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SPECIAL POLYNOMIALS IN ORTHOMODULAR LATTICES

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Abstract: In this paper the set MF_n of all meet-Frattini polynomials and the set of all join-Frattini polynomials are studied. In particular, it is shown that the upper commutator belongs to MF_n. Some properties of friendly pairs of polynomials are established. Also quite complete information regarding the commutativity relation in the free orthomodular lattice F_2 is given and, as a by-product, a simple description of the quotient set corresponding to the equivalence relation defined by friendly pairs of polynomials in two variables is obtained.

Key words: Commutativity relation, free orthomodular lattice with two generators, commutator, Frattini polynomial, friendly pairs of polynomials.

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1. Preliminaries

If a,b are elements of an orthomodular lattice $L = (L, \vee, \wedge, ', 0, 1)$, we say that a and b <u>commute</u> and write aCb, provided $a = (a \wedge b) \vee (a \wedge b')$.

Recall the following result (cf., e.g., [1]):

Lemma 1.1. In every orthomodular lattice,

(i) $aCb \iff aCb' \iff bCa$;

(ii) $(aCb \;\&\; aCc) \Rightarrow aCb \wedge c$;

(iii) (aCb & aCc) $\Rightarrow$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

For our purposes here, we need the fact that C has an exchange property of the following type:

Lemma 1.2. For any elements a,b,c of an orthomodular lattice,

$$(aCb \wedge c \;\&\; bCc) \Rightarrow a \wedge bCc.$$

For a proof, see [2].

Convention. In what follows, L will always denote an orthomodular lattice.

The 96-element lattice which represents the free orthomodular lattice F_2 with two generators was studied in [4]. It should be noted that its elements can be decomposed in a natural way in six different Boolean algebras B_1 - B_6, where

$B_1 = [0;\ \underline{com}\ (x,y)]$,

$B_2 = [x \wedge (x' \vee y) \wedge (x' \vee y');\ x \vee (x' \wedge y) \vee (x' \wedge y')]$,

$B_3 = [y \wedge (y' \vee x) \wedge (y' \vee x');\ y \vee (y' \wedge x) \vee (y' \wedge x')]$,

$B_4 = [y' \wedge (y \vee x') \wedge (y \vee x);\ y' \vee (y \wedge x') \vee (y \wedge x)]$,

$B_5 = [x' \wedge (x \vee y') \wedge (x \vee y);\ x' \vee (x \wedge y') \vee (x \wedge y)]$,

$B_6 = [\overline{com}\ (x,y);\ 1]$.

For more about this and the basic properties of F_2 the reader may consult [1].

The set of all the polynomials in $\wedge, \vee$ and $'$ of n variables $x_1, x_2, \dots, x_n$ will be denoted by P_n. To simplify notation we shall denote the value $p(a_1, a_2, \dots, a_n)$ of a polynomial $p = p(x_1, x_2, \dots, x_n)$ in $a_1, a_2, \dots, a_n \in L$ by $p(a_1, \bullet)$. A similar formalism will be retained also for $p(x_1,x_2, \dots ,x_n)$. Two polynomials $p(x_1,\bullet)$ and $q(x_1,\bullet)$ of P_n are said to <u>commute</u> if and only if for every `L and for every choice of elements $a_1,a_2, \dots ,a_n$ in L the element $p(a_1,\bullet)$ commutes with $q(a_1,\bullet)$.

Let a be an element of L. We define $a^1 = a$ and $a^{-1} = a'$. Now it is easy to recall the concept of a commutator due to [3]. The <u>upper commutator</u> of $a_1,a_2, \dots ,a_n \in L$ is defined by

$$\overline{\text{com}}\,(a_1,a_2, \dots ,a_n) = \bigwedge (a_1^{e(1)} \vee a_2^{e(2)} \vee \dots \vee a_n^{e(n)}),$$

where e runs over all the mappings $e:\{1,2, \dots ,n\} \to \{-1,1\}$. The <u>lower commutator</u> of $a_1,a_2, \dots ,a_n$ is defined dually, i.e.,

$$\underline{\text{com}}\,(a_1,a_2, \dots ,a_n) = \bigvee (a_1^{e(1)} \wedge a_2^{e(2)} \wedge \dots \wedge a_n^{e(n)}).$$

2. Frattini polynomials

A polynomial $f \in P_n$ is said to be <u>meet-Frattini</u> if and only if it has the following property: For every p, $q \in P_n$ and for every $a_1,a_2, \dots ,a_n$ of any `L the element $p(a_1,\bullet)$ commutes with $q(a_1,\bullet) \wedge f(a_1,\bullet)$ if and only if $p(a_1,\bullet)$ commutes with $q(a_1,\bullet)$. A <u>join-Frattini</u> polynomial f is defined dually by the condition

$$p(a_1,\bullet)Cq(a_1,\bullet) \vee f(a_1,\bullet) \iff p(a_1,\bullet)Cq(a_1,\bullet).$$

We shall denote the set of all meet-Frattini polynomials of P_n and the set of all join-Frattini polynomials of P_n by MF_n and JF_n, respectively.

Our first result is a technical lemma about polynomials in P_n which will be useful later.

Lemma 2.1. Let $p \in P_n$ and let $a_1, a_2, \ldots, a_n \in L$. If e maps $\{1, 2, \ldots, n\}$ into $\{-1, 1\}$, then either

$$p(a_1, a_2, \ldots, a_n) \leqq a_1^{e(1)} \vee a_2^{e(2)} \vee \ldots \vee a_n^{e(n)}$$

or

$$p'(a_1, a_2, \ldots, a_n) \leqq a_1^{e(1)} \vee a_2^{e(2)} \vee \ldots \vee a_n^{e(n)}.$$

Proof: Use induction on the rank of p.

Lemma 2.2. For any $e: \{1, 2, \ldots, n\} \to \{-1, 1\}$,

$$x_1^{e(1)} \vee x_2^{e(2)} \vee \ldots \vee x_n^{e(n)} \in MF_n$$

and

$$x_1^{e(1)} \wedge x_2^{e(2)} \wedge \ldots \wedge x_n^{e(n)} \in JF_n.$$

Proof: First note that

(1) $p(a_1, \bullet) C q(a_1, \bullet) \wedge (a_1^{e(1)} \vee \bullet)$

is equivalent to

(2) $p'(a_1, \bullet) C q(a_1, \bullet) \wedge (a_1^{e(1)} \vee \bullet)$.

Now, $a_1^{e(1)} \vee \bullet$ commutes with $q(a_1, \bullet)$ and with $p^d(a_1, \bullet)$, where $d = \pm 1$. Thus, by Lemma 1.2, (1) is equivalent to

(3) $p^d(a_1, \bullet) \wedge (a_1^{e(1)} \vee \bullet) C q(a_1, \bullet)$.

From Lemma 2.1 we infer that (3) is equivalent to

(4) $p^d(a_1, \bullet) C q(a_1, \bullet)$.

Consequently, it follows by Lemma 1.1 that (1) is equivalent to $p(a_1, \bullet) C q(a_1, \bullet)$.

Similar reasoning yields the remainder of the proof.

As a direct consequence of Lemma 2.2 we have the following useful proposition.

Proposition 2.3. For any $n \in \underline{N}$,

$$\overline{\underline{\text{com}}}\,(x_1, x_2, \ldots, x_n) \in MF_n$$

and

$$\underline{\text{com}}\,(x_1, x_2, \ldots, x_n) \in JF_n.$$

3. Friendly pairs of polynomials

Let $p,q,r,s \in P_n$. The pairs (p,q) and (r,s) are said to be friendly (written $(p,q) \sim (r,s)$) if and only if the following condition is satisfied for any L and any $a_1, a_2, \ldots, a_n \in L$: The element $p(a_1, \bullet)$ commutes with $q(a_1, \bullet)$ if and only if the element $r(a_1, \bullet)$ commutes with $s(a_1, \bullet)$.

Our next lemma gives information regarding the relation $\sim$.

Lemma 3.1. Let $p,q,r,s \in P_n$. Then

(i) $[(p,q) \sim (r,s)] \Leftrightarrow [(q,p) \sim (r,s)] \Leftrightarrow [(r,s) \sim (p,q)]$.

(ii) The relation $\sim$ is an equivalence relation on P_n^2.

Proof: Obvious.

Proposition 3.2. Let $p,q \in P_n$, let e_i, f_j, E_u, F_v $(1 \leq i \leq a,\ 1 \leq j \leq b,\ 1 \leq u \leq c,\ 1 \leq v \leq d)$ be mappings of $\{1,2, \ldots, n\}$ into $\{-1,1\}$ and let $a,b,c,d \in \underline{N}_0$. If $w,s \in \{-1,1\}$ and

$$r(x_1,x_2, \ldots ,x_n) = [p^w(x_1,x_2,\ldots,x_n) \wedge \bigwedge_{i=1}^{n} (x_1^{e_i(1)} \vee x_2^{e_i(2)} \vee \ldots$$
$$\ldots \vee x_n^{e_i(n)})] \vee [\bigvee_{j=1}^{b} (x_1^{f_j(1)} \wedge x_2^{f_j(2)} \wedge \ldots \wedge x_n^{f_j(n)})]$$

$$s(x_1,x_2,\dots,x_n)=[q^z(x_1,x_2,\dots,x_n)\wedge \bigwedge_{u=1}^{c} (x_1^{E_u(1)} \vee x_2^{E_u(2)} \vee \dots$$

$$\dots \vee x_n^{E_u(n)})] \vee [\bigvee_{v=1}^{d} (x_1^{F_v(1)} \wedge x_2^{F_v(2)} \wedge \dots \wedge x_n^{F_v(n)})],$$

then the pairs $(r(x_1,x_2, \dots ,x_n),s(x_1,x_2, \dots ,x_n))$ and $(p(x_1,x_2, \dots ,x_n),q(x_1,x_2, \dots ,x_n))$ are friendly.

Proof: Let

$$A_1 = \bigwedge_{i=1}^{a} (x_1^{e_i(1)} \vee \bullet), \qquad {}^{\backprime}A = \bigwedge_{i=1}^{a} (a_1^{e_i(1)} \vee \bullet);$$

$$B_1 = \bigvee_{j=1}^{b} (x_1^{f_j(1)} \wedge \bullet), \qquad {}^{\backprime}B = \bigvee_{j=1}^{b} (a_1^{f_j(1)} \wedge \bullet);$$

$$C_1 = \bigwedge_{u=1}^{c} (x_1^{E_u(1)} \vee \bullet), \qquad {}^{\backprime}C = \bigwedge_{u=1}^{c} (a_1^{E_u(1)} \vee \bullet);$$

$$D_1 = \bigvee_{v=1}^{d} (x_1^{F_v(1)} \wedge \bullet), \qquad {}^{\backprime}D = \bigvee_{v=1}^{d} (a_1^{F_v(1)} \wedge \bullet);$$

$${}^{\backprime}P = p(a_1,\bullet), \quad {}^{\backprime}Q = q(a_1,\bullet).$$

Now, ${}^{\backprime}B C {}^{\backprime}P \wedge {}^{\backprime}A$. This, together with the dual of Lemma 1.2, implies that

(5) $[({}^{\backprime}P \wedge {}^{\backprime}A) \vee {}^{\backprime}B] C [({}^{\backprime}Q \wedge {}^{\backprime}C) \vee {}^{\backprime}D]$

is equivalent to

(6) $({}^{\backprime}P \wedge {}^{\backprime}A) C ({}^{\backprime}Q \wedge {}^{\backprime}C) \vee {}^{\backprime}D \vee {}^{\backprime}B.$

From Lemma 2.2 we infer that (6) is equivalent to

(7) $({}^{\backprime}P \wedge {}^{\backprime}A) C [({}^{\backprime}Q \wedge {}^{\backprime}C) \vee {}^{\backprime}D \vee {}^{\backprime}B] \wedge ({}^{\backprime}D \vee {}^{\backprime}B)'.$

However, $({}^{\backprime}D \vee {}^{\backprime}B) C ({}^{\backprime}Q \wedge {}^{\backprime}C)$ and $({}^{\backprime}D \vee {}^{\backprime}B) C ({}^{\backprime}D \vee {}^{\backprime}B)'$. It then follows from Lemma 1.1 that

$$[(`Q \wedge `C) \vee `D \vee `B] \wedge (`D \vee `B)' = (`Q \wedge `C) \wedge (`D \vee `B)'.$$

Note that, by Lemma 2.2, $D_1 \vee B_1 \in MF_n$. Therefore, (7) is equivalent to

(8) $(`P \wedge `A)C(`Q \wedge `C)$.

But the polynomials A_1, C_1 are also meet-Frattini. Thus, (8) is equivalent to $`PC`Q$.

4. The commutativity relation in the free orthomodular lattice F_2

Similarly as in [1], let x,y denote the free generators of the free orthomodular lattice F_2.

Given two polynomials p,q of the infinite set P_2, one can ask what means the condition "p commutes with q". An answer to the question is evidently given, provided we can characterize what means the condition

(9) $p(x,y)Cq(x,y)$

in F_2.

Since F_2 has exactly 96 elements, we have $\binom{96}{2} =$ $= 48.95 = 4{,}560$ possibilities how to choose the couples (p,q) in (9). However, we shall see that no computer is needed to give a complete survey of the corresponding situations.

The next two lemmas are of critical importance for what follows but are also of independent interest.

Lemma 4.1. Let $p \in P_2$. If $p(x,y) \in B_1 \cup B_6$, then $p(x,y)Cq(x,y)$ for every $q \in P_2$.

Proof: Suppose $p(x,y) \in B_6$. Then $p(x,y)$ is equal to a meet of some elements $x^{e_i} \vee y^{f_i}$ $(e_i, f_i \in \{-1,1\},\ i \in I)$. Since $x^{e_i} \vee y^{f_i}$ belongs to the center of F_2, $x^{e_i} \vee y^{f_i}$ commutes with $q(x,y)$. By Lemma 1.1, $p(x,y)Cq(x,y)$.

A similar argument can be used if $p(x,y) \in B_1$.

Lemma 4.2. Let $p(x,y)$ and $q(x,y)$ be elements of B_i, where $1 \leq i \leq 6$. Then $p(x,y)Cq(x,y)$.

Proof: By Lemma 4.1, the assertion holds whenever $i = 1$ or $i = 6$. In the sequel we suppose that $2 \leq i \leq 5$.

Using the information found in Figure 18 of [1], we can see that

$$p(x,y) = [z_i \wedge \overline{com}\ (x,y)] \vee d(x,y)$$

and

$$q(x,y) = [z_i \wedge \overline{com}\ (x,y)] \vee e(x,y),$$

where $d(x,y), e(x,y) \in B_1$ and where $z_2 = x$, $z_3 = y$, $z_4 = x'$, $z_5 = y'$. Therefore, by Proposition 3.2, $p(x,y)Cq(x,y)$ is equivalent to $z_i C z_i$ which is always true.

Theorem 4.3. Let $2 \leq i < j \leq 5$ and let $p(x,y) \in B_i$, $q(x,y) \in B_j$. Then $p(x,y)Cq(x,y)$ if and only if either

$$i = 2 \quad \& \quad j = 5$$

or

$$i = 3 \quad \& \quad j = 4.$$

Proof: Similarly as in the proof of Lemma 4.2 we have

$$(10) \qquad p(x,y) = [z \wedge \overline{com}\ (x,y)] \vee d(x,y)$$

and

$$(11) \qquad q(x,y) = [v \wedge \overline{com}\ (x,y)] \vee e(x,y),$$

where $d(x,y), e(x,y) \in B_1$ and $\{z,v\} \subset \{x,x',y,y'\}$. Hence $p(x,y)Cq(x,y)$ if and only if zCv, i.e., if and only if either $\{z,v\} = \{x,x'\}$ or $\{z,v\} = \{y,y'\}$.

Remark 4.4. Figure 1 indicates all the relations of commutativity in F_2. The edge joining B_3 and B_4 means that any two elements $p \in B_3$, $q \in B_4$ commute. No two elements $p_1 \in B_2$, $p_2 \in B_3$ commute and, therefore, there is no edge joining B_2 and B_3. The loop at B_i means that p_3Cp_4 whenever $p_3, p_4 \in B_i$.

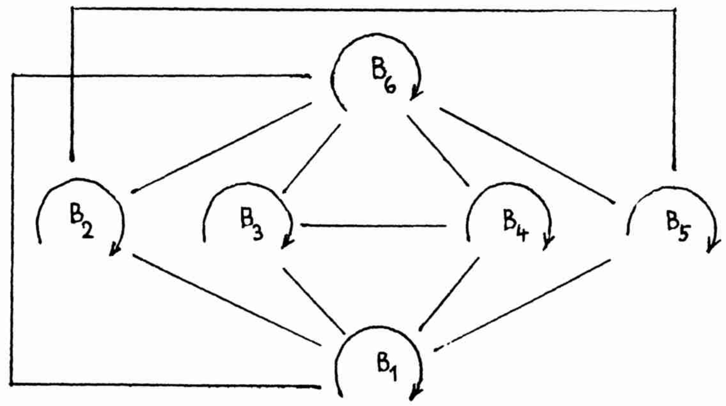

Fig. 1

Theorem 4.5. Two polynomials $p(x_1,x_2)$ and $q(x_1,x_2)$ of P_2 either commute or in any $'L$ the element $p(a_1,a_2)$ commutes with $q(a_1,a_2)$ $(a_1,a_2 \in L)$ if and only if a_1Ca_2.

Proof: Suppose there exists an orthomodular lattice $'T$ and elements $b_1,b_2 \in T$ such that $p(b_1,b_2)$ does not commute with $q(b_1,b_2)$. Then the elements $p(x,y), q(x,y)$ do not belong to $B_1 \cup B_6$. Moreover, by Lemma 4.2 and Remark 4.4 neither $\{p(x,y),q(x,y)\} \subset B_i$ nor $\{p(x,y),q'(x,y)\} \subset B_i$.

Hence we may assume that $p(x,y)$ and $q(x,y)$ are of the form given in (10) and (11). Therefore, if $a_1, a_2 \in L$, then

$$p(a_1,a_2) = [z_0 \wedge \overline{\text{com}}\,(a_1,a_2)] \vee d(a_1,a_2),$$

$$q(a_1,a_2) = [v_0 \wedge \overline{\text{com}}\,(a_1,a_2)] \vee e(a_1,a_2),$$

where $\{z_0, v_0\} \subset \{a_1, a_1', a_2, a_2'\}$ and $v_0 \neq z_0 \neq v_0'$. Without loss of generality we may assume that $z_0 = a_1$ and $v_0 = a_2$. From Proposition 3.2 it follows that $p(a_1,a_2)Cq(a_1,a_2)$ if and only if $z_0 C v_0$, i.e., if and only if $a_1 C a_2$.

As a direct consequence of Theorem 4.6 we have the following result.

Corollary 4.6. For any $p,q \in \bar{P}_2$ either $(p,q) \sim (0,1)$ or $(p,q) \sim (x_1, x_2)$.

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