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A NOTE ON THE MARTINGALE CENTRAL LIMIT THEOREM

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Abstract. The purpose of this paper is to show that McLeish s Central Limit Theorem (see [1],p. 58) for the martingale differences is valid without assuming their square integrability.

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Theorem. Let $(S_{nk}, A_{nk}, k = 1,\dots,k_n, n \in N)$ be a zero-mean martingale array with differences X_{nk}. Suppose that

1) $E \max \{|X_{nk}| \mid k = 1,\dots,k_n\} \to 0$,

2) $\sum_{k=1}^{k_n} X_{nk}^2 \xrightarrow{p} \eta^2$, where η^2 is an a.s. finite random variable,

3) the σ-fields are nested:

$A_{nk} \subset A_{n+1,k}$ for $k = 1,\dots,k_n$, $n \in N$.

Then $S_{nk_n} \xrightarrow{d} S$ (stably), where the r.v. S has the characteristic function $E \exp(-\frac{1}{2} t^2 \eta^2)$.

Proof: A detailed examination of the proof in [1] (Theorem 3.2, p. 58-63) shows that we have only to prove that

$\prod_{k=1}^{k_n}(1 + itX_{nk}) \longrightarrow 1$ weakly in L^1 for all real t

assuming that $\sum_{k=1}^{J_n-1} X_{nk}^2 \leq C$ and $X_{nj} = 0$ for $j=J_n+1,\dots,k_n$.

Fix real t and put $M_n = \max\{|X_{nk}| \mid k=1,\dots,k_n\}$,

$T_{nk} = \prod_{j=1}^{k}(1 + itX_{nj})$ and $T_n = T_{nk_n}$.

a) We have $|T_{nk}| \leq \prod_{j=1}^{J_n} \sqrt{1 + t^2X_{nj}^2} \leq$

$\leq (1+|t| M_n)\exp(\frac{1}{2} t^2 \sum_{j=1}^{J_n-1} X_{nj}^2) \leq (1+|t| M_n)\exp(\frac{1}{2} t^2C)$.

Consequently $(T_{nk}, k=1,\dots,k_n, n \in N)$ is uniformly integrable by (1).

b) Fix $j \in N$ and f a bounded function which is A_{jk_j}-measurable. Then we have

$$ET_nf = E\{T_{nk_j} f\, E[\prod_{k=k_j+1}^{k_n}(1+ itX_{nk})/A_{nk_j}]\} = E\, T_{nk_j} f$$

for $n \geq j$ as X_{nk} are martingale differences.

It follows from (1) that $T_{nk_j} \xrightarrow{n} 1$, hence

$$E\, T_nf = E\, T_{nk_j} f \longrightarrow Ef \text{ by (a).}$$

c) Let f be an arbitrary measurable bounded function, such that $|f| \leq D$.

Denote $B = \sigma(\bigcup_{n=1}^{+\infty} \bigcup_{k=1}^{+\infty} A_{nk})$ and observe that

$B = \sigma(\bigcup_{n=1}^{+\infty} A_{nk_n})$ as the σ-fields are nested. For a fixed $j \in N$ we have

$$|E\{(T_n-1)E[f/B]\}| \leq E\{|T_n-1|\,|E[f/B]- E[f/A_{jk_j}]|\} +$$
$$+|E\{(T_n-1)E[f/A_{jk_j}]\}|$$

and by (a)

$$E\{|T_n-1|\,|E[f/B]-E[f/A_{jk_j}]|\} \leq 2D\exp(\tfrac{1}{2}t^2C)|t|M_n +$$

$$+(1+\exp(\tfrac{1}{2}t^2C))\,E|E[f/B]-E[f/A_{jk_j}]| .$$

Using (b) we get

$$\limsup_{n\to+\infty} |E(T_n-1)f| \leq (1+\exp(\tfrac{1}{2}t^2C))\,E|E[f/B]-E[f/A_{jk_j}]|$$

for all $j \in N$.

As $E[f/A_{jk_j}] \xrightarrow[j\to+\infty]{} E[f/B]$ a.s. it follows that $T_n \longrightarrow 1$ weakly in L^1. □

As a consequence to our Theorem we shall prove the law of large numbers for a zero-mean martingale with Feller-Lindeberg type condition.

Corollary: Let $(S_n, n\in N)$ be a zero-mean martingale with differences X_n for which the following assumptions hold:

$E|X_n| \leq D$ for all $n \in N$ and

$\frac{1}{n}\sum_{k=1}^{n} E\{|X_k|\,I(|X_k| \geq \varepsilon n)\} \longrightarrow 0$ for any $\varepsilon > 0$.

Then $\frac{1}{n}S_n \xrightarrow{w} 0$.

Proof: Denote $X_{nk} = \frac{1}{n}X_k$, $A_{nk} = \sigma(X_j, j=1,\dots,k)$, $k_n = n$ and $M_n = \max\{|X_k| \mid k=1,\dots,n\}$. Then $(X_{nk}, k=1,\dots,n)$ are martingale differences. It is enough to check the other assumptions of Theorem.

1) For $\varepsilon > 0$ we can write

$$E\max\{|X_{nk}| \mid k=1,\dots,n\} \leq \varepsilon + \frac{1}{n}E\{M_n\,I(M_n \geq \varepsilon n)\} \leq$$

$$\leq \frac{1}{n}\sum_{k=1}^{n} E\{|X_k|\,I(|X_k| \geq \varepsilon n)\} + \varepsilon .$$

Hence $E\max\{|X_{nk}| \mid k=1,\dots,n\} \longrightarrow 0$.

2) For B, $\varepsilon > 0$, we have

$$P\left(\sum_{k=1}^{n} X_{nk}^2 \geq \varepsilon\right) = P\left(\sum_{k=1}^{n} X_k^2 \geq \varepsilon n^2, \sum_{k=1}^{n} |X_k| \leq Bn\right) +$$

$$+ P\left(\sum_{k=1}^{n} X_k^2 \geq \varepsilon n^2, \sum_{k=1}^{n} |X_k| > Bn\right) \leq$$

$$\leq P\left(M_n \sum_{k=1}^{n} |X_k| \geq \varepsilon n^2, \sum_{k=1}^{n} |X_k| \leq Bn\right) + \frac{D}{B} \leq$$

$$\leq P\left(M_n \geq \frac{n}{B}\varepsilon\right) + \frac{D}{B}.$$

Using (1) we get $\limsup_{n\to+\infty} P\left(\sum_{k=1}^{n} X_{nk}^2 \geq \varepsilon\right) \leq \frac{D}{B}$

and consequently $\sum_{k=1}^{n} X_{nk}^2 \xrightarrow{n} 0$.

3) It is evident that the σ-fields are nested.

The required result then follows from Theorem. □

R e f e r e n c e s

[1] HALL,P., HEYDE C.C.: Martingale Limit Theory and Its Application, Academic Press, New York, 1980.

[2] McLEISH D.L.: Dependent central limit theorem and Invariance principles, Ann. Probab. 2(1974), 620-628.

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