) - 1/2\,\|v(0)\|^2_{L^2(\Omega)} \geq$

$$\geq \langle f_m \vee T_m,\ v-z\rangle$$

$$\forall v \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \cap L^\infty(\Omega),$$

$$u \geq v \geq u-1$$

$$z \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(\Omega),\ u \geq z \geq u-1 .$$

By the methods of [3],[13] we can prove that (7.5) has a unique solution.

Using the penalized problems and a regularization of f_m and $f_m \wedge T_m$, we can prove (by methods substantially analogous to the one used in [10] for the elliptic case) that

$$u \leq z.$$

Then we have $u=z$.

From our variational inequality we have

(7.6) $$u_t + Au + H(\cdot,\cdot,u,D_x u) \geq 0 .$$

From variational inequality (7.5) we have

(7.7) $u_t + Au + H_m(\cdot,\cdot,u,D_x u) + \lambda_m u \leq$

$$\leq (H_m(\cdot,\cdot,u,D_x u) - H(\cdot,\cdot,u,D_x u) + \lambda_m u) \vee$$

$$(\Psi_t + A\Psi + H(\cdot,\cdot,\Psi,D_x\Psi) + \lambda_m \Psi)$$

which, being $u \geq \Psi$, implies

(7.8) $u_t + Au + H(\cdot,\cdot,u,D_x u) \leq$

$$0 \vee (\Psi_t + A\Psi + H(\cdot,\cdot,\Psi,D_x\Psi) + \sigma_m),$$

where $\sigma_m = H(\cdot,\cdot,u,D_x u)-H_m(\cdot,\cdot,u,D_x u)-H(\cdot,\cdot,\Psi,D_x\Psi)+$

$$+H_m(\cdot,\cdot,\Psi,D_x\Psi).$$

Passing to the limit as $m \to +\infty$ in (7.8) and taking into account (7.6) we have the result.

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