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THE FARKAS LEMMA OF GLOVER
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Abstract: We use standard functional analysis techniques to establish a result of Glover which he employs to obtain a nonlinear version of the classical Farkas Lemma.

Key words: Farkas Lemma, convex functions, subgradients, Krein-Smulian Theorem.

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In this brief note we present a proof of a theorem which has been used in optimization to establish a nonlinear version of the Farkas Lemma ([1], Lemma 3) and to establish Kuhn-Tucker Theorems ([3] 2.1, [4] 2.3, 2.4). The proof given by Glover in [1] uses machinery from set-valued mappings; we present a proof below which only employs standard topics from functional analysis, namely, the Krein-Smulian Theorem.

Let X and Y be locally convex spaces with S a closed convex cone in Y . Let $g: X \rightarrow Y$ be positive homogeneous, S -convex and such that $s' \circ g = s'g$ is lower semi-continuous for each $s' \in S^*$, where $S^* = \{s' \in Y': (s', s) \geq 0 \forall s \in S\}$ is the dual cone of S . As usual we write $\partial f(0)$ for the subgradient of a convex function $f: X \rightarrow \mathbb{R}$ at 0 ([7]). Glover shows that if $A = \bigcup_{s' \in S^*} \partial(s'g)(0)$, then $-(g^{-1}(-S))^* = \bar{A}$, where the closure is in the weak* topology of Y' ([1] Lemma 1), and then uses this result to establish a

general nonlinear Farkas Lemma ([1] Theorem 1). Glover's Farkas Lemma contains the linear Farkas Lemmas of Zalinescu ([6]) and Schirotzek ([5]). In order to obtain a sharper form of the Farkas Lemma, Glover gives sufficient conditions which guarantee that the set A above is weak* closed ([1] Lemma 3). We state and prove a version of this result which uses only standard functional analysis techniques whereas Glover's proof uses results of Robinson on set-valued mappings. Our proof also covers the case when X is only metrizable and not necessarily a normed space, but we must assume that the range space is barrelled although not necessarily normed.

Theorem 1. Let X be complete, metrizable and let Y be barrelled and suppose that $g(X) + S = Y$. Then $-(g^{-1}(-S))^* = A$ so in particular A is weak* closed.

Proof: By Lemma 1 of [1] it suffices to show A is weak* closed and by the Krein-Smulian Theorem ([2], 3.40.2) it suffices to show that $A \cap U^0$ is weak* closed for each closed, absolutely convex neighborhood of 0, U , in X . Let (x'_j) be a net in $A \cap U^0$ such that (x'_j) is weak* convergent to x' . It suffices to show that $x' \in A \cap U^0$. Let p be the Minkowski functional of U . Choose $s'_j \in S^*$ such that $x'_j \in \partial(s'_j, g)(0)$ and let $y \in Y$. By hypothesis, $y = g(x) + s$ for some $x \in X$, $s \in S$. Then

$$(1) \quad \langle s'_j, y \rangle = \langle s'_j, g(x) \rangle + \langle s'_j, s \rangle \geq \langle x'_j, x \rangle \geq -p(x).$$

Also $-y = g(\bar{x}) + \bar{s}$ for some $\bar{x} \in X$, $\bar{s} \in S$ so $\langle s'_j, -y \rangle = \langle s'_j, g(\bar{x}) \rangle + \langle s'_j, \bar{s} \rangle \geq \langle x'_j, \bar{x} \rangle \geq -p(\bar{x})$ and

$$(2) \quad \langle s'_j, y \rangle \leq p(\bar{x}).$$

Thus, if $r = \max\{p(x), p(\bar{x})\}$, (1) and (2) imply that

$|\langle s'_j, y \rangle| \leq r$. Hence, $\{s'_j\}$ is weak* bounded and, therefore, relatively weak* compact by the barrelledness ([2], 3.6.2).

Thus, $\{s'_j\}$ has a subnet, which we continue to denote by $\{s'_j\}$, which is weak* convergent to some $y' \in Y'$. Since $\langle s'_j, s \rangle \geq 0$ for $s \in S$, $\langle y', s \rangle \geq 0$ so that $y' \in S^*$. For $x \in X$, we have $\langle y', g(x) \rangle = \lim \langle s'_j, g(x) \rangle \geq \lim \langle x'_j, x \rangle = \langle x', x \rangle$ so $x' \in \partial(y'g)(0)$ and $x' \in A \cap U^0$ since U^0 is weak* closed.

In Glover's version he assumes that X is a Banach space and Y is a normed space, but there is no completeness assumption on Y .

If $f: X \rightarrow \mathbb{R}$ is lower semicontinuous and sublinear and $x' \in X'$, then under the assumptions of Theorem 1 Glover's Farkas Lemma ([1], Theorem 1) is: $x' \in \partial f(0) \cap A$ iff $-g(x) \in S$ implies $f(x) \geq \langle x', x \rangle$. For the case when f and g are linear, this yields the Farkas Lemmas of Zalinescu ([6]) and Schirotzek ([5]).

R e f e r e n c e s

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