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SIMPLE ESTIMATORS OF THE PARAMETERS OF GENERALIZED TUKEY'S λ-FAMILY

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Abstract: Tukey (1960) introduced a one-parameter family of symmetric distributions which appeared useful in the representation of data when the underlying model is unknown.
Ramberg and Schmeiser (1972, 1974) extended this family to a four-parameter family containing distributions both symmetric and skewed (to the left or to the right).

In the present paper simple estimators of the unknown parameters based on order statistics are developed and their asymptotic properties are investigated.

Key words: generalized λ-family, robust estimators, order statistics

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1. Introduction. Tukey (1960) introduced the family of distributions $\{F(.,\lambda);\lambda\in R_1\}$ defined by their quantile function

$$F^{-1}(u;\lambda) = (u^{\lambda}-(1-u)^{\lambda})/\lambda \qquad u\in(0,1), \tag{1.1}$$

where $\lambda\in R_1$ is a parameter. This family is known as (Tukey's) λ-family and is useful in the representation of data when the underlying model is unknown. It contains distributions ranging from light-tailed ones ($\lambda>0$) to heavy tailed ones ($\lambda<0$); $\lambda=0$ corresponds to the logistic distribution, $\lambda=1$ or $=2$ corresponds to the uniform distribution, $\lambda=0,135$ corresponds to the standard normal distribution.

Ramberg and Schmeiser (1974) considered a generalized

<u>Tukey's λ-family</u> to a four-parameter family $\{F(.;\lambda_1,\lambda_2,\lambda_3,\lambda_4);\ \lambda_1\in R_1,(\lambda_2,\lambda_3,\lambda_4)\in\Lambda\}$ with $\Lambda=\{\lambda_i>0, i=2,3,4\}\cup\{\lambda_i<0, i=2,3,4\}\cup\{\lambda_2>1, \lambda_3<-1,\lambda_4<0\}\cup\{\lambda_2<-1,\lambda_3>1,\lambda_4<0\}\cup\{\lambda_2=0,\lambda_3\cdot\lambda_4>0\}\cup\{\lambda_3=0,\lambda_2\cdot\lambda_4>0\}$, defined through the quantile function $F^{-1}(.;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as follows:

$$F^{-1}(u;\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \lambda_1 + (u^{\lambda_2}-(1-u)^{\lambda_3})\ \lambda_4,\quad u\in(0,1) \tag{1.2}$$

where $\lambda_1\in R_1$ is the location parameter, $|\lambda_4|$ is the scale parameter, λ_2 , λ_3 are the shape parameters. This family contains the original Tukey's λ-family and, moreover, distributions skewed to the right and to the left.

The concept that the distribution is defined by its quantile is convenient in Monte-Carlo simulation studies (if U is a random variable with the uniform (0,1)-distribution and F is a distribution function then $F^{-1}(U)$ has the distribution function F).

Moreover, it can be useful in nonparametric statistics, e.g. in construction adaptive R- or L-estimators.

One should remark that generally the distribution function $F(.;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ corresponding to $F^{-1}(.;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ defined by (2.1) is not expressible in a "simple closed form".

Tukey's λ-family was studied by several authors, e. g. Joiner and Rosenblatt (1971), Ramberg and Schmeiser (1972, 1974).

The properties of generalized Tukey's λ-family were treated e. g. by Ramberg and Schmeiser (1974). It was shown how to determine the parameters of the distribution using the first four moments and how to fit the resulting distribution. For selected values of skewness and curtosis with expectation 0 and unit dispersion the tables of $\lambda_1,\ldots,\lambda_4$ are given.

The problem of estimation of the location parameter λ_1 was widely studied. Filiben (1969) proposed to use the trimmed mean and the Winsorized mean where censoring proportion was suitably chosen. Chan and Rhodin (1980) developed a robust estimator of λ_1 expressible as a linear combination of a finite number (≤ 5) of order statistics with coefficients suitably chosen.

Jones (1979) considering Tukey's λ-family proposed an estimator of λ based on the ordered sample and utilized it to develope the adaptive rank test for the symmetry problem.

The aim of this paper is to develop simple estimators of $\lambda_1,\dots,\lambda_4$ based on the ordered sample, to investigate their properties and to discuss the possibility of applications.

2. Estimators of the unknown parameters. Let $X_1,\dots,X_n$ be a random sample from the distribution function $\{F(\cdot,\lambda_1,\lambda_2,\lambda_3,\lambda_4);\ \lambda_1\in R_1,\ (\lambda_2,\lambda_3,\lambda_4)\in\mathcal{L}\}$ with the quantile function $F^{-1}(u,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, $u\in(0,1)$, given by (1.2) and let $Z_1\leq\dots\leq Z_n$ be the corresponding ordered sample.

We shall focus on the problem of estimation of λ_2. The estimators of λ_3 can be developed quite analogously. These estimators will be utilized to construct estimators of λ_1 and λ_4.

Jones (1979) towards construction of an adaptive rank statistic proposed to estimate the parameter λ of Tukey's λ-family defined by (1.1) with $\lambda<1$ by the statistics:

$$\hat{\lambda}=\frac{1}{\log 2}\log\frac{Z_{4M}-Z_{2M}}{Z_{2M}-Z_{M}}, \tag{2.1}$$

where M is suitably chosen and showed its consistency.

A slight generalization leads to the estimator $\hat{\lambda}_2(M;a,b,s)$ of λ_2 in the family $\{F(\cdot;\lambda_1,\lambda_2,\lambda_3,\lambda_4);\ \lambda_1\in R_1,\ (\lambda_2,\lambda_3,\lambda_4)\in\mathcal{L},\ \lambda_4<0,\ \lambda_2<1\}$:

$$\hat{\lambda}(M;a,b,s)=\frac{1}{\log\frac{1}{s}}\log\frac{Z_{[aM]}-Z_{[bM]}}{Z_{[asM]}-Z_{[bsM]}} \tag{2.2}$$

where

$$(2.3)\quad \begin{array}{l} a,b,s>0,\ s\neq 1,\ M \text{ is a positive integer fulfilling} \\ \max(a,b,as,bs)<n/2),\ a\neq b, \end{array}$$

and [A] denotes the largest integer not exceeding A.

Similarly, the parameter λ_3 in the family $\{F(\cdot;\lambda_1,\lambda_2,\lambda_3,\lambda_4);$ $\lambda_1\in R_1,\ \lambda_3<1,\ (\lambda_2,\lambda_3,\lambda_4)\in\Lambda\}$ can be estimated by

$$(2.4)\qquad \hat{\lambda}_3(M;a,b,s) = \frac{1}{\log s}\log\frac{Z_{n-[aM]}-Z_{n-[bM]}}{Z_{n-[asM]}-Z_{n-[bsM]}},$$

where a,b,s,M fulfil (2.3).

According to Theorem 4.3 as $n\to\infty$, $M\to\infty$, $M/n\to\infty$

$$(2.5)\qquad \hat{\lambda}_2(M;a,b,s) = \lambda_2 + O_P(\max(M^{-1/2},(M/n)^{1-\lambda_2}),$$

which means that this is a consistent estimator of λ_2 if $\lambda_2<1$. The highest order of consistency (in the considered class of estimators) is reached for

$$(2.6)\qquad M = M_{\lambda_2} = O(n^{\frac{2(1-\lambda_2)}{3-2\lambda_2}});$$

the corresponding rate of consistency is

$$(2.7)\qquad n^{-\frac{1-\lambda_2}{3-2\lambda_2}}.$$

Another class of simple estimators of λ_2 of the family $\{F(\cdot;\lambda_1,\lambda_2,\lambda_3,\lambda_4);\lambda_i\in R_1,\ i=1,3;\ \lambda_2\leq 2,\ \lambda_4>0\}$ can be defined as follows:

$$(2.8)\qquad \hat{\lambda}_2(M;a,b,c,d,s) = \log\left\{\frac{Z_{[aM]}-Z_{[bM]}+Z_{[cM]}-Z_{[dM]}}{Z_{[asM]}-Z_{[bsM]}+Z_{[csM]}-Z_{[dsM]}}\right\}\frac{1}{\log s^{-1}}$$

where

$$(2.9)\quad \begin{array}{l} a,b,c,d,s>0;\ s\neq 1,\ a-b+c-d=0;\ c\neq b,d;\ a\neq b,d, \\ M \text{ is a positive integer fulfilling} \\ \max(a,b,c,d,as,bs,cs,ds)M<n/2. \end{array}$$

According to Theorem 4.4

$$(2.10)\quad \hat{\lambda}_2(M;a,b,c,d,s) = \lambda_2+O_P(\max(M^{-1/2},(M/n)^{2-\lambda_2},M^{-1/2}(M/n)^{1-\lambda_2}))$$

as $n\to\infty$, $M\to\infty$, $m/n\to 0$, hence it is a consistent estimator of λ_2 if $\lambda_2 < 0$.

The choice

$$(2.11)\qquad M = M^*_{\lambda_2} = O(\max(n^{\frac{2(2-\lambda_2)}{5-2\lambda_2}}, n^{2/3}))$$

leads the highest order consistency (in the considered class of estimators) which is

$$(2.12)\qquad O_P(\min(n^{-(2-\lambda_2)/3}, n^{-(2-\lambda_2)(5-2\lambda_2)^{-1}})).$$

Table 1 below presents choices of M and the highest rate of consistency for some particular value of λ_2

Table 1

λ_2	M_{λ_2}	corr. rate of conv.	$M^*_{\lambda_2}$	corr. rate of conv.
0	$n^{2/3}$	$n^{-1/3}$	$n^{4/5}$	$n^{-2/5}$
1/2	$n^{1/2}$	$n^{-1/4}$	$n^{3/4}$	$n^{-3/8}$
1	-	-	$n^{2/3}$	$n^{1/3}$

If λ_2 is unknown we cannot find optimal choice of M_{λ_2} and $M^*_{\lambda_2}$. But we can proceed as follows:

1. choose M satisfying (2.3) and compute $\hat{\lambda}_2(M;a,b,s)$ (which is consistent);
2. compute M_{λ_2} according to (2.6) with λ_2 replaced by $\hat{\lambda}_2(M;a,b,s)$;
3. compute $\hat{\lambda}_2(M_{\hat{\lambda}_2(M;a,b,s)};a,b,s)$.

None of the introduced classes of estimators attains the highest possible consistency ($n^{-1/2}$ if the Fisher information is finite). This rate is reached e. g. by the class of estimators $(\tilde\lambda_2, \tilde\lambda_3)$ defined implicitly as the solution of the equations:

$$\frac{Z_{[a_i n]} - Z_{[b_i n]}}{Z_{[c_i n]} - Z_{[d_i n]}} = \frac{a_i^{\tilde\lambda_2} - (1-a_i)^{\tilde\lambda_3} - b_i^{\tilde\lambda_2} + (1-b_i)^{\tilde\lambda_3}}{c_i^{\tilde\lambda_2} - (1-c_i)^{\tilde\lambda_3} - d_i^{\tilde\lambda_2} + (1-d_i)^{\tilde\lambda_3}}, \quad i=1,2, \tag{2.13}$$

where $1 > a_i, b_i, c_i, d_i > 0$, $(a_i, b_i) \neq (c_i, d_i), (d_i, c_i)$, $i = 1,2$, $(a_1, b_1, c_1, d_1) \neq (a_2, b_2, c_2, d_2), (c_2, d_2, a_2, b_2)$, $a_i \neq b_i$, $c_i \neq d_i$. To find the estimators means in fact to solve transcendent equations which can sometimes bring computational problems.

Now, we turn to the problem of simple estimators of the scale parameter λ_4. By Lemma 4.2 one has

$$\frac{Z_{[an]} - Z_{[bn]}}{a^{\lambda_2} - (1-a)^{\lambda_3} - b^{\lambda_2} + (1-b)^{\lambda_3}} = \lambda_4 + O_p(n^{-1/2}) \text{ as } n \to \infty \tag{2.14}$$

where $a \neq b \in (0,1)$, and for $\lambda_2 > 0$, $\lambda_3 > 0$

$$\frac{Z_{n-M} - Z_N}{2} = \lambda_4 + O_p\left(\max\left(\frac{M}{n}, \left(\frac{M}{n}\right)^{\lambda_2}, \frac{N}{n}, \left(\frac{N}{n}\right)^{\lambda_3}\right)\right) \tag{2.15}$$

as $n \to \infty$, $M/n \to 0$, $N/n \to 0$ (M and N can be as fixed as n tend to infinity). Hence we can introduce the following estimator of λ_4:

$$\hat\lambda_4(a,b) = \frac{Z_{[an]} - Z_{[bn]}}{a^{\hat\lambda_2} - (1-a)^{\hat\lambda_3} + (1-b)^{\hat\lambda_3} - b^{\hat\lambda_2}} \tag{2.16}$$

where $a \neq b \in (0,1)$, $\hat\lambda_2$, $\hat\lambda_3$ are some of the estimators of λ_2 and λ_3, respectively, introduced above. If $\lambda_2 > 0$ and $\lambda_3 > 0$ simpler estimator of λ_4 can be proposed:

$$\hat\lambda_2(M,N) = \frac{Z_{n-M} - Z_N}{2}. \tag{2.17}$$

At last, the parameter λ_1 can be estimated by (for motivation see Lemma 4.1):

$$(2.18)\qquad \hat{\lambda}_1(a,b) = \tfrac{1}{2}(Z_{[an]}+Z_{[bn]}) - \hat{\lambda}_4(a^{\hat{\lambda}_2}-(1-a)^{\hat{\lambda}_3}+b^{\hat{\lambda}_2}+(1-b)^{\hat{\lambda}_3})$$

where $a \neq b \in (0,1)$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, $\hat{\lambda}_4$ are some of the estimators of the respective parameters introduced above. If we succeed to find $\hat{a}$, $\hat{b}$ $\in (0,1)$ such that

$$(2.19)\qquad \hat{a}^{\hat{\lambda}_2} - (1-\hat{a})^{\hat{\lambda}_3} + \hat{b}^{\hat{\lambda}_2} - (1-\hat{b})^{\hat{\lambda}_3} = 0$$

then

$$(2.20)\qquad \hat{\lambda}_1(\hat{a},\hat{b}) = \tfrac{1}{2}(Z_{[\hat{a}n]} + Z_{[\hat{b}n]}).$$

If $\lambda_2 > 0$, $\lambda_3 > 0$ then

$$(2.21)\qquad \hat{\lambda}_1(M,N) = \tfrac{1}{2}(Z_{n-M} + Z_N)$$

can be used as an estimator λ_1.

All considered estimators of λ_1, λ_2, λ_3, λ_4 are very simple, easy to compute (except, may be, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$). In practice one should choose M and N small with respect to n; around 0,01n to 0,1n according to how large is n.

As for the asymptotic properties of these estimators, they are consistent of order n^{α}, $\alpha > 0$ but in case of finite Fisher's information are not asymptotically optimal (i. e. asymptotically unbiased with the asymptotic variance attaining the Cramér-Rao lower bound). To obtain such optimal estimators the developed estimators as $(\hat{\lambda}_2, \hat{\lambda}_3)$ as $(\hat{\lambda}_2, \hat{\lambda}_3)$ can serve as preliminary estimators.

The estimators $(\hat{\lambda}_2, \hat{\lambda}_3)$ (or $(\hat{\lambda}_2, \hat{\lambda}_3)$) can be also utilized to construct various adaptive rank statistics along the line done by Jones.

In case of λ_2, $\lambda_3 > 1/2$ the estimators $\hat{\lambda}_1(M,N)$ and $\hat{\lambda}_4(M,N)$ are quite satisfactory for they attain the highest order of convergence, i. e. $n^{-\min(\lambda_2,\lambda_3)}$ if $\min(\lambda_2,\lambda_3) \leq 1$ and n^{-1} if $\lambda_2, \lambda_3 \geq 1$, and coincide with the estimators considered by Akahira and Takeuchi (1981).

3. Estimators in case $\lambda_2 = \lambda_3 = \lambda$. When $\lambda_2 = \lambda_3 = \lambda$ the corresponding density is symmetric around λ_1. One can use the estimators suggested above or utilizing the symmetry property and apply the following ones:

$$(3.1) \qquad \hat{\lambda}(M,a,b,s) = \frac{1}{2}(\hat{\lambda}_2(M;a,b,s)+\hat{\lambda}_3(M;a,b,s)), \quad \lambda < 1,$$

where a,b,s,M satisfy (2.3), $\hat{\lambda}_2$ and $\hat{\lambda}_3$ are done by (2.2) and (2.4), respectively, or

$$(3.2) \quad \hat{\hat{\lambda}}(M;a,b,c,d,s) = \frac{1}{2}(\hat{\hat{\lambda}}_2(M;a,b,c,d,s)+\hat{\hat{\lambda}}_3(M;a,b,c,d,s)), \quad \lambda < 2,$$

where a,b,c,d,s,M satisfy (2.9) and $\hat{\lambda}_2$ is defined by (2.8). At last, putting $a_1 = 1-b_1$, $c_1 = 1-d_1$ in (2.13) one can define the estimator $\tilde{\lambda}(a_1,c_1)$ of λ as the solution of the equation:

$$(3.3) \qquad \frac{Z_{[a_1n]}-Z_{[(1-a_1)n]}}{Z_{[c_1n]}-Z_{[(1-c_1)n]}} = \frac{a_1^{\lambda} - (1-a_1)^{\lambda}}{c_1^{\lambda} - (1-c_1)^{\lambda}}$$

where $a_1 \neq c_1 \in (0,1)$.

Similarly, the parameters λ_4, λ_1 can be estimated by

$$(3.4) \qquad \hat{\lambda}_4(a) = \frac{Z_{[an]}-Z_{[(1-a)n]}}{2(a^{\lambda}-(1-a)^{\lambda})}$$

and

$$(3.5) \qquad \hat{\lambda}_1(a) = (Z_{[an]}+Z_{[(1-a)n]})/2$$

respectively, where $a \in (0,1)$. If $\lambda > 0$ then one can also use

$$\hat{\lambda}_4(M) = (Z_{n-M} - Z_M)/2$$

or

$$\hat{\lambda}_1(M) = (Z_{n-M}+Z_M)/2,$$

where $M < n/2$.

We could observe that the estimators of the unknown parameters are simpler when we have the situation $\lambda_2 = \lambda_3 = \lambda$, i. e. when the distribution in symmetric around the location parameter

λ_1. Hence it could be of interest to have a test for testing problem:

$$H: \lambda_2 = \lambda_3 = \lambda \quad \text{against} \quad A: \lambda_2 \neq \lambda_3.$$

By Theorem 4.6 one has

$$(3.6) \qquad 0 \leq T_n = \frac{Z_{[aM]} - Z_{[bM]}}{Z_{n-[bM]} - Z_{n-[aM]}} = O_P((M/n)^{\lambda_2 - \lambda_3})$$

$$= 1 + o_P(1) \text{ if } \lambda_2 = \lambda_3 = \lambda$$

as $n \to \infty$, $M \to \infty$, $M/n \to 0$, which means that the values of T_n close to 1 indicate the validity of H and the values either close to 0 or large values indicate that H fails to be true.

According to the results in Section 4 under H $\sqrt{M}(T_n - 1)$ has asymptotically normal distribution with the parameters $(0, 2\lambda^2 b^{2\lambda-1}|a-b|a^{-1}(a^\lambda - b^\lambda)^{-2})$ and $(0, 2ba^{-1}|a-b|^{-1})$ if $\lambda \leq 1$ and $\lambda > 1$, respectively (notice (4.12)).

If we establish the test on the asymptotic distribution of T_n we reject the hypothesis H on the level α if

$$(3.7) \quad |\sqrt{M}\, T_n - 1| \geq \Phi^{-1}(1-\alpha/2)\hat{\lambda}\, b^{\hat{\lambda}} |a^{\hat{\lambda}} - b^{\hat{\lambda}}|^{-1} (2|a-b| a^{-1} b^{-1})^{1/2}$$

where Φ^{-1} is the quantile function of the standard normal distribution and

$$\hat{\lambda} = (\hat{\lambda}_2(M;a,b) + \hat{\lambda}_3(M;a,b))/2.$$

<u>4. Properties of the proposed estimators.</u> If $Z_1 \leq \ldots \leq Z_n$ is the ordered sample from the distribution $F(x;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ defined by its quantile function (1.2) then

$$(4.1) \qquad Z_i = \lambda_1 + \lambda_4(U_i^{\lambda_2} - (1-U_i)^{\lambda_3}), \quad 1 \leq i \leq n,$$

(i. e. $Z_i = F^{-1}(U_i;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, $1 \leq i \leq n$), where $U_1,\ldots,U_n$ is the ordered sample (of size n) from the uniform (0,1)-distribution.

It is known that

(4.2) $$EU_i = \frac{i}{n+1}, \quad \operatorname{cov}(U_i,U_j) = \frac{i(n-j+1)}{(n+1)^2(n+2)}, \quad 1 \le i \le j \le n,$$

(4.3) $$U_i = \frac{i}{n+1} + O_p\left(\left(\frac{i(n-i+1)}{n^3}\right)^{1/2}\right).$$

Further, the random vector $(\mathcal{U}_1,\ldots,\mathcal{U}_n)$ has the same distribution as the vector

$$(\mathcal{U}_M V_{1,M-1},\ldots,\mathcal{U}_M V_{M-1,M-1},\mathcal{U}_M,\mathcal{U}_M+(\mathcal{U}_N-\mathcal{U}_M)W_{1,N-M-1},\ldots,$$
$$\mathcal{U}_M+(\mathcal{U}_N-\mathcal{U}_M)W_{N-M-1,N-M-1},\mathcal{U}_N,\mathcal{U}_N+(1-\mathcal{U}_N)Y_{1,n-N},\ldots,\mathcal{U}_N+(1-\mathcal{U}_N)Y_{n-M,n-M}),$$

where $0 \le M \le N \le n+1$, $\mathcal{U}_0 = 0$, $\mathcal{U}_{n+1} = 0$, the vectors $(V_{1,M-1},\ldots V_{M-1,M-1})$ $(W_{1,N-M-1},\ldots,W_{N-M-1,N-M-1})$ and $(Y_{1,n-N},\ldots,Y_{n-M,n-M})$ are given (Z_M,Z_N) the independent ordered samples of size $M-1$, $N-M-1$, $n-M$, respectively, from the uniform (0,1)-distribution.

In the following we shall omit the second indices in $V_{i,M-1}$, $W_{j,N-M-1}$, $Y_{k,n-M}$, whenever it causes no confusion.

Combining suitably these results one directly obtains:

Lemma 4.1. a) For $a \in (0,1)$, $n \to \infty$

(4.4) $$\frac{Z_{[an]}-\lambda_1}{4} = a^{\lambda_2}-(1-a)^{\lambda_3}+(U_{[an]}-a)(\lambda_2 a^{\lambda_2-1}+\lambda_3(1-a)^{\lambda_3-1})+O_p(n^{-1})$$
$$= a^{\lambda_2} + 1-a^{\lambda_3} + O_p(n^{-1/2}).$$

b) For $n \to \infty$, $M \to \infty$, $M/n \to 0$

(4.5) $$\frac{Z_M-\lambda_1}{\lambda_4} = -1 + \lambda_3 U_M + U_M^{\lambda_2} + O_p((M/n)^2)$$
$$= -1 + O_p(\max((M/n)^{\lambda_2},M/n)) \quad \text{if } \lambda_2 \ge 0$$
$$= U_M^{\lambda_2} + O_p(1) \quad \text{if } \lambda_2 < 0 .$$

c) For $n \to \infty$, $M \to \infty$, $M/n \to 0$

$$(4.6) \quad \frac{Z_{n-M}-\lambda_1}{\lambda_4} = -(1-U_{n-M})^{\lambda_3} + 1 + \lambda_3(U_{n-M}-1) + O_P((M/n)^2)$$

$$= 1 + O_P(\max(M/n, (M/n)^{\lambda_3})) \qquad \lambda_3 \geq 0$$

$$= -(1-U_{n-M})^{\lambda_3} + O_P(1) . \qquad \lambda_3 < 0$$

Lemma 4.2. a) For $A > a$, $b > 0$, $n \to \infty$, $M \to \infty$, $M/n \to 0$

$$(4.7) \quad (Z_{[aM]}-Z_{[bM]})/\lambda_4 = U_{[AM]}^{\lambda_2}\Big\{A^{-\lambda_2}\Big\{a^{\lambda_2}-b^{\lambda_2} +$$

$$+ (V_{[aM],[AM]-1}-\tfrac{a}{A})\lambda_2 a^{\lambda_2-1} A - (V_{[bM],[AM]-1}-\tfrac{b}{A})\lambda_2 b^{\lambda_2-1} A\Big\}$$

$$+ \lambda_3 U_{[AM]}^{1-\lambda_2}(a-b)/A + O_P(\max(M^{-1},(M/n)^{1-\lambda_2} M^{-1/2})\Big\}.$$

b) For $A > a,b,c,d > 0$, $a-b+c-d = 0$, $n \to \infty$, $M \to \infty$, $M/n \to 0$

$$(4.8) \quad (Z_{[aM]}-Z_{[bM]}+Z_{[cM]}-Z_{[dM]})/\lambda_4 =$$

$$= U_{[AM]}^{\lambda_2}\Big\{A^{-\lambda_2}\Big\{a^{\lambda_2}-b^{\lambda_2}+c^{\lambda_2}-d^{\lambda_2} +$$

$$+ A\lambda_2(V_{[aM],[AM]-1}-\tfrac{a}{A})a^{\lambda_2-1} - A\lambda_2(V_{[bM],[AM]-1}-\tfrac{b}{A})b^{\lambda_2-1}$$

$$+ A\lambda_2(V_{[cM],[AM]-1}-\tfrac{c}{A})c^{\lambda_2-1} - A\lambda_2(V_{[dM],[AM]-1}-\tfrac{d}{A})d^{\lambda_2-1}$$

$$+ \lambda_3(\lambda_3-1)\, U_{[AM]}^{2-\lambda_2}(-a^2+b^2-c^2-d^2)/2 + O_P(\max(M^{-1}(M/n)^{2-\lambda_2}M^{-1/2})).$$

Proof. Clearly,

$$(4.9) \quad (Z_{[aM]}-Z_{[bM]})/\lambda_4 = U_{[aM]}^{\lambda_2}-(1-U_{[aM]})^{\lambda_3}-U_{[bM]}^{\lambda_2}+(1-U_{[bM]})^{\lambda_3} =$$

$$= U_{[AM]}^{\lambda_2}(V_{[aM]}^{\lambda_2} - V_{[bM]}^{\lambda_2})-(1-U_{[AM]}V_{[aM]})^{\lambda_3}+(1-U_{[AM]}V_{[bM]})^{\lambda_3}.$$

Applying the Taylor expansion together with (4.3) and (4.3) implies the assertion a).

b) It can be proved quite analogously noticing:
that $a-b+c-d = 0$ together with (4.3) and (4.4) implies

$$V_{[aM]} - V_{[bM]} + V_{[cM]} - V_{[dM]} = O_P(M^{-1/2}) \quad \text{as } M \to \infty .$$

Q.E.D.

Now, we are in a position to state the main theorems on properties of the estimators of λ_2 (those of λ_3 are quite analogous thus they are omitted).

<u>Theorem 4.3.</u> If (2.3), $\lambda_2 < 1$ holds then

$$(4.10) \qquad (\log s^{-1})(\hat{\lambda}_2(M;a,b,s)-\lambda_2) = (a^{\lambda_2}-b^{\lambda_2})^{-1} \cdot A\lambda_2$$
$$\cdot \Big\{(V_{[aM],[AM]-1}-\tfrac{a}{A})a^{\lambda_2-1} - (V_{[bM],[AM]-1}-\tfrac{b}{A})b^{\lambda_2-1} -$$
$$- (V_{[asM],[AM]-1}-\tfrac{as}{M})a^{\lambda_2-1}s^{-1}+(V_{[bsM],[AM]-1}-\tfrac{bs}{A})b^{\lambda_2-1}s^{-1}\Big\}$$
$$+ (M/n)^{1-\lambda_2}\gamma_1 + O_P(\max(M/n)^{1-\lambda_2} M^{-1/2}, M^{-1}, (M/n)^{2-\lambda_2}))$$

as $n \to \infty$, $M \to \infty$, $M/n \to 0$, where $\hat{\lambda}_2$ is defined by (2.2) and

$$(4.11) \qquad \gamma_1 = \lambda_3(1-s^{1-\lambda_2})(a-b)(a^{\lambda_2}-b^{\lambda_2})^{-1},$$

then

$$(4.12)\ \mathcal{L}(M^{1/2}(\hat{\lambda}_2(M;a,b,s)-\lambda_2-\gamma_1(M/n)^{1-\lambda_2}(\log s^{-1})^{-1})\sigma_{\lambda_2}^{-1}) \to N(0,1)$$

as $n \to \infty$, $M \to \infty$, $M/n \to 0$, where

$$\sigma_{\lambda_2}^2 = (\log s^{-1})^{-2}\lambda_2^2\Big\{(a^{2\lambda_2-1}+b^{2\lambda_2-1})\,|1-s^{-1}|-(1+s^{-1})\min(a,b) +$$
$$+ s^{-1}\min(a,bs)+\min(b,as)\Big\}\,(a^{\lambda_2}-b^{\lambda_2})^{-2}.$$

<u>Proof.</u> Applying the Taylor expansion (of the function $h(x,y) = \log \frac{x}{y}$) and Lemma 4.2 we easily obtain the first part of assertion.

As for the latter, according the proved part theorem it suffices to show that

$$\sigma^2_{\lambda_2} = (\log s^{-1})^2 \, (a^{\lambda_2} - b^{\lambda_2})^{-2} \, A^2 \lambda_2^2 \, .$$

$$\operatorname{var}\Big\{ (V_{[aM]} - \tfrac{a}{A}) \, a^{\lambda_2 - 1} - (V_{[bM]} - \tfrac{b}{A}) \, b^{\lambda_2 - 1} -$$

$$- (V_{[asM]} \tfrac{as}{A}) \, a^{\lambda_2 - 1} \, s^{-1} + (V_{[bsM]} - \tfrac{bs}{A}) \, b^{\lambda_2 - 1} \, s^{-1} \Big\}.$$

The assertion can be concluded from this relation and (4.2). Q.E.D.

Going carefully through the proof of Lemma 4.2 and Theorem 4.3 we find that

$$(4.12) \qquad \hat{\lambda}_2(M;a,b,s) = \lambda_2 + o_P(1) \qquad \lambda_2 \leq 1$$
$$= 1 + o_P(1) \qquad \lambda_2 > 1$$

as $n \to \infty$, $M \to \infty$, $M/n \to 0$.

<u>Theorem 4.4.</u> If (2.9), $\lambda_2 < 2$, $\lambda_2 \neq 1$ is fulfilled then

$$(4.13) \quad (\log s^{-1})(\hat{\lambda}_2(M;a,b,c,d,s) - \lambda_2) = (a^{\lambda_2} - b^{\lambda_2} + c^{\lambda_2} - d^{\lambda_2})^{-1}$$

$$\cdot \Big\{ A\lambda_2 ((V_{[aM],[AM]-1} - \tfrac{a}{A}) a^{\lambda_2 - 1} - (V_{[bM],[AM]-1} - \tfrac{b}{A}) b^{\lambda_2 - 1} +$$

$$+ (V_{[cM],[AM]-1} - \tfrac{c}{A}) c^{\lambda_2 - 1} - (V_{[dM],[AM]-1} - \tfrac{d}{A}) d^{\lambda_2 - 1} -$$

$$- A\lambda_2 s^{-1} ((V_{[asM],[AM]-1} - \tfrac{sa}{A}) a^{\lambda_2 - 1} - (V_{[bsM],[AM]-1} - \tfrac{sb}{A}) b^{\lambda_2 - 1}$$

$$+ (V_{[csM],[AM]-1} - \tfrac{sc}{A}) c^{\lambda_2 - 1} - (V_{[dsM],[AM]-1} - \tfrac{sd}{A}) d^{\lambda_2 - 1}) \Big\}$$

$$+ (M/n)^{2-\lambda_2} \gamma_2 + O_P(\max(M^{-1}, (M/n)^{2-\lambda_2} M^{-1/2}))$$

as $n \to \infty$, $M \to \infty$, $M/n \to 0$, where $\hat{\lambda}_2$ is defined by (2.8) and

$$\gamma_2 = 2^{-1} \lambda_3 (\lambda_3 - 1)(a^2 - b^2 + c^2 - d^2)(-1 + s^{-\lambda_2})(a^{\lambda_2} - b^{\lambda_2} + c^{\lambda_2} - d^{\lambda_2})^{-1}$$

then

$$\mathscr{L}(M^{-1/2}(\hat{\lambda}_2(M;a,b,c,d,s) - \lambda_2 - (M/n)^{2-\lambda_2} \gamma_2 (\log s^{-1})^{-1}) / \sigma^*_{\lambda_2}) \to N(0,1)$$

as $n \to \infty$, where $\sigma^{*2}_{\lambda_2}$ is the variance of the linear combinations of order statistics from the right hand-side of (4.13) which under the additional assumption $s > 1$ and

for every pair (α,β), $\alpha,\beta \in \{a,b,c,d\}$ $\alpha < \beta$ implies $s\alpha < \beta$

can be expressed as follows:

$$\sigma^{*2}_{\lambda_2} = \lambda_2^2 \left|1-s^{-1}\right| (a^{2\lambda_2-1}+b^{2\lambda_2-1}+c^{2\lambda_2-1}+d^{2\lambda_2-1}).(a^{\lambda_2}-b^{\lambda_2}+c^{\lambda_2}-d^{\lambda_2})^{-2}.$$

The <u>proof</u> is quite analogous to this of Theorem 4.4 (but more tedious computations are needed) hence is omitted. Q.E.D.

<u>Corollary 4.5.</u> a) For $a,b \in (0,1)$

$$\hat{\lambda}_4(a,b) = O_P(n^{-1/2}) + O_P(\hat{\lambda}_2-\lambda_2) + O_P(\hat{\lambda}_3-\lambda_3) \tag{4.14}$$

and

$$\hat{\lambda}_1(a,b) = O_P(n^{-1/2}) + O_P(\hat{\lambda}_2-\lambda_2) + O_P(\hat{\lambda}_3-\lambda_3), \tag{4.15}$$

where $\hat{\lambda}_4(a,b)$ and $\hat{\lambda}_1(a,b)$ are defined by (2.16) and (2.18), respectively.

b) If $\lambda_2 > 0$, $\lambda_3 > 0$ and $n \to \infty$, $M/n \to 0$, $N/n \to 0$ then

$$\begin{aligned} \hat{\hat{\lambda}}_4(M,N) &= \tfrac{1}{2}\{2-(1-U_{n-M})^{\hat{\lambda}_3}-U_N^{\hat{\lambda}_2}+\hat{\lambda}_3(U_{n-M}-1-U_N)+O_P((M/n)^2) \\ &= 1 + O_P(\max((M/n)^{\lambda_3}, (N/n)^{\lambda_2}, M/n, N/n)) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \hat{\hat{\lambda}}_1(M,N) &= \tfrac{1}{2}\{2\lambda_1-(1-U_{n-M})^{\hat{\lambda}_3}+U_N^{\hat{\lambda}_2}+\hat{\lambda}_3(U_{n-M}-1+U_N)+O_P((M/n)^2) \\ &= \lambda_1 + O_P(\max((M/n)^{\lambda_3}, (N/n)^{\lambda_2}, M/n, N/n)), \end{aligned} \tag{4.17}$$

where $\hat{\hat{\lambda}}_4(M,N)$ and $\hat{\hat{\lambda}}_1(M,N)$ are defined by (2.17) and (2.21), respectively.

Theorem 4.6. Under H

$$\mathcal{L}(\sqrt{M}(T_n-1)) \rightarrow N(0,\sigma^2) \quad \text{as } M\to\infty\ ,\ N\to\infty\ ,\ M/n\to 0,\ N/n\to 0$$

where T_n is defined by (3.6) and

$$\sigma^2 = \frac{2\lambda^2\, b^{2\lambda}|a-b|}{ab(a^\lambda-b^\lambda)^2} \qquad \text{if } \lambda < 1$$

$$= \frac{2b}{a|a-b|} \qquad \text{if } \lambda > 1.$$

Proof. By Lemma 4.2 and the Taylor expansion one can easily arrive at

$$T_n = \left(\frac{U_{[AM]}}{1-U_{n-[AM]}}\right)^\lambda \Big\{1 + \frac{A\lambda}{a^\lambda-b^\lambda}$$

$$\Big\{(V_{[aM],[AM]-1}-\tfrac{a}{A})a^{\lambda-1} - (V_{[bM],[AM]-1}-\tfrac{b}{A})b^{\lambda-1} -$$

$$-(W_{[aM],[AM]-1}-\tfrac{a}{A})a^{\lambda-1} + (W_{[bM],[AM]-1}-\tfrac{b}{A})b^{\lambda-1}\Big\} +$$

$$+ \frac{\lambda A^\lambda}{a^\lambda-b^\lambda}\cdot\frac{a-b}{A}\,(U^{1-\lambda}_{[AM]}-(1-U_{n-[AM]})^{1-\lambda}) + O_P(\max(M^{-1},(M/n)^{1-\lambda}M^{-1/2}))\Big\}.$$

Clearly,

$$\frac{U_{[AM]}}{1-U_{n-[AM]}} = 1 + O_P(\tfrac{M}{n}\,M^{-1/2}).$$

Hence for $\lambda < 1$

$$T_n = 1 + \frac{\lambda A}{a^\lambda-b^\lambda}((V_{[aM],[AM]-1} - Y_{[aM],[AM]})a^{\lambda-1} -$$

$$- (V_{[bM],[AM]-1} - Y_{[bM],[AM]})b^{\lambda-1}) + O_P(\max(M^{-1},(M/n)^{1-\lambda}M^{-1/2}))$$

The result for $\lambda < 1$ follows directly recalling (4.2) and that the vectors $\{V_{i,[AM]-1},\ i=1,\dots,[AM]-1\}$ and $\{Y_{j,[AM]},\ j=1,\dots,[AM]\}$ are given $U_{[AM]}$ and $U_{n-[MA]}$ independent.

As for $\lambda > 0$, by (4.5), (4.6), (4.16) and (4.17) one has

$$T_n = \frac{U_{[aM]} - U_{[bM]} + O_P(\max((M/n)^2,(M,n)^\lambda))}{U_{n-[bM]} - U_{n-[aM]} + O_P(\max((M/n)^2,(M/n)^\lambda)}\ ,$$

which, using the Taylor expansion, can be rewritten as follows

$$T_n = 1 - \frac{1}{a-b}\frac{n}{M}\Big\{U_{[aM]} - U_{[bM]} - U_{n-[bM]} + U_{n-[aM]} + \\ + O_p(\max((M/n)^2, (M/n)^{\lambda}))\Big\} + O_p(M^{-1}).$$

Hence T_n is asymptotically equivalent to the linear combination of order statistics and regarding (4.2) the assertion follows also for $\lambda \geq 1$.

Q.E.D.

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