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A NOTE TO E. MIERSEMANN'S PAPERS ON HIGHER EIGENVALUES OF VARIATIONAL INEQUALITIES

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Abstract: An improvement of E. Miersemann's result on higher eigenvalues of variational inequalities and some examples, for which the obtained criterion is sharp, are given.

Key words: variational inequality, eigenvalue problem

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1. INTRODUCTION

Let H be a real separable Hilbert space and $K \subset H$ a closed convex cone with its vertex at zero (see [3]). Let $A:H \rightarrow H$ be a linear, completely continuous, symmetric and positive operator. Let $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \dots > 0$ be the eigenvalues of the operator A and let the corresponding eigenvectors $u_1, u_2, u_3, \dots$ form an orthonormal basis of H.

We are interested in the eigenvalue problem for the variational inequality

(1) $\quad u \in K: \quad (\lambda u - Au, v-u) \geqq 0 \quad$ for all $v \in K$,

where λ is a real eigenvalue parameter and we look for non-trivial solutions u of (1).

We shall denote $\sigma_K(A)$ the set of all eigenvalues to (1).

2. E. MIERSEMANN'S RESULT

Denote E_n the linear hull of $\{u_1,\dots,u_n\}$, L_n the eigenspace to λ_n, $B=\{u\in H;\ \|u\|\leq 1\}$, $S=\{u\in H;\ \|u\|=1\}$, $S_n=E_n\cap S$. Further let P be an orthogonal projection of H onto E_n.

In [1,2,3] the following assertions are proved:

<u>Theorem 1</u>. Let $\tilde{H}\subset H$ be a closed subspace, $\tilde{H}\subset K$. Denote $\tilde{P}$ the orthogonal projection of H onto $\tilde{H}$. We consider the equation

$$u\in\tilde{H}:\quad \tilde{P}Au=\tilde{\lambda}u$$

and assume that there exist at least n positive eigenvalues $\tilde{\lambda}_1\geq\tilde{\lambda}_2\geq\dots\geq\tilde{\lambda}_n$. Let

$$\tilde{\lambda}_n>\lambda_{n+1}\,. \tag{2}$$

Then there exists an eigenvalue $\lambda\in\sigma_K(A)\cap(\lambda_{n+1},\lambda_n\rangle$.

<u>Theorem 2</u>. Let $V=\{v\in E_n^{\perp};\ u+v\in K \text{ for all } u\in S_n\}$ be nonempty and suppose

$$\lambda_n>\lambda_{n+1}+\inf_{v\in V}\{\lambda_{n+1}\|v\|^2-(Av,v)\}\,. \tag{3}$$

Then there exists $\lambda\in\sigma_K(A)\cap(\lambda_{n+1},\lambda_n\rangle$.

<u>Remark 1</u>. The assumptions of Theorem 2 are fulfilled, if e.g. $u_{n+1}\in K^o$ (= interior of K).

<u>Theorem 3</u>. Let the assumptions of Theorem 1 or Theorem 2 be fulfilled and let, moreover, $L_n\not\subset K$.

Then there exists $\lambda\in\sigma_K(A)\cap(\lambda_{n+1},\lambda_n)$.

The idea of the proof is following:

Define N_α the class of all compact sets $F \subset K \cap S$ such that

(a) $\min_{u \in F} (Au,u) \geq \lambda_{n+1} + \alpha$

(b) F is not contractible within the set $R = \{u \in H;\ Pu \neq 0\}$.

Using (2) or (3) it is proved that the class N_α is nonempty for a suitable $\alpha > 0$ and then using some topological technique (see [1]) it is proved that there exists $u \in K \cap S$ such that

$$(Au,u) = \sup_{F \in N_\alpha} \min_{v \in F} (Av,v)$$

which is also a solution of the variational inequality (1) with corresponding eigenvalue $\lambda \in \langle \lambda_{n+1} + \alpha, \lambda_n \rangle$ (resp. $\lambda \in \langle \lambda_{n+1} + \alpha, \lambda_n)$).

3. IMPROVEMENT OF E. MIERSEMANN'S RESULT

We shall weaken conditions (2),(3) in Theorems 1,2. A slight weaker version of Theorem 4 was obtained also by prof. Miersemann (personal communication).

Lemma. Let $A_k : H \to H$ be linear continuous operators, $A_k \to A$ in the operator norm (the operator A is supposed to satisfy the assumptions from Section 1). Let $u^k \in K \cap S$, $\lambda^k \in \langle c_1, c_2 \rangle$ (where c_1, c_2 are positive constants) and

$$(\lambda^k u^k - A_k u^k, v - u^k) \geq 0 \quad \text{for all} \quad v \in K .$$

Then there exists a subsequence (we denote it as before) such that $\lambda^k \to \lambda$, $u^k \to u$ and

$$(\lambda u - Au, v-u) \geq 0 \quad \text{for all} \quad v \in K .$$

Proof. We may suppose $\lambda^k \longrightarrow \lambda$, $u^k \to u \in K \cap B$.
Then $\lambda^k = (A_k u^k, u^k) \longrightarrow (Au,u)$, hence

(4) $\quad \lambda = (Au,u), \quad u \neq 0$.

Further $0 \leqq (\lambda^k u^k - A_k u^k, v) \longrightarrow (\lambda u - Au, v)$ for all $v \in K$, thus

(5) $\quad (\lambda u - Au, v) \geqq 0$ for all $v \in K$.

Putting $v=u$ in (5) and using (4) we get

$$\lambda \|u\|^2 \geqq (Au,u) = \lambda ,$$

thus $u \in K \cap S$ and $u^k \to u$.

<u>Theorem 4</u>. Suppose that $E_n^{\perp} \cap K \cap S \neq \emptyset$ and put
$c_n = \sup\limits_{u \in E_n^{\perp} \cap K \cap S} (Au,u)$. Assume instead of the conditions (2), (3) in Theorems 1,2 the conditions

(2*) $\quad \tilde{\lambda}_n > c_n$

(3*) $\quad \lambda_n \geqq c_n + \inf\limits_{v \in V^*} \{ c_n \|v\|^2 - (Av,v) \}$,

where $V^* = \{ v \in E_n^{\perp} ; \ u+v \in K \text{ for all } u \in S_n^* \}$,

$$S_n^* = \left\{ u \in E_n - \{0\} ; \ \|u\|^2 = \frac{\lambda_n - c_n}{\frac{(Au,u)}{\|u\|^2} - c_n} \right\}$$

and V^* is supposed to be nonempty.

Then there exists $\lambda \in \sigma_K(A) \cap \langle c_n, \lambda_n \rangle$ (and $\lambda < \lambda_n$ if $L_n \not\subset K$).

<u>Remark 2</u>. Obviously $c_n \leqq \lambda_{n+1}$ and it can be easily proved $V \subset V^*$, hence (2) $\Rightarrow$ (2*), (3) $\Rightarrow$ (3*).

If $V^* \neq \emptyset$ then $E_n^{\perp} \cap K \cap S \neq \emptyset$.

Proof of Theorem 4.

1. First suppose (2^*) or that in (3^*) strong inequality holds. Define N^*_α the class of all compact sets $F \subset K \cap S$ such that

(a^*) $$\min_{u \in F} (Au,u) \geq c_n + \alpha$$

(b) F is not contractible within the set $R = \{u \in H;\ Pu \neq 0\}$.

If (2^*) holds, then $S \cap \tilde{E}_n \in N^*_\alpha$ for some $\alpha > 0$ ($\tilde{E}_n$ denotes the linear hull of the first n eigenvectors of the equation $\tilde{P}Au = \tilde{\lambda} u$). If in (3^*) strong inequality holds, then the set $F = \left\{ \frac{u+v}{\|u+v\|} ;\ u \in S^*_n \right\}$ belongs to N^*_α for a suitable $v \in V^*$ and $\alpha > 0$ (cf. [2,3]). Hence in both cases $N^*_\alpha \neq \emptyset$ for some $\alpha > 0$ and the remaining part of the proof is nearly the same as in [1].

2. If (3^*) holds and $\lambda_n = c_n + \inf_{v \in V^*} \{ c_n \|v\|^2 - (Av,v) \}$,

then put $A_k u = (1+ \frac{1}{k}) APu + A(I-P)u$, use the proved part of Theorem 4 for A_k and then use Lemma.

<u>Theorem 5</u>. Let $u_n \notin K$ and let the set V (see Theorem 2) be nonempty. Choose $v \in V$ and put

(6) $$d_n = \inf_{0 \leq s < \frac{1}{\sqrt{1+\|v\|^2}}} c_n(s) , \quad \text{where} \quad c_n(s) = \sup_{\substack{u \in S \cap K \\ Pu = su_n}} (Au,u)$$

Suppose

(3^{**}) $$\lambda_n > d_n + d_n \|v\|^2 - (Av,v) .$$

Then there exists $\lambda \in \sigma_K(A) \cap (d_n, \lambda_n)$.

<u>Remark 3</u>. Obviously $d_n \leq c_n = c_n(0)$.

<u>Remark 4</u>. The assumption $v \in V$ guarantees that the set

$\{u \in S \cap K \ ; \ Pu = su_n\}$ is nonempty for all $|s| \leq \frac{1}{\sqrt{1+\|v\|^2}}$.

Remark 5. In (6) we could put $d_n = \inf\limits_{\substack{z \in E_n \\ \|z\|^2 < \frac{1}{1+\|v\|^2}}} \sup\limits_{\substack{u \in S \cap K \\ Pu = z}} (Au,u)$,

but then we would loose the estimate $\lambda \leq \lambda_n$.

Remark 6. There can be stated an analogous condition (2^{**}).

Idea of the proof of Theorem 5:

There exists $s \in \langle 0, \frac{1}{\sqrt{1+\|v\|^2}})$ such that

$$\lambda_n > c_n(s) + c_n(s)\|v\|^2 - (Av,v) .$$

We define N_α^{**} the class of all compact sets $F \subset K \cap S$ such that

(a^{**}) $\min\limits_{u \in F} (Au,u) \geq c_n(s) + \alpha$

(b^{**}) F is not contractible within the set $R(s) = \{u \in H ; \ Pu \neq su_n\}$.

Then the set $F = \{ \frac{u+v}{\|u+v\|} \ ; \ u \in S_n \}$ belongs to N_α^{**} for a suitable $\alpha > 0$ and one can use the technique from [1] to obtain the desired result.

4. EXAMPLES AND REMARKS

Example 1. Let $H = R_3$, $A([x_1,x_2,x_3]) = [\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3]$, $\lambda_1 > \lambda_2 > \lambda_3 > 0$, $K = \{u \in H ; \ (u,w_1) \geq 0, \ (u,w_2) \geq 0\}$, where $w_1 = [M(a-1),-1,a]$, $w_2 = [M(a-1),a,-1]$ $(a > 1, \ M > 0)$.

Let us fix $a > 1$. Using elementary calculus we get that the problem (1) has an eigenvalue $\lambda \neq \lambda_1$ if and only if

$$M \leq M_1 = \sqrt{c(A)\,\frac{a+1}{a-1} - \frac{1}{2}}\ , \quad \text{where} \quad c(A) = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{2(\lambda_2 - \lambda_3)}$$

(and it has exactly two eigenvalues different from λ_1 iff $M < M_1$).

Theorem 5 is available (with n=1) if $M < M_1$, using Lemma we get the positive result also for $M = M_1$.

Theorem 4 is available if $M \leq M_2 = \sqrt{c(A)\,\frac{a^2+1}{a^2-1} - \frac{1}{2}}\ (< M_1)$,

Theorem 3 is available only for $M < M_3 = \sqrt{c(A) - \frac{1}{2}}\ \ (< M_2)$.

Unfortunately, using our variational approach we get (for $M < M_1$) only one of two existing eigenvalues different from λ_1. We do not get the eigenvector $u \in \partial K \cap S$, where the functional (Au,u) attains a local minimum on $\partial K \cap S$ (∂K denotes the boundary of K).

Example 2. Let H,K,A satisfy the general assumptions from Section 1. Let $u_1, \dots, u_n \in K^o$, $\lambda_n > \lambda_{n+1}$ and $\{u_1, \dots, u_n\}^{\perp} \cap K^o = \emptyset$ $\ (\Leftrightarrow V = \emptyset)$. Suppose $u_k \notin K$ for $k > n$. Then the problem (1) has no eigenvalue λ with $\lambda < \lambda_n$.

Proof. Suppose $\lambda < \lambda_n$, $u \in K$, $(\lambda u - Au, v-u) \geq 0$ for all $v \in K$. Let us write $u = \sum_{i=1}^{n} \alpha_i u_i + w$, where $w \in E_n^{\perp}$.

Putting $v = u + u_i$ we get $(\lambda - \lambda_i)\alpha_i \geq 0$, thus $\alpha_i \leq 0$ $(i = 1, \dots, n)$. Suppose $\alpha_i < 0$ for some i, then

$-\sum_{i=1}^{n} \alpha_i u_i \in K^o$, $\quad w = u - \sum_{i=1}^{n} \alpha_i u_i \in K^o$, which gives us a contradiction. Thus $\alpha_i = 0$ for all $i = 1, \dots, n$, $u = w$.

Putting $v=u+u_1+\tilde{w}$, $\tilde{w}\in E_n^\perp$ arbitrary (but small), we get $\lambda u=Au$. Since $u_k\notin K$ for $k>n$, we have $u=0$.

Remark 7. Let $f:H\to R$ be a weakly continuous functional of the class C^2, $f'(0)=0$ and let the second Fréchet derivative f'' be bounded (on bounded sets). Denote $A=f''(0)$ and suppose that A fulfils the assumptions of Section 1. Then the eigenvalue λ to (1), which we get in Theorems 1-5, is also a bifurcation point for the variational inequality

(7) $\quad u\in K: \quad (\lambda u - f'(u),\ v-u) \geq 0 \quad$ for all $v\in K$

(see [1]). The following example shows that a general eigenvalue λ to (1) (which is not an eigenvalue of the operator A) need not be a bifurcation point for (7).

Example 3. Let $H = R_3$, let $A:H\to H$ be a symmetric linear operator with eigenvalues $\lambda_1>\lambda_2>\lambda_3>0$ and corresponding eigenvectors u_1,u_2,u_3. Put

$K=\{u\in H;\ (u,u_1)\geq 0,\ (u,u_3-u_2)\geq 0\}$, $\quad f(u)=\frac{1}{2}(Au,u)+\|u\|^2(u,u_1)$.

Then $u=u_2+u_3$ is an eigenvector to (1) with $\lambda=\frac{\lambda_2+\lambda_3}{2}$, since $(\lambda u-Au,v-u)=\frac{1}{2}(\lambda_2-\lambda_3)(u_3-u_2,v)\geq 0$ for all $v\in K$.

Suppose $u\in K$, $\lambda\leq\lambda_1$ and $(\lambda u-f'(u),v-u)\geq 0$ for all $v\in K$. Putting $v=u+u_1$ we get

$$0\leq(\lambda u-f'(u),u_1)=(\lambda u-Au-\|u\|^2u_1-2u(u,u_1),u_1)=$$

$$=-\|u\|^2+(u,u_1)(\lambda-\lambda_1-2(u,u_1))\leq-\|u\|^2\ ,$$

thus $u=0$. Hence $\lambda=\frac{1}{2}(\lambda_2+\lambda_3)$ is not a bifurcation point for (7).

Remark 8. Suppose that the assumptions from Section 1 are fulfilled. Then the set $\sigma_K(A)$ is nonempty and closed in $R^+ = \{\lambda \in R; \lambda > 0\}$. It would be interesting to investigate the general structure of $\sigma_K(A)$. Example 2 shows that this set may consist only of one point (also for $\dim H = \infty$), Theorems 1-5 assure the existence of higher eigenvalues to (1). There can be constructed examples in R_3, for which the set $\sigma_K(A)$ has infinitely many accumulation points (see Example 5). Nevertheless, it can be proved that for $H=R_3$ the set $\sigma_K(A) \subset R$ has Lebesgue measure zero (this is not true for A nonsymmetric). It is also an open problem (to the author) to find reasonable assumptions on A and K (for $\dim H = \infty$) which would guarantee that the set $\sigma_K(A)$ consists of a sequence of eigenvalues which converge to zero (cf. the following example).

Example 4. Let H be the Hilbert space $W_0^{1,2}(0,\pi)$ with the inner product $(u,v) = \int_0^\pi u'(x)v'(x)\,dx$, let $A:H \to H$ be defined by $(Au,v) = \int_0^\pi u(x)v(x)\,dx$. Let $M \subsetneqq \langle 0,\pi \rangle$ be a closed set and put $K = \{u \in H;\ u \geq 0 \text{ on } M\}$. Then it can be shown that the eigenvalues to (1) form a sequence converging to zero.

Example 5. Let $H = R_3$, let $A:H \to H$ be a symmetric linear operator with eigenvalues $\lambda_1 = \lambda_2 > \lambda_3 > 0$ and corresponding eigenvectors u_1, u_2, u_3.

Put $w_n = \sqrt{1-\frac{1}{n}}\,u_1 + \sqrt{\frac{1}{n}}\,u_2$, $\quad v_n = u_3 + \frac{w_n + w_{n+1}}{1+(w_n, w_{n+1})}$,

$K = \{u \in H;\ (u, u_3 - w_n) \geq 0 \text{ for each } n=1,2,3,\dots\}$.

Then $v_n \in K$, $\lambda^n = \frac{(Av_n, v_n)}{\|v_n\|^2} = \frac{2\lambda_1 + \lambda_3(1+(w_n, w_{n+1}))}{3+(w_n, w_{n+1})} \searrow \frac{1}{2}(\lambda_1+\lambda_3)$,

$$(\lambda^n v_n - Av_n, v) = \frac{\lambda_1-\lambda_3}{3+(w_n, w_{n+1})}(2u_3 - w_n - w_{n+1}, v) \geq 0 \quad \text{for all} \quad v \in K,$$

hence $\lambda^n \in \sigma_K(A)$ and $\sigma_K(A)$ contains a non-zero accumulation point.

If we put $w_{n,k} = r_k\sqrt{1-\frac{1}{k}-\frac{1}{n}}\,u_1 + \sqrt{\frac{1}{k}+\frac{1}{n}}\,u_2$,

where $n > k^2$ and $k > 1$ are natural numbers, $r_2 = 1$,

$r_{k+1}^2 = r_k^2 + \frac{1}{8(k+1)^5}$, and $v_{n,k} = u_3 + \frac{w_{n,k}+w_{n+1,k}}{1+(w_{n,k}, w_{n+1,k})}$,

$K = \{u \in H;\ (u, u_3 - w_{n,k}) \geq 0 \ \text{ for } n > k^2 > 1\}$, then again $v_{n,k}$ is an eigenvector to (1) and $\sigma_K(A)$ contains infinitely many accumulation points $\lambda(k)$, where

$$\lambda(k) = \lim_{n\to\infty} \lambda^{n,k} = \frac{\lambda_1(\frac{1}{k}+r_k^2(1-\frac{1}{k}))+\lambda_3}{\frac{1}{k}+r_k^2(1-\frac{1}{k})+1} \nearrow \frac{\lambda_1 r^2+\lambda_3}{r^2+1}$$

$(r = \lim_{n\to\infty} r_k)$.

Similar example can be constructed also for $\lambda_1 > \lambda_2 > \lambda_3$ (we start with $w_n = c\sqrt{1-\frac{1}{n^2}}\,u_1 + \frac{1}{cn}u_2$, where $c^2(\lambda_2-\lambda_3) = \lambda_1-\lambda_3$).

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ARCHIMEDEAN AND GEODETICAL BIEQUIVALENCES

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Abstract: This paper contributes to the topological problematics in the AST. The central role in it is due to the concept of a biequivalence introduced in [G-Z 1] . A metrization theorem for biequivalences is established. Two properties of biequivalences bearing upon the connectedness of galaxies named in the title are formulated and characterized. The notions of a path and a motion of point appear as powerful tools in formulations and proofs of the results.

Key words: Biequivalence, path, motion, compact, connected, metric, galaxy, Archimedean, direct, geodetical.

Classification: Primary 54J05
Secondary 54D05, 54E35

This paper is a direct continuation of [G-Z 1] contributing to the topological problematics in the AST. The central role in it is played by the concept of a biequivalence introduced in [G-Z 1] . The article joins results of two areas of "biequivalence problematics" originally occurring rather independent.

The first part is devoted to the characterization of biequivalences $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ such that for each mean bound R and each pair $x \overset{+}{\leftrightarrow} y$ there is a finite R-path from x to y (Archimedean biequivalences). The formulation and the proof of the result itself are preceeded by a section dealing with paths and motions.

The second part of our work was iniciated by the unexpectedly easy (using the results of [M 2]) proof of the metrization theorem for arbitrary biequivalences. From this theorem some results, analogous to those from the classical topology, on embeddings of a biequivalence with a set domain u into the linear space RN^u endowed with the componentwise biequivalence easily follow. If one would like to generalize these results to arbitrary biequivalences, he will find unavoidable to extend suitably the field of rational numbers. That's why we sketch the construction of hyperreal numbers in the AST.

Keeping the fact that every biequivalence is induced by some metric, in mind, there arise several questions under what conditions such a metric abundant in some further useful properties can be found. One particular problem of this type is solved in the paper. Namely, biequivalences which can be induced by a metric H such that each pair of accessible points can be joined by a direct motion with respect to H are characterized (geodetical biequivalences).

Both the notions of Archimedean and geodetical biequivalences illustrate the "restriction principle to galaxies" mentioned in [G-Z 1] . Via the concept of a motion of point they bear upon some questions concerning the connectedness of galaxies of a biequivalence, as well.

The reader is assumed to be acquainted with [V] and [G-Z 1] . Most of the notions and results from these two sources will be used even without any explicit referring to them.

1. Paths, motions and connectedness

Z denotes the set-theoretically definable class of all integers and FZ stands for finite integers. Variables $\alpha, \beta, \gamma, \lambda, \mu, \nu, \ldots$ (k,m,n) are used sometimes for arbitrary (finite) integers, not just natural numbers. The interval of integers between μ, ν is denoted by $[\mu, \nu] = \{\lambda \in Z;\ \mu \leq \lambda \leq \nu\}$. In particular $[\mu, \nu] = \emptyset$

if $\mu > \nu$, $[\mu,\mu] = \{\mu\}$ and $\nu = [0, \nu - 1]$ for each $\nu \in N$.

In the whole paper $\langle \doteq, \dot{\leftrightarrow} \rangle$ denotes the usual biequivalence on the class RN of all rational numbers (see [G-Z 1] Example 3). For any set u $\langle \doteq^u, \dot{\leftrightarrow}^u \rangle$ denotes the biequivalence on $RN^u = \{f;\ dom(f) = u \ \& \ rng(f) \subseteq RN\}$ arising from $\langle \doteq, \dot{\leftrightarrow} \rangle$ componentwise ([G-Z 1] Example 5). For rational numbers a,b we put $a \dot{\leq} b \equiv a < b \vee a \doteq b$ and $a <\cdot b \equiv a < b \ \& \ a \not\doteq b$. The formulation of the basic properties of the relations $\dot{\leq}$ and $<\cdot$ is left to the reader.

Let us record a result for the future.

Lemma. Let R be a π-relation and $u \subseteq dom(R)$ be a set. Then there is a set function $f \subseteq R$ such that $dom(f) = u$.

Proof. If R is set-theoretically definable, the statement can be easily proved by induction. Let $\{R_n;\ n \in FN\}$ be a decreasing sequence of set-theoretically definable relations whose intersection is R. For each n there is a function $f_n \subseteq R_n$ with domain u. Then the result follows by the axiom of prolongation.

Let R be an arbitrary relation. A (set) function p such that $dom(p) = [\eta, \vartheta]$ is a nonvoid interval of integers is called an R-path provided for each $\alpha \in [\eta, \vartheta - 1]$ holds $\langle p(\alpha), p(\alpha+1) \rangle \in R$. Then the set $rng(p)$ is called the trace of p. If $x = p(\eta)$ and $p(\vartheta) = y$ then p is called an R-path from x to y. In most cases the domains of the paths considered will be of form $[0, \vartheta] = \vartheta + 1$ where $\vartheta \in N$; in such a case p will be called an R-path in the time ϑ. Thus $\langle x,y \rangle \in R^{\vartheta}$ iff there is an R-path from x to y in the time ϑ.

If $\doteq$ is a π-equivalence then any ($\doteq$)-path is called a motion of point in $\doteq$. If $\{R_n;\ n \in FN\}$ is a generating sequence of $\doteq$ then obviously p is a motion of point in $\doteq$ iff p is an R_n-path for each n. We will frequently say "a motion" instead of "a motion of point in $\doteq$", mainly in the case when the $\mathfrak{T}$-equivalence $\doteq$ will be clear from the context.

From the results in [V] it follows directly:

<u>Theorem 1</u>. Let $\doteq$ be a $\mathfrak{T}$-equivalence and u be a set. The following conditions are equivalent:

(1) for each nonempty proper subset v of u there are two points $x \in v$, $y \in u - v$ such that $x \doteq y$;

(2) there is a motion p such that $u = \mathrm{rng}(p)$;

(3) for all $x, y \in u$ there is a motion p from x to y such that $\mathrm{rng}(p) \subseteq u$.

According to Theorem 1 we addopt the following definition: A class X is connected in the $\mathfrak{T}$-equivalence $\doteq$ if for all $x, y \in X$ there is a motion p from x to y such that $\mathrm{rng}(p) \subseteq X$.

<u>Theorem 2</u>. Let $\doteq$ be a $\mathfrak{T}$-equivalence and X be a class. If X is connected then Fig(X) is also connected. If X is a $\mathfrak{T}$-class and Fig(X) is connected then X is connected, as well.

<u>Proof</u>. Let X be connected and $a \doteq x$, $b \doteq y$ where $x, y \in X$. If p is a motion from x to y in the time ϑ such that $\mathrm{rng}(p) \subseteq X$, then $q = p \cup \{\langle a,-1\rangle , \langle b,\vartheta+1\rangle\}$ is a motion from a to b and $\mathrm{rng}(q) \subseteq \mathrm{Fig}(X)$. Now, let X be a $\mathfrak{T}$-class with connected figure and $x, y \in X$. There is a motion p from x to y in the time ϑ such that $\mathrm{rng}(p) \subseteq \mathrm{Fig}(X)$. Put

$$R = \{\langle x,0\rangle , \langle y,\vartheta\rangle\} \cup \{\langle z,\alpha\rangle ; 0 < \alpha < \vartheta \;\&\; z \in X \;\&\; z \doteq p(\alpha)\}.$$

Then R is a $\mathfrak{T}$-relation and $[0,\vartheta] = \mathrm{dom}(R)$. The choice function $f \subseteq R$ with domain $[0,\vartheta]$ existing by the virtue of the Lemma is a motion from x to y and $\mathrm{rng}(f) \subseteq X$.

Each motion p in the $\mathcal{S}$-equivalence $\doteq$ induces two $\mathcal{S}$-equivalences on its domain. The first one seems to be prima facie more natural: $\alpha \underset{(p)}{\doteq} \beta \equiv p(\alpha) \doteq p(\beta)$.

<u>Theorem 3</u>. Let p be a motion in $\doteq$. The $\mathcal{S}$-equivalence $\underset{(p)}{\doteq}$ is compact iff $\mathrm{rng}(p)$ is compact in $\doteq$.

<u>Proof</u>. Clearly, $\underset{(p)}{\doteq}$ is a $\mathcal{S}$-equivalence. Let $\doteq$ be compact and $u \subseteq \mathrm{rng}(p)$ be infinite. If for all $\alpha, \beta \in p^{-1}$"$u = v$ $p(\alpha) \neq p(\beta)$ implied $p(\alpha) \not\doteq p(\beta)$, v would be an infinite set of pairwise discernible elements contradicting the compactness of $\doteq$. Thus there are $\alpha, \beta \in v$ such that $p(\alpha) \neq p(\beta)$ and $p(\alpha) \doteq p(\beta)$. Now, assume that $\mathrm{rng}(p)$ is compact in $\doteq$ and $v \subseteq \mathrm{dom}(p)$ is infinite. If p"v is finite then there are at least two elements $\alpha, \beta \in v$ such that even $p(\alpha) = p(\beta)$. If p"v is infinite then there are two $\alpha, \beta \in v$ such that $p(\alpha) \doteq p(\beta)$.

The second $\mathcal{S}$-equivalence induced by the motion p in $\doteq$ on its domain is even of more importance:

$$\alpha \underset{p}{\doteq} \beta \equiv (\forall \gamma, \delta \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}]\ )\ p(\gamma) \doteq p(\delta).$$

The motion p is called compact if the $\mathcal{S}$-equivalence $\underset{p}{\doteq}$ is compact. Obviously, $\underset{p}{\doteq}$ is finer than $\underset{(p)}{\doteq}$, thus the compactness of $\underset{p}{\doteq}$ implies the compactness of $\underset{(p)}{\doteq}$.

Let us recall from [V] that a motion p oscillates between points x and y (sets u and v) if $x \not\doteq y$ ($\mathrm{Fig}(u) \cap \mathrm{Fig}(v) = \emptyset$) and there are sequences $\{\alpha_n;\ n \in FN\}$, $\{\beta_n;\ n \in FN\}$ of elements of $\mathrm{dom}(p)$ such that for each n holds $\alpha_n < \beta_n < \alpha_{n+1}$ and $p(\alpha_n) \doteq x$, $p(\beta_n) \doteq y$ ($p(\alpha_n) \in u$, $p(\beta_n) \in v$).

We omit the proof of the following slight generalization of the result from [V]:

<u>Theorem 4</u>. The following conditions are equivalent for any

motion p in the $\mathfrak{X}$-equivalence $\doteq$:

(1) p is compact;

(2) the trace of p is compact in $\doteq$ and for each point x p^{-1}"Mon(x) is compact in $\underset{p}{\doteq}$;

(3) p has a compact trace and there are no points x and y (sets u and v) such that p oscilates between x and y (u and v).

Remark. Given a $\mathfrak{X}$-equivalence $\doteq$ and any set-theoretically definable function F the relation $a \underset{(F)}{\doteq} b \equiv F(a) \doteq F(b)$ is still a $\mathfrak{X}$-equivalence on its domain. Theorem 3 remains valid without the assumption that F is a motion, as well. Similarly, if F is a set-theoretically definable function and $\leq$ is a set-theoretical lattice ordering of dom(F), the definitions of the $\mathfrak{X}$-equivalence $\underset{F}{\doteq}$ on dom(F) and of the oscillation extend directly. A careful analysis of the proofs in [V] shows that Theorem 4 still holds for such an F.

Thus particularly Theorems 3 and 4 apply to arbitrary set functions defined on intervals of integers (V^2-paths). Given such a function p and a $\mathfrak{X}$-equivalence $\doteq$ we put for $\alpha, \beta \in \mathrm{dom}(p)$

$$\alpha \underset{p}{\stackrel{\pm}{\leq}} \beta \equiv \alpha < \beta \vee \alpha \underset{p}{\doteq} \beta \quad \text{and}$$
$$\alpha \underset{p}{<+} \beta \equiv \alpha < \beta \ \& \ \alpha \underset{p}{\not\doteq} \beta .$$

2. Archimedean biequivalences

For each upper bound R of the $\mathfrak{X}$-equivalence $\doteq$ the least equivalence

$$[R] = \bigcup \{R^n ; n \in FN\}$$

containing R raises to a biequivalence $\langle \doteq, [R] \rangle$. This construction was already used in the proof of Theorem 10 in [G-Z 1]. Similarly, for each mean bound R of the biequivalence $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ one obtains a biequivalence $\langle \overset{+}{=}, [R] \rangle$ which is tighter than $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$. A biequivalence is called Archimedean if for each its mean bound R holds $[R] = (\overset{+}{\leftrightarrow})$.

<u>Theorem 5</u>. Let $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ be a biequivalence. The following conditions are equivalent:

(1) $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ is Archimedean;

(2) for each mean bound R of $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ and all x,y such that $x \overset{+}{\leftrightarrow} y$ there is a finite R-path from x to y;

(3) for all x,y such that $x \overset{+}{\leftrightarrow} y$ and each infinite natural number ν there is a motion p from x to y such that $p \gtrsim \nu$;

(4) $\overset{+}{\leftrightarrow}$ is the least σ-equivalence with respect to inclusion containing $\overset{+}{=}$;

(5) $\overset{+}{\leftrightarrow}$ is a minimal σ-equivalence with respect to inclusion containing $\overset{+}{=}$.

<u>Proof</u>. $(1) \equiv (2)$ and $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$ are trivial.
$(2) \Rightarrow (3)$: Let $x \overset{+}{\leftrightarrow} y$, $\nu \in N-FN$ and $\{R_n; n \in FZ\}$ be a bi-generating sequence of $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$. For each $n \leqslant 0$ there is a finite R_n-path p_n from x to y. By the prolongation axiom there is a motion p from x to y such that $p \gtrsim \nu$.
$(3) \Rightarrow (2)$: Let R be a mean bound of $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ and $x \overset{+}{\leftrightarrow} y$. Then the set-theoretically definable class $\{\nu \in N; \langle x,y \rangle \in R^{\nu}\}$ contains all infinite natural numbers. Hence it contains also a finite n.

None of conditions (4) and (5) ensures that for an Archimedean biequivalence $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ there does not exist any biequivalence strictly tighter than $\langle \stackrel{+}{=}, \stackrel{+}{\longleftrightarrow} \rangle$. The reader will easily find examples of biequivalences on RN which are strictly tighter than the compatible Archimedean biequivalence $\langle \dot{=}, \stackrel{\cdot}{\longleftrightarrow} \rangle$.

Condition (3) suggests that the Archimedean property is a kind of compactness concerning connectedness of galaxies of a biequivalence.

<u>Theorem 6</u>. Let $\langle \stackrel{+}{=}, \stackrel{+}{\longleftrightarrow} \rangle$ be a biequivalence such that for any accessible x,y there is a motion with compact trace from x to y. Then $\langle \stackrel{+}{=}, \stackrel{+}{\longleftrightarrow} \rangle$ is Archimedean and has connected galaxies.

<u>Proof</u>. We will prove that $\langle \stackrel{+}{=}, \stackrel{+}{\longleftrightarrow} \rangle$ satisfies condition (2) of Theorem 5. Let R be a mean bound of $\langle \stackrel{+}{=}, \stackrel{+}{\longleftrightarrow} \rangle$, $x \stackrel{+}{\longleftrightarrow} y$, and p be a motion with compact trace from x to y in the time ϑ . We put $G(0) = x$ and $G(\alpha+1) = p(\mu)$ where $\mu = \max \{\gamma \leq \vartheta;\ \langle G(\alpha), p(\gamma)\rangle \in R\}$ if $G(\alpha) \neq y$, and $G''\{\alpha+1\} = \emptyset$ if $G(\beta) = y$ for some $\beta \leq \alpha$. Then G is a set-theoretically definable function and its domain is a section in the linearly ordered class $\langle N, \leq \rangle$. For any $\alpha, \beta \in \mathrm{dom}(G)$ $\alpha < \beta$ implies $G(\alpha) \not\stackrel{+}{=} G(\beta)$ (with perhaps one exception $\beta = \alpha + 1$ and $G(\beta) = y$). Since $\mathrm{rng}(G) \subseteq \mathrm{rng}(p)$ and the latter is compact, the former has to be a finite set. As G is one-one, it is a finite set function and a finite R-path from x to y. Essentially the same argument works to establish that for any motion p with compact trace and for all $\alpha, \beta \in \mathrm{dom}(p)$ holds $p(\alpha) \stackrel{+}{\longleftrightarrow} p(\beta)$. Thus $\langle \stackrel{+}{=}, \stackrel{+}{\longleftrightarrow} \rangle$ has connected galaxies.

<u>Corollary</u>. Every compatible biequivalence with connected galaxies is Archimedean.

As a byproduct of the proof of Theorem 6 one obtains:

<u>Theorem 7</u>. Let $\langle \stackrel{+}{=}, \stackrel{+}{\longleftrightarrow} \rangle$ be a biequivalence and X be a pseudocompact connected class in $\stackrel{+}{=}$. Then for all $x,y \in X$ holds

$x \overset{+}{\leftrightarrow} y$.

Notice that both the notions of pseudocompactness and connectedness are defined purely in terms of the σ-equivalence $\overset{+}{=}$. Thus Theorem 7 applies to any σ-equivalence $(\overset{+}{\leftrightarrow}) \supseteq (\overset{+}{=})$.

<u>Example 1</u>. Put $A = \{ \langle x,y\rangle \in RN^2;\ x^2y^2 = 1 \}$. Then the biequivalence $\langle \overset{\bullet}{=}^2 \upharpoonright A, \overset{\bullet}{\leftrightarrow}^2 \upharpoonright A \rangle$ is compatible, Archimedean and its domain A is connected. However, $\mathrm{Gal}(\langle 1,1\rangle)$ is not connected. Moreover, there is no motion with compact trace from $\langle -1,1\rangle$ to $\langle 1,1\rangle$.

<u>Example 2</u>. For every set u the biequivalence $\langle \overset{\bullet}{=}^u, \overset{\bullet}{\leftrightarrow}^u \rangle$ on RN^u is Archimedean with connected galaxies and for any pair $f \overset{\bullet}{\leftrightarrow}^u g$ there is even a compact motion from f to g (it can be defined for any $\nu \in N - FN$ by $p(\alpha)(x) = (\alpha/\nu) f(x) + (1-\alpha/\nu) g(x)$ for $\alpha \in \nu + 1$, $x \in u$). Nevertheless, for infinite u $\langle \overset{\bullet}{=}^u, \overset{\bullet}{\leftrightarrow}^u \rangle$ is not compatible.

<u>Example 3</u>. Let α be an infinite natural number. Put $R_n = \{ \langle x,y\rangle \in RN^2;\ |x - y| < \alpha^n \}$ for each $n \in FZ$. Then the biequivalence with the bigenerating sequence $\{ R_n;\ n \in FZ \}$ has connected galaxies, connected domain and is neither compatible nor Archimedean.

<u>Example 4</u>. Let $\nu \in N - FN$. Let us endow $RN^{\nu+1}$ with the structure of a linear space over RN in the obvious way defining the vedtor addition and the multiplication by scalars componentwise. Put

$$A = \{ f \in RN^{\nu+1};\ f(\nu) = 1 \ \&\ \mathrm{rng}(f) \subseteq \{-1,1\} \},$$

$$B = \{ tf + (1-t)g;\ t \in RN \ \&\ 0 \leq t \leq 1 \ \&\ f,g \in A \ \&\ \{ \lambda \leq \nu;\ f(\lambda) \neq g(\lambda) \} \mathrel{\hat{\approx}} 1 \},$$

$$T = \{ \langle g,f \rangle \in RN^{\nu+1} \times RN^{\nu+1};\ g(\nu) \geq 1 \ \& \ (\forall \lambda < \nu)\ g(\lambda)\ g(\nu) = f(\lambda) \} .$$

Then the set-theoretically definable class $X = B \cup T''A$ consists of the edges of the ν-dimensional hypercube with vertices coordinates ± 1 situated in the hyperplane $f(\nu) = 1$ in $RN^{\nu+1}$ and of parts of arcs of the hyperbolas running through the vertices of the cube to the common assymptote $f(0) = \dots = f(\nu-1) = 0$. Then the biequivalence $\langle \doteq^{\nu+1} \upharpoonright X, \overset{\cdot}{\leftrightarrow}^{\nu+1} \upharpoonright X \rangle$ is Archimedean with connected galaxies and connected domain. However, there is no motion with compact trace from $\{1\} \times (\nu + 1)$ to $(\{-1\} \times \nu) \cup \{\langle 1, \nu \rangle\}$ in X.

<u>Example 5</u>. In this Example $[a,b] = \{x \in RN;\ a \leq x \leq b\}$ always denotes the interval of rationals between $a,b \in RN$. Put $I_0 = [0,1] \times \{0\}$ and $I_\alpha = [0,1] \times \{1/\alpha\}$ for $\alpha \in N - \{0\}$,

$$J_\alpha = \begin{cases} \{0\} \times [1/(\alpha+1),\ 1/\alpha] & \text{for even } \alpha \in N - \{0\} \\ \{1\} \times [1/(\alpha+1),\ 1/\alpha] & \text{for odd } \alpha \in N - \{0\} \end{cases}$$

Finally $A = I_0 \cup \bigcup \{ I_\alpha \cup J_\alpha;\ \alpha \in N - \{0\}\}$ is a set-theoretically definable class. Then $\doteq^2 \upharpoonright A$ is a compact π-equivalence (the biequivalence $\langle \doteq^2 \upharpoonright A, A^2 \rangle$ is compatible) with connected domain A (its single galaxy). Thus each motion in $\doteq^2 \upharpoonright A$ has a compact trace and, in particular, $\langle \doteq^2 \upharpoonright A, A^2 \rangle$ is an Archimedean biequivalence. Nevertheless, there is no compact motion from $\langle 0,0 \rangle$ to $\langle 0,1 \rangle$ in $\doteq^2 \upharpoonright A$ since every such a motion has to oscillate between the points $\langle 0,0 \rangle$ and $\langle 1,0 \rangle$.

3. The metrization theorem and embeddings of biequivalences

For the rest of the article Sd_V^* denotes a fixed revealment of the codable class Sd_V of all set-theoretically definable classes (see [S-V 2]). Everything one needs to know is that Sd_V^* is a fully revealed codable class (i.e. there is a code $\langle K,S \rangle$ of Sd_V^* such that the class $K \times \{0\} \cup S \times \{1\}$ is fully revealed) satisfying the following conditions:

(1) $Sd_V \subseteq Sd_V^*$ and each class $X \in Sd_V^*$ is fully revealed;

(2) for each set u and each $X \in Sd_V^*$ $u \cap X$ is a set;

(3) for each normal formula $\varphi(x_0,X_0)$ of the language FL_V and each $X \in Sd_V^*$ holds $\{x; \varphi(x,X)\} \in Sd_V^*$;

(4) if $X \in Sd_V^*$ and $X \cap N \neq \emptyset$ then there is the least element of $X \cap N$ in the natural ordering of N;

(5) if $X \in Sd_V^*$, $\emptyset \in X$ and $(\forall x,y)(x \in X \Rightarrow x \cup \{y\} \in X)$ then $X = V$ (induction);

(6) if $X \in Sd_V^*$ and $(\forall x)(x \subseteq X \Rightarrow x \in X)$ then $X = V$ ($\in$-induction);

(7) if $\{X_n; n \in FN\}$ is a sequence of classes from Sd_V^* then there is an $R \in Sd_V^*$ such that $R''\{n\} = X_n$ for each $n \in FN$ (prolongation).

According to (1) - (7), Sd_V^* should be understood as a "well behaved" system of "well behaved" classes conveniently extending Sd_V admitting "well behaved" prolongations of countable sequences of its members.

On the base of Sd_V^* such notions as $^*\pi$-class, $^*\sigma$-class, $^*\pi$- and $^*\sigma$-equivalence, *generating sequence of a $^*\pi$- or of a $^*\sigma$-equivalence, *bigenerating sequence, et cetera, can be defined in the obvious way. (E.g. X is a $^*\pi$-class if there is a sequence $\{X_n;\ n \in FN\}$ of classes from Sd_V^* such that $X = \bigcap \{X_n;\ n \in FN\}$; or a *bigenerating sequence is a sequence $\{R_n;\ n \in FZ\}$ of reflexive and symetric relations from Sd_V^* such that for each n holds $R_n \circ R_n \subseteq R_{n+1}$.) The reader should think over that any result concerning the "star-free" notions from [V], [G-Z 1] or from this article remains true under an appropriate "starification". Finally, notice that the restriction of a *biequivalence to a set is always a biequivalence.

For the sake of transparency we shall deal also with *biequivalences with domains different from V. A triple $\langle X, \overset{+}{=}, \overset{+}{\longleftrightarrow} \rangle$ where $\langle \overset{+}{=}, \overset{+}{\longleftrightarrow} \rangle$ is a *biequivalence and $\emptyset \neq X \in Sd_V^*$ is its domain will be called a *biequivalence space.

Let $\{R_n;\ n \in FZ\}$ be a *bigenerating sequence (of some *biequivalence $\langle \overset{+}{=}, \overset{+}{\longleftrightarrow} \rangle$). The sequence $\{R_\nu;\ \nu \in [\sigma, \tau]\}$ where $-\sigma, \tau \in N - FN$ is called a prolongation of the *bigenerating sequence $\{R_n;\ n \in FZ\}$ in Sd_V^* if the class $S = \bigcup \{R_\nu \times \{\nu\};\ \nu \in [\sigma, \tau]\}$ belongs to Sd_V^*, for each $\nu \in [\sigma, \tau]$ $R_\nu \in Sd_V^*$ is a reflexive and symetric relation and $R_\nu \circ R_\nu \subseteq R_{\nu+1}$, $S''\{n\} = R_n$ holds for each $n \in FZ$, and $R_\sigma = Id$, $R_\tau = V^2$.

The following theorem is a direct consequence of the prolongation condition (7):

<u>Theorem 8</u>. Every *bigenerating sequence has a prolongation in Sd_V^*.

Let X be a nonempty class, H be a function with domain X^2 and $rng(H) \subseteq RN$. Then H is called a metric on X if for all $x, y, z \in X$ holds $H(x,y) \geq 0$, $H(x,y) = H(y,x)$, $H(x,y) + H(y,z) \geq H(x,z)$ and $H(x,y) = 0 \equiv x = y$. The weakening of the last condition to mere $H(x,x) = 0$ leads to the notion of pseudometric. When also metrics taking values in other ordered fields than RN will be considered, we will refer to the notion just defined as to a rational metric.

If H is a metric on X then the pair $\langle X,H \rangle$ is called a (rational) metric space. Given a metric space $\langle X,H \rangle$ we put for $x,y \in X$

$$x \doteq_H y \equiv H(x,y) \doteq 0 \quad \text{and}$$

$$x \overset{\cdot}{\leftrightarrow}_H y \equiv H(x,y) \overset{\cdot}{\leftrightarrow} 0.$$

The next theorem shows that the AST succeeded in a natural way completely to exclude the pathologies of nonmetrizable spaces from our study and to recure the balance between the topology and "measuring of distances" both on the discernibility and accessibility horizons. From this point of view the indiscernibility and accessibility equivalences occur as mere certain invariants of metric spaces.

__Theorem 9__. If $\langle X,H \rangle \in Sd_V^*$ is metric space (i.e. $H \in Sd_V^*$ is a metric on X) then $\langle \doteq_H, \overset{\cdot}{\leftrightarrow}_H \rangle$ is a *biequivalence with domain X. Conversely, for every *biequivalence space $\langle X, \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ there is a metric $H \in Sd_V^*$ on X such that $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle = \langle \doteq_H, \overset{\cdot}{\leftrightarrow}_H \rangle$.

__Proof__. The first assertion is trivial. The converse follows directly from the existence of a *bigenerating sequence for $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ (see [G-Z 1]) and from the γ- and/or σ-valuation lemma ([M 2]). All one has to do is to take a suitable prolongation of an appropriate *bigenerating sequence of $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ in Sd_V^*. We omit the precious proof which is, in fact, implicitly contained in [M 2] . The reader will be made amends in the next section where for a more specific class of (*)biequivalences a metric subject to some additional properties will be constructed using ideas similar to the dropped ones.

The *biequivalence $\langle \doteq_H, \leftrightarrow_H \rangle$ will be called the *biequivalence induced by the metric $H \in Sd_V^*$ (of course, $\langle \doteq_H, \leftrightarrow_H \rangle$ can be induced by many different metrics). Obviously, every metric $H \in Sd_V$ induces a biequivalence on its domain, though the converse is not true: there are biequivalences which cannot be induced by any set-theoretically definable metric.

The reader can easily verify that for any set u the biequivalence $\langle \doteq^u, \dot{\leftrightarrow}^u \rangle$ is induced by the set-theoretically definable metric

$$D(f,g) = \max \{ |f(t) - g(t)| ;\ t \in u \} .$$

A function $E: X \to X'$ is called an isometry of the metric space $\langle X,H \rangle$ into the metric space $\langle X',H' \rangle$ if for all $x,y \in X$ holds $H(x,y) = H'(E(x),E(y))$.

The following result is fairly expected in the light of the classical topology:

<u>Theorem 10</u>. Let $\langle u,h \rangle$ be a metric space (u and h are sets). Then the function $e: u \to RN^u$ given by $e(x)(t) = h(x,t)$ for $x,t \in u$ is an isometry of $\langle u,h \rangle$ into $\langle RN^u, D \rangle$, and for each $x \in u$ the function $e(x) \in RN^u$ satisfies

$$t \doteq_h z \Rightarrow e(x)(t) \doteq e(x)(z),$$

$$t \dot{\leftrightarrow}_h z \Rightarrow e(x)(t) \dot{\leftrightarrow} e(x)(z)$$

for all $t,z \in u$.

<u>Proof</u>. The fact that e is an isometry follows from the computation

$$D(e(x),e(y)) = \max \{ |h(x,t) - h(y,t)| ;\ t \in u \}$$

$$\leq h(x,y) = |e(x)(y) - e(y)(y)|$$
$$\leq D(e(x),e(y))$$

The rest of the Theorem follows from the inequality

$$|e(x)(t) - e(x)(z)| \leq e(t,z).$$

Thus in particular according to the results from [G-Z 1] each function $e(x)$ is uniformly continuous from $\doteq_h$ to $\doteq$.

Let $\langle X,\stackrel{+}{=},\stackrel{+}{\longleftrightarrow}\rangle$, $\langle X',\acute{=},\acute{\longleftrightarrow}\rangle$ be two *biequivalence spaces. A one-one map $E: X \longrightarrow X'$ is called an embedding of $\langle X,\stackrel{+}{=},\stackrel{+}{\longleftrightarrow}\rangle$ into $\langle X',\acute{=},\acute{\longleftrightarrow}\rangle$ iff for all $x,y \in X$ holds

$$x \stackrel{+}{=} y \equiv E(x) \acute{=} E(y) \quad \text{and}$$
$$x \stackrel{+}{\longleftrightarrow} y \equiv E(x) \acute{\longleftrightarrow} E(y).$$

An embedding of a *π-equivalence space $\langle X,\stackrel{+}{=}\rangle$ into another *π-equivalence space $\langle X',\acute{=}\rangle$ can be treated as an embedding of the *biequivalence space $\langle X,\stackrel{+}{=},X^2\rangle$ into the *biequivalence space $\langle X',\acute{=},X'^2\rangle$. Obviously, every isometry of the metric space $\langle X,H\rangle \in Sd_V^*$ into the metric space $\langle X',H'\rangle \in Sd_V^*$ is an embedding of the *biequivalence space $\langle X,\doteq_H,\stackrel{\cdot}{\longleftrightarrow}_H\rangle$ into the *biequivalence space $\langle X',\doteq_{H'},\stackrel{\cdot}{\longleftrightarrow}_{H'}\rangle$.

Then Theorems 9 and 10 have the following consequence:

<u>Theorem 11</u>. For every biequivalence space $\langle u,\stackrel{+}{=},\stackrel{+}{\longleftrightarrow}\rangle$ there is an embedding e of $\langle u,\stackrel{+}{=},\stackrel{+}{\longleftrightarrow}\rangle$ into $\langle RN^u,\doteq^u,\stackrel{\cdot}{\longleftrightarrow}^u\rangle$ such that all the functions $e(x) \in RN^u$ $(x \in u)$ are uniformly continuous from $\stackrel{+}{=}$ to $\doteq$.

Let $\stackrel{x}{=}$ be the π-equivalence on $RN \times u$ given by $\langle a,x\rangle \stackrel{x}{=} \langle b,y\rangle \equiv a \doteq b \mathbin{\&} x \stackrel{+}{=} y$. We also put for $f \in RN^u$

$$\|f\| = \sum_{t\in u}|f(t)|/Card(u).$$

The function e from Theorem 10 can be also regarded as an embedding of the $\mathfrak{X}$-equivalence space $\langle u, \stackrel{\pm}{=} \rangle$ into various $\mathfrak{X}$-equivalence spaces with domain RN^u using the results from [G-Z 1] .

Theorem 12. Let $\langle u, \stackrel{\pm}{=} \rangle$ be a $\mathfrak{X}$-equivalence space. Then there is a one-one set-function $e: u \rightarrow RN^u$ such that for all $x, y \in u$ the following conditions are equivalent:

(1) $x \stackrel{\pm}{=} y$; (2) $e(x) \doteq^u e(y)$; (3) $\mathrm{Fig}^*(e(x)) = \mathrm{Fig}^*(e(y))$.

If $u \in N$ and $\alpha \stackrel{\pm}{=} \beta \equiv \alpha/u \doteq \beta/u$ holds for all $\alpha, \beta \in u$ then the conditions (1) - (3) are equivalent to

(4) $\| e(x) - e(y) \| \doteq 0$.

If one would like to generalize the above embedding results to arbitrary *biequivalences (however, to deal with biequivalences with domain V is quite sufficient), he will find unavoidable to extend the ordered field of all rational numbers in such a way that every nonempty subclass of RN belonging to Sd_V^* which has an upper bound in RN had the supremum in the extension.

A nonempty proper subclass C of RN is called a cut of $\langle RN, \leqq \rangle$ if it is a section of $\langle RN, \leqq \rangle$ without the greatest element.

Then the fully revealed codable class HR of all cuts in RN belonging to Sd_V^* can be given the structure of an ordered field in the obvious way. It will be called the field of all hyperreal numbers. Using an appropriate coding of HR, of the equality relation on HR and of the operations and order relation on HR, one can work with it as if it were a class from the extended universe. RN can be naturally embedded as an ordered subfield into HR. Likewise HR can be endowed with a pair of relations $\langle \doteq, \leftrightarrow \rangle$ behaving as a biequivalence with domain HR prolonging the biequivalence from RN denoted by the same symbol in such a way that $HR = Fig(RN)$. A hyperreal number will be called set-theoretically definable if it is determined by a set-theoretically definable cut. The set-theoretically definable hyperreals contain all rationals and form an ordered subfield of the hyperreals. Each nonempty subclass of RN with an upper bound belonging to Sd_V has the supremum in that field. The reason why the field of all set-theoretically definable hyperreal numbers is an unsatisfactory extension of RN is that one cannot apply the prolongation technics in it. HR can be also obtained as a revealment of the field of all set-theoretically definable cuts in RN.

Each nonempty class $X \in Sd_V^*$, $X \subseteq RN$, with an upper (lower) bound in RN has the supremum sup X (infimum inf X) in HR. The ordered field HR is determined by its properties with respect to Sd_V^* uniquely up to an isomorphism. Also the hyperreal numbers constructed on the base of another revealment of Sd_V, say $Sd_V^{\sim}$, are isomorphic to "our" HR via the automorphism of the universe mapping Sd_V^* onto $Sd_V^{\sim}$ (see [S-V 2]).

Let us denote just for a moment

$$RN^X = \{F \in Sd_V^*;\ dom(F) = X \ \&\ rng(F) \subseteq RN \ \& \\ (\exists a \in RN)(\forall x \in X)\ |F(x)| < a\}$$

the codable class of all bounded rational functions with domain X belonging to Sd_V^* (clearly $RN^X \neq \emptyset$ iff $X \in Sd_V^*$). Then for each $X \in Sd_V^*$, RN^X can be converted into a metric space endowed with a hyperreal metric

$$D(F,G) = \sup\{|F(x) - G(x)|;\ x \in X\}$$

Notice that for X being a set RN^X and D coincide with the original ones.

The generalization of Theorems 10 – 12 to arbitrary *biequivalences using the hyperreal numbers is quite straightforward, now. It is left to the reader.

From the matter just indicated it should follow that the hyperreal numbers will play rather an auxiliary role of a technically convenient extension of the rationals in our study. From this point of view the irrational numbers in HR, and the more, the not set-theoretically definable ones, seem much more curious and odder than the infinitesimally small and infinitely large rationals.

4. Geodetical biequivalences

Let $\langle \stackrel{\pm}{=}, \stackrel{\pm}{\longleftrightarrow} \rangle$ be a biequivalence (with domain V). We already know that there is a metric $H \in Sd_V^*$ on V inducing $\langle \stackrel{\pm}{=}, \stackrel{\pm}{\longleftrightarrow} \rangle$. Using the metric H a ternary relation "t lies between x and y" can be defined by the equality $H(x,t) + H(t,y) = H(x,y)$. Similarly, one can define the ternary relation "t lies nearly between x and y" by $H(x,t) + H(t,y) \doteq H(x,y)$. According to some results in [G] concerning classical metric spaces one can show that the biequivalence $\langle \stackrel{\pm}{=}, \stackrel{\pm}{\longleftrightarrow} \rangle$ can be induced by a metric $H \in Sd_V^*$ such that for all x,y,t holds t lies between x and y iff $t = x$ or $t = y$, and t lies nearly between x and y iff $t \stackrel{\pm}{=} x$ or $t \stackrel{\pm}{=} y$. The reader will probably agree that such a metric is rather a "bad" one. According to a "good" metric H at least for any accessible pair x,y there should be a connected set u containing both x and y such that each $t \in u$ lies between (or at least nearly between) x and y. This section is devoted to the precisation of the notion of a "good" metric and to the characterisation of biequivalences which can be induced by such metrics.

Let $H \in Sd_V^*$ be a rational metric on V and p be a path (i.e. a V^2-path) with domain $[q,\vartheta]$. The rational number

$$L_H(p) = \sum_{\alpha=\eta}^{\vartheta-1} H(p(\alpha),p(\alpha+1))$$

is called the length of the path p with respect to the metric H. Then p is called a direct (nearly direct) path with respect to H if $L_H(p) = H(p(\eta),p(\vartheta))$ $(L_H(p) \doteq H(p(\eta),p(\vartheta)))$. When the metric H is clear from the context the attribute "with respect to H" can be omitted from the notions just introduced. The length of the path p will be denoted $L(p)$ in such a case.

In the following three theorems $H \in Sd_V^*$ denotes a fixed metric on V and $\langle \doteq, \overset{+}{\leftrightarrow} \rangle$ is the *biequivalence induced by it.

Theorem 13. Let p be a path in the time ϑ.

(1) p is a direct path iff for all $\alpha \leq \beta \leq \vartheta$ holds
$H(p(\alpha),p(\beta)) = L(p \restriction [\alpha, \beta])$;

(2) p is a nearly direct path iff for all $\alpha \leq \beta \leq \vartheta$ holds
$H(p(\alpha),p(\beta)) \doteq L(p \restriction [\alpha, \beta])$.

Proof. We will prove only the second claim which is a bit less trivial. Let p be nearly direct. Then $H(p(\alpha),p(\beta)) \leq L(p \restriction [\alpha, \beta])$ for all $\alpha \leq \beta \leq \vartheta$. Assume that $H(p(\alpha),p(\beta)) <\cdot\, L(p \restriction [\alpha, \beta])$ for some α, β. Then

$$\begin{aligned} L(p) &= L(p \restriction [0,\alpha]) + L(p \restriction [\alpha,\beta]) + L(p \restriction [\beta,\vartheta]) \\ &\cdot\!> H(p(0),p(\alpha)) + H(p(\alpha),p(\beta)) + H(p(\beta),p(\vartheta)) \\ &\geq H(p(0),p(\vartheta)). \end{aligned}$$

Thus p were not nearly direct. The remaining implication is trivial.

Corollary. If p is a (nearly) direct path from x to y then each $t \in rng(p)$ lies (nearly) between x and y.

Theorem 14. Let p be a path in the time ϑ and $\alpha, \beta \in \vartheta+1$.

(1) If p is a direct path then

$$\alpha \leq \beta \equiv H(p(0),p(\alpha)) \leq H(p(0),p(\beta))$$
$$\alpha = \beta \equiv H(p(0),p(\alpha)) = H(p(0),p(\beta)).$$

(2) If p is a nearly direct path then

$$\alpha \underset{p}{\dot{\leq}} \beta \equiv H(p(0),p(\alpha)) \dot{\leq} H(p(0),p(\beta))$$

$$\alpha \underset{p}{\doteq} \beta \equiv H(p(0),p(\alpha)) \doteq H(p(0),p(\beta)).$$

__Proof.__ We will prove only (2) again. Let p be nearly direct and $\alpha \underset{p}{\dot{\leq}} \beta$. Then either $\alpha \leq \beta$ and by the preceeding Theorem

$$\begin{aligned} H(p(0),p(\alpha)) &\leq H(p(0),p(\alpha)) + H(p(\alpha),p(\beta)) \\ &\doteq L(p \upharpoonright [0,\alpha]) + L(p\upharpoonright[\alpha,\beta]) \\ &= L(p \upharpoonright [0,\beta]) \doteq H(p(0),p(\beta)), \end{aligned}$$

or $\alpha > \beta$, $p(\alpha) \doteq p(\beta)$ and

$$\begin{aligned} H(p(0),p(\alpha)) &\doteq L(p \upharpoonright [0,\alpha]) = L(p\upharpoonright[0,\beta]) + L(p\upharpoonright[\beta,\alpha]) \\ &\doteq H(p(0),p(\beta)) + H(p(\beta),p(\alpha)) \\ &\doteq H(p(0),p(\beta)). \end{aligned}$$

Now assume that $H(p(0),p(\alpha)) \dot{\leq} H(p(0),p(\beta))$. If $\alpha \leq \beta$, there is nothing to be proved. So let $\alpha > \beta$. Then as already proved also $H(p(0),p(\beta)) \dot{\leq} H(p(0),p(\alpha))$. If there were a $\gamma \in [\beta,\alpha]$ such that $p(\beta) \not\doteq p(\gamma)$, the following computation would yield a contradiction:

$$\begin{aligned} H(p(0),p(\beta)) &\doteq H(p(0),p(\alpha)) \doteq L(p \upharpoonright [0,\alpha]) \\ &= L(p\upharpoonright[0,\beta]) + L(p\upharpoonright[\beta,\gamma]) + L(p\upharpoonright[\gamma,\alpha]) \\ &\geq H(p(0),p(\beta)) + H(p(\beta),p(\gamma)) + H(p(\gamma),p(\alpha)) \\ &\dot{>} H(p(0),p(\beta)). \end{aligned}$$

The second equivalence in (2) is a direct consequence of the first one.

Corollary. Let p be a nearly direct path. Then

(1) the equivalences $\overset{+}{\underset{(p)}{=}}$ and $\overset{+}{\underset{p}{=}}$ on dom(p) coincide;

(2) p is a compact path (i.e. $\overset{+}{\underset{p}{=}}$ is compact) iff p has a compact trace iff $L(p) \overset{\bullet}{\leftrightarrow} 0$.

Theorem 15. Let p be a nearly direct path. Then rng(p) is a connected set iff p is a motion.

Proof. Obviously, the trace of a motion is a connected set. Assume that p is a nearly direct path in the time ϑ which is not a motion. Then there is an $\alpha < \vartheta$ such that $p(\alpha) \overset{+}{\not=} p(\alpha+1)$. Then for all $\beta, \gamma \leq \vartheta$ $\beta \leq \alpha$ and $\alpha < \gamma$ imply $\beta \overset{+}{\underset{p}{\not=}} \gamma$. By the previous results $p(\beta) \overset{+}{\not=} p(\gamma)$. Thus rng(p) is not connected.

Theorems 13 - 15 and their Corollaries justify the following definition:

A metric $H \in Sd^{*}_{V}$ is called (nearly) geodetical if for all x,y such that $x \overset{\bullet}{\leftrightarrow}_H y$ there is a (nearly) direct motion (with respect to H) from x to y. A biequivalence $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ is called (nearly) geodetical if it can be induced by a (nearly) geodetical metric.

An immediate consequence of this definition and of the preceeding results is the following:

Theorem 16. Let $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ be a nearly geodetical biequivalence. Then for every pair $x \overset{+}{\leftrightarrow} y$ there is a compact motion from x to y. In particular $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ is Archimedean and has connected galaxies.

Theorem 17. Let $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ be a biequivalence. The following conditions are equivalent:

(1) $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ is geodetical;

(2) $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ is nearly geodetical;

(3) there is a *bigenerating sequence $\{R_n;\ n \in FZ\}$ of $\langle \doteq, \overset{+}{\leftrightarrow} \rangle$ such that for each n hold

$$R_{n+1} \subseteq R_n \circ R_n \circ (\doteq) \quad \text{and} \quad (\doteq) \circ R_n = R_n \circ (\doteq);$$

(4) there is a *bigenerating sequence $\{S_n;\ n \in FZ\}$ of $\langle \doteq, \overset{+}{\leftrightarrow} \rangle$ such that for each n holds $S_n \circ S_n = S_{n+1}$;

(5) for some $d \in FN$, $d \geq 2$, there is a *bigenerating sequence $\{S_n;\ n \in FZ\}$ such that for each n holds $S_n^d = S_{n+1}$.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) are trivial.

(2) $\Rightarrow$ (3): Let $H \in Sd_V^*$ be a nearly geodetical metric inducing $\langle \doteq, \overset{+}{\leftrightarrow} \rangle$. We put $R_n = \{\langle x,y\rangle ;\ H(x,y) \leq 2^n\}$. Obviously, $\{R_n;\ n \in FZ\}$ is a *bigenerating sequence of $\langle \doteq, \overset{+}{\leftrightarrow} \rangle$. Let $\langle x,y\rangle \in R_{n+1}$ and p be a nearly direct motion from x to y in the time ϑ . Let $\alpha \leq \vartheta$ be the greatest natural number such that $\langle x,p(\alpha)\rangle \in R_n$, and $\beta \in [\alpha, \vartheta]$ be the greatest natural number such that $\langle p(\alpha),p(\beta)\rangle \in R_n$. It is routine to check that $p(\beta) \doteq y$ since p is a nearly direct motion. Thus $R_{n+1} \subseteq R_n \circ R_n \circ (\doteq)$. Now, assume that $\langle x,y\rangle \in (\doteq) \circ R_n$. Let p be a nearly direct motion from x to y in the time ϑ and $\alpha \leq \vartheta$ be the greatest natural number such that $\langle x,p(\alpha)\rangle \in R_n$. The reader can easily verify that $p(\alpha) \doteq y$. Thus $(\doteq) \circ R_n \subseteq R_n \circ (\doteq)$. The remaining inclusion follows by a symetric argument.

(3) $\Rightarrow$ (4): One can easily verify that

$$(\forall n \in FZ)(\forall k \in FN)\ R_n^{2^k} \subseteq R_{n+k} \subseteq R_n^{2^k} \circ (\doteq).$$

Therefore

$$(\forall m,n \in FZ)(m \leq n \Rightarrow R_m^{2^{n-m}} \subseteq R_n \subseteq R_m^{2^{n-m}+1}).$$

Hence there is a prolongation $\{R_\nu;\ \nu \in [\sigma-1, \tau+1]\}$ of the *bigenerating sequence $\{R_n;\ n \in FZ\}$ in Sd_V^* such that for each $\nu \in [\sigma, \tau]$ holds

$$R_{\sigma}^{2^{\nu-\sigma}} \subseteq R_{\nu} \subseteq R_{\sigma}^{2^{\nu-\sigma}+1} .$$

Finally, we put $S_n = R_{\sigma}^{2^{n-\sigma}}$ for each $n \in FZ$. Then $\{S_n ; n \in FZ\}$ is a *bigenerating sequence of $\langle \stackrel{+}{=}, \stackrel{+}{\leftrightarrow} \rangle$ and for each n holds $S_n \circ S_n = S_{n+1}$.

(5) $\Rightarrow$ (1): Let $\{S_n ; n \in FZ\}$ be a *bigenerating sequence of $\langle \stackrel{+}{=}, \stackrel{+}{\leftrightarrow} \rangle$ such that $(\forall n)\ S_n^d = S_{n+1}$ where $d \in FN - \{0,1\}$, and $\{S_\nu ; \nu \in [\sigma-1, \tau+1]\}$ be a prolongation of this sequence in Sd_V^* such that for each $\nu \in [\sigma, \tau-1]$ holds $S_\nu^d = S_{\nu+1}$.

Then for each $\nu \in [\sigma, \tau]$ holds $S_\sigma^{d^{\nu-\sigma}} = S_\nu$. Then the function

$$H(x,y) = d^{\sigma} \min(\{\mu ; \langle x,y \rangle \in S_{\sigma}^{\mu}\} \cup \{d^{\tau-\sigma+1}\})$$

obviously belongs to Sd_V^* and is a metric on V. Let us show that the *biequivalence induced by H is indeed $\langle \stackrel{+}{=}, \stackrel{+}{\leftrightarrow} \rangle$. For all x,y the following conditions are equivalent:

$x \stackrel{+}{=} y$; $(\forall n)\ \langle x,y \rangle \in S_n = S_{\sigma}^{d^{n-\sigma}}$; $(\forall n)\ H(x,y) \leq d^n$; $H(x,y) \doteq 0$.

Similarly, changing "$\forall n$" to "$\exists n$", one obtains

$(\forall x,y)(x \stackrel{+}{\leftrightarrow} y \equiv H(x,y) \stackrel{+}{\leftrightarrow} 0)$.

It remains to prove that H is geodetical. But from the construction of H it follows even more. Namely, for every pair $\langle x,y \rangle \in S_\tau$ there is a direct S_σ-path from x to y.

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