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A NOTE ON SPATIAL REPRESENTATION OF GRAPHS

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Abstract: The purpose of this remark is to provide a solution of two problems of Sachs [1] on representation of graphs in E_3. One of them uses a recent solution of Wagner's conjecture by Seymour and Robertson.

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Let $G = (V,E)$ be a graph (loops and multiple edges are allowed). By a (spatial) representation of G (in E_3) we mean a rule which assigns to each vertex of G a point in E_3 in such a way that the existing incidencies are preserved and no new intersections are created. Clearly every circuit in G corresponds to a closed curve homeomorphic to a circle.

Let us recall some notions introduced in [1]. Given a representation of G, we say that two disjoint circuits in G are concatenated if they cannot be embedded in disjoint topological closed balls. Otherwise the cycles are called discatenated. The degree of concatenation of two disjoint circquits of G is the minimal number of permeations which are necessary to discatenate them. Here the permeation of two arcs of the same cycle is counted with multiplicity 2.

The degree of concatenation of a representation of a graph is the sum of the degrees of concatenation over all unordered pairs of disjoint cycles. Finally, the degree of concatenation of a graph G is the minimal degree of concatenation of a representation of G. G is <u>discatenable</u> if its degree of concatenation is zero.

In [1] Sachs considered the following problem:

Let $\mathfrak{D}$ be the class of all discatenable graphs. Can the class $\mathfrak{D}$ be characterized by a finite set of forbidden subgraphs ?

Explicitly: Do there exist graphs $A_1, \dots, A_k$ such that $G \in \mathcal{K}$ iff G does not contain a subgraph homeomorphic to a graph A_i, $1 \leq i \leq k$?

(Clearly $\mathfrak{D}$ is closed under homeomorphism.) It is proved in [1] that $K_6 \notin \mathfrak{D}$, $K_{4,4} \notin \mathfrak{D}$ and in fact conjectured that the answer is negative. In this note we observe that the above problem has a positive solution even in a stronger sense.

Given an integer $k \geq 0$ denote by $\mathfrak{D}_k$ the class of all graphs with degree of concatenation $\leq k$. We use the following lemma.

<u>Lemma</u>. (i) Let $G \in \mathfrak{D}_k$ and let H be a subgraph of G. Then $H \in \mathfrak{D}_k$.

(ii) Let $G \in \mathfrak{D}_k$, $e \in E(G)$. Put $H = G.e$ (i.e. the resulting graph after the contraction of the edge e). Then $H \in \mathfrak{D}_k$.

(iii) $K_{5k+6} \notin \mathfrak{D}_k$.

<u>Proof</u>. (i) and (ii) are obvious and follow from the physical meaning of deletion and contraction of an edge. (iii) follows by induction from [1]; it is $K_6 \notin \mathfrak{D} = \mathfrak{D}_0$, and clearly (using (i)) $K_{5(k-1)+6} \notin \mathfrak{D}_{k-1}$ implies $K_{5k+6} \notin \mathfrak{D}_k$.

<u>Corollary</u>: For each non-negative integer k the class $\mathfrak{D}_k$ closed on minors.

(A graph H is a minor of the graph G if there exists a subgraph G' of G which can be contracted onto H.)

The following outstanding result (known as Wagner s conjecture) was recently proved by P. Seymour and N. Robertson.

Theorem (Seymour, Robertson [2]): Every class $\mathcal{K}$ of finite graphs which is closed on minors can be determined by a finite set of forbidden minors.

Explicitly: Suppose that a class $\mathcal{K}$ of graphs has the following property:

If $G \in \mathcal{K}$ and H is a minor of G then $H \in \mathcal{K}$. Then there exists a finite set $A_1, \dots, A_n$ of graphs such that the following two statements are equivalent:

1) $G \in \mathcal{K}$

2) no A_i is a minor of G.

Using this theorem and the above corollary we have the following:

Corollary I: Given a nonnegative integer k there exists a set $A_1^k, \dots, A_{n(k)}^k$ of graphs with the following property:

$$G \in \mathfrak{D}_k \text{ iff no } A_i^k \text{ is a minor of } G.$$

This seems to be not directly related to the above problem of Kuratowski-type. However, it is well known and easy to prove that a class of graphs closed on minors (and thus closed on homeomorphisms) can be characterized by a finite set of forbidden minors iff it can be characterized by a finite set of subgraphs. Thus we have

Corollary II: Given a non-negative integer k there exists a set $B_1^k, \dots, B_{m(k)}^k$ of graphs with the following property:

$G \in \mathfrak{D}_k$ iff no B_i^k is homeomorphic to a subgraph of G.

Presently, no bound can be deduced from the Seymour-Robertson argument (which is based on the theory of well-quasiorderings). This may be a very difficult problem.

Sachs mentions another problem which may be solved as follows:

Corollary III. Given a non-negative integer k there exists an integer K(k) such that

$$\chi(G) \leq K(k)$$

for every graph $G \in \mathfrak{D}_k$.

Proof. According to the above lemma $K_{5k+6} \notin \mathfrak{D}_k$. It is well known that if the chromatic number of a graph G is at least 2^{5k+6} then K_{5k+6} is a minor of G. This proves $K(k) \leq 2^{5k+6}$. In fact by a result of Mader [3] there is a polynomial relation between k and K(k).

The problem of the maximal chromatic number of a graph which belongs to the class $\mathfrak{D}_k$ is related to Hadwiger conjecture. Particularly, $\chi(G) \leq 5$ for every discatenable graph if K_6 is a minor of every graph which fails to be 5-colourable.
(Note that K_5 is discatenable; using a result of Jacobsen [4] we know that $\chi(G) \leq 6$ for every discatenable graph G only.)

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