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AN APPLICATION OF A FIXED POINT PRINCIPLE OF SADOVSKII TO DIFFERENTIAL EQUATIONS ON THE REAL LINE

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Abstract: In this note we consider the existence of solutions for the differential equation $x' = f(t,x)$ on the half-line $t \geq 0$ via a fixed point theorem of Sadovskii. Here f is a continuous function with values in a Banach space satisfying some regularity condition expressed in terms of the measure of noncompactness α.

Key words: Differential equations in Banach spaces, existence of solutions on the half-line $t \geq 0$, measure of noncompactness α, fixed point theorem.

Classification: 34G20

Let $J = [0,\infty)$, let E be a Banach space with norm $\|\cdot\|$, and let f be an E-valued function defined on $J \times E$. Suppose f is continuous and $\|f(t,x)\| \leq G(t, \|x\|)$ for $(t,x) \in J \times E$, where the function G is continuous on $J \times J$ and monotonically nondecreasing in the second variable.

Let $x_0 \in E$. By (PC) we shall denote the problem of finding a solution of the differential equation

$$x' = f(t,x)$$

satisfying the initial condition $x(0) = x_0$.

Using the fixed point theorem of Sadovskii ([6], Th. 3.4.3)we shall prove the existence of solutions of (PC) provided some regularity condition expressed in terms of the Kuratowski measure of noncompactness α.

The measure of noncompactness $\alpha(A)$ of a nonempty bounded subset A of E is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter $\leq \varepsilon$. For the properties of α the reader is referred to [2] - [4], [6].

Denote by C(J,E) the family of all continuous functions from J to E. The set C(J,E) will be considered as a vector space endowed with the topology of almost uniform convergence. Further we will use standard notations. The closure of a set A and its closed convex hull be denoted, respectively, by $\bar{A}$ and $\overline{\mathrm{conv}}\, A$. For $X \subset C(J,E)$ we denote by X(t) the set of all x(t) with $x \in X$.

Let S_∞ be the set of all nonnegative real sequences. For u = $= (u_n)$, $v = (v_n) \in S_\infty$ we write $u < v$ if $u \leq v$ (that is, $u_n \leq v_n$ for $n = 1,2,\ldots$) and $u \neq v$.

Let us state our fixed point theorem in the following form.

Sadovskii s fixed point principle. Let Q be a closed convex subset of C(J,E). Let Φ be a function which assigns to each nonempty subset X of Q a sequence $\Phi(X) \in S_\infty$ with the following properties:

1^0 $\Phi(\{x\} \cup X) = \Phi(X)$ for $x \in Q$;

2^0 $\Phi(\overline{\mathrm{conv}}\, X) = \Phi(X)$;

3^0 if $\Phi(X) = \Theta$ (the zero sequence) then $\bar{X}$ is compact.

Assume that $F: Q \longrightarrow Q$ is a continuous mapping satisfying $\Phi(F[X]) < \Phi(X)$ for an arbitrary subset X of Q with $\Phi(X) > \Theta$ Then F has a fixed point in Q.

Our result reads as follows.

Theorem. Let

$$\alpha(f[I \times X]) \leq \sup\{L(t, \alpha(X)) : t \in I\}$$

for any compact subset I of J and each bounded subset X of E, where E is a nonnegative function. Suppose that the scalar differential equation

$$g' = G(t,g),\ g(0) = \|x_0\|$$

has a solution g_0 existing on J. Assume in addition that $L(t,0) \equiv 0$ on J, $t \mapsto L(t,r)$ is continuous on J for each fixed r in J, and

(+) $$\sup \{ \int_0^t L(s, r) ds : t \in J\} < r$$

for all $r > 0$.

Under the above hypotheses there exists a solution of (PC) such that $\|x(t)\| \leq g_0(t)$ for $t \in J$.

Proof. Denote by Q the set of all $x \in C(J,E)$ such that $\|x(t)\| \leq g_0(t)$ on J, and $\|x(t') - x(t'')\| \leq |\int_{t'}^{t''} G(s,g_0))ds|$ for t',t'' in J. We define a continuous map F of Q into itself by

$$(Fx)(t) = x_0 + \int_0^t f(s,x(s))ds \text{ for } x \in C(J,E).$$

Let n be a positive integer and X a nonempty subset of Q. We prove that

(*) $$\sup_{0 \leq t \leq n} \alpha(F[X](t)) \leq \sup_{t \in J} \int_0^t L(s, \sup_{0 \leq \sigma \leq n} \alpha(X(\sigma)))ds.$$

To this end, fix t in $[0,n]$. Put $Z = \bigcup \{X(\sigma): 0 \leq \sigma \leq n\}$. Since $s \mapsto L(s,\alpha(Z))$ is uniformly continuous on $[0,t]$, for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|s' - s''| < \delta$ with $s',s'' \in [0,t]$ implies $|L(s',\alpha(Z)) - L(s'',\alpha(Z))| < \varepsilon$. Now, we divide the interval $[0,t]$ into m parts $t_0 = 0 < t_1 < \ldots < t_m = t$ in such a way that $|t_i - t_{i-1}| < \delta$. Denote by I_i $(i = 1,2,\ldots,m)$ the interval $[t_{i-1},t_i]$; let s_i be a point in I_i such that $L(s_i,\alpha(Z)) \geq L(s,\alpha(Z))$ for $s \in I_i$.

For continuous vector valued functions the integral mean value theorem may be stated as $\int_a^b h(s)ds \in (b-a)\overline{\text{conv}}(\{h(\sigma): a \leq \sigma \leq b\})$. Therefore

$$\alpha(F[X](t)) \leq$$
$$\leq \alpha(\sum_{i=1}^{m} (t_i - t_{i-1})\overline{\text{conv}}(\{f(s,x(s)): s \in I_i\})) \leq$$

$$\le \sum_{i=1}^{m}(t_i-t_{i-1})\alpha(f[I_i\times Z])\le \sum_{i=1}^{m}(t_i-t_{i-1})L(s_i,\alpha(Z)) \le$$

$$\le \sum_{i=1}^{m}\int_{I_i}|L(s,\alpha(Z)) - L(s_i,\alpha(Z))|ds + \sum_{i=1}^{m}\int_{I_i} L(s,\alpha(Z))ds <$$

$$< \varepsilon t + \int_0^t L(s,\alpha(Z))ds.$$

Since $X|_{[0,n]}$ is equicontinuous and bounded, we can apply Lemma 2.2 of [1] to get

$$\alpha(F[X](t)) < \varepsilon t + \int_0^t L(s, \sup_{0\le\sigma\le m}\alpha(X(\sigma)))ds$$

and our claim is proved.

Define:

$$\Phi(X) = (\sup_{0\le t\le 1}\alpha(X(t)), \sup_{0\le t\le 2}\alpha(X(t)),\dots)$$

for any nonempty subset of Q. Evidently, $\Phi(X)\in S_\infty$. By the corresponding properties of α the function Φ satisfies the conditions 1^0 - 3^0 listed above. From (+) and ($*$) it follows that $\Phi(F[X]) <$ $< \Phi(X)$ whenever $\Phi(X) > \Theta$. Thus all assumptions of Sadovskii´s Fixed Point Prinxiple are satisfied, F has a fixed point in Q and the proof is complete.

References

[1] A. AMBROSETTI: Un teorema di esistenza per le equazioni differenziali negli spazi di Banach, Rend. Sem. Mat. Univ. Padova, 39(1967), 349-360.

[2] J. DANEŠ: On densifying and related mappings and their application in nonlinear functional analysis, Theory of Nonlinear Operators, Akademie-Verlag, Bwelin 1974, 15-46.

[3] K. DEIMLING: Ordinary Differential Equations in Banach Spaces, Lect. Notes in Math. 596, Springer-Verlag, Berlin 1977.

[4] R.H. MARTIN, Jr.: Non linear Operators and Differential Equations in Banach Spaces, John Wiley and Sons, New York, 1976.

[5] B. RZEPECKI: An existence theorem for bounded solutions of differential equations in Banach spaces, Rend. Sem. Mat. Univ. Padova 73(1984).

[6] B.N. SADOVSKII: Limit compact and condensing operators, Russian Math. Surveys 27(1972), 86-144.

[7] A. STOKES: The application of a fixed-point theorem to a variety of nonlinear stability problems, Proc. Nat. Acad. Sci. U.S.A. 45(1959), 231-235.

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