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ON THE EXISTENCE OF BOUNDED SOLUTIONS OF DIFFERENTIAL
EQUATIONS IN BANACH SPACES
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Abstract: In this note we shall give sufficient conditions for the existence of bounded solutions of the differential equation $y' = f(t,y)$, $y(0) = x_0$, on the half-line $t \geq 0$. Here f is a function with values in a Banach space satisfying some conditions expressed in terms of an axiomatic measure of noncompactness μ . The proof of our theorem is suggested by the paper of Stokes [7] concerning finite dimensional vector differential equations.

Key words: Ordinary differential equations in Banach spaces, fixed point, measure of noncompactness.

Classification: 34G20, 47H09

Introduction: Let $(E, \|\cdot\|)$ be a Banach space. The closure of a subset A of E , its convex hull and its closed convex hull will be denoted, respectively, by \bar{A} , $\text{conv } A$ and $\overline{\text{conv } A}$. If A and B are subsets of E and t, s are real numbers, the $t \cdot A + s \cdot B$ is the set of all $t \cdot x + s \cdot y$, where $x \in A$ and $y \in B$. Further let \mathcal{M}_E denote the family of all nonempty and bounded subsets of E and \mathcal{K}_E - the family of all relatively compact and nonempty subsets of E .

A function $\mu: \mathcal{M}_E \rightarrow [0, +\infty)$ is said to be a measure of noncompactness if it satisfies the following conditions:
1° the family $\mathcal{P} = \{A \in \mathcal{M}_E: \mu(A) = 0\}$ is nonempty and $\mathcal{P} \subset \mathcal{K}_E$,
2° $\mu(\{x\}) = 0$ for all $x \in E$,
3° $A \subset B \implies \mu(A) \leq \mu(B)$,

- $4^\circ \quad \mu(\bar{A}) = \mu(A),$
 $5^\circ \quad \mu(\text{conv } A) = \mu(A),$
 $6^\circ \quad \mu(t \cdot A) = |t| \cdot \mu(A) \text{ for every } t \in \mathbb{R},$
 $7^\circ \quad \mu(A + B) \leq \mu(A) + \mu(B),$
 $8^\circ \quad \mu(A \cup B) \leq \max(\mu(A), \mu(B)).$

We put

$$\|A\| = \sup \{\|x\| : x \in A\}, \quad K(0,1) = \{x \in E : \|x\| \leq 1\}.$$

The following property of the function μ is true:

Lemma 1. If $A \in \mathcal{M}_E$ then $\mu(A) \leq \|A\| \cdot \mu(K(0,1))$.

Now let $J = [0, +\infty)$ and denote by $C(J)$ the set of all continuous functions from J to E . The set $C(J)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of J .

Let us put $X(t) = \{x(t) : x \in X\}$, $X_t = \cup \{X(s) : 0 \leq s \leq t\}$ for $t \in J$ and $X \subset C(J)$. We have

Lemma 2. If $X \subset C(J)$ is bounded and almost equicontinuous then $\mu(X_t) = \sup \{\mu(X(s)) : 0 \leq s \leq t\}$ for $t \in J$.

For properties of μ see [1],[2],[3],[4].

The Ascoli theorem we state as follows: $X \subset C(J)$ is conditionally compact if and only if X is almost equicontinuous and $X(t)$ is compact for each $t \in J$.

We shall use the following fixed-point theorem of Sadovskii type (see [3],[5],[6]):

Let \mathcal{K} be a nonempty closed convex subset of $C(J)$. Let $\Phi : 2^{\mathcal{K}} \rightarrow [0, +\infty)$ be a function with the following properties:

- (1) $\Phi(X) = 0 \Rightarrow \bar{X}$ is compact,
 (2) $\Phi(\overline{\text{conv } X}) = \Phi(X),$

$$(3) \quad \Phi(X \cup \{x\}) = \Phi(X)$$

for every subset X of \mathcal{X} and for each $x \in \mathcal{X}$.

Suppose that T is a continuous mapping of \mathcal{X} into itself and $\Phi(T[X]) < \Phi(X)$ for $\Phi(X) > 0$. Then T has a fixed point in \mathcal{X} .

Main result.

Theorem. Assume that $f: J \times E \rightarrow E$ is a function satisfying the following conditions:

- 1° for each fixed $x \in E$ the mapping $t \mapsto f(t, x)$ is measurable;
- 2° for each fixed $t \in J$ the mapping $x \mapsto f(t, x)$ is continuous;
- 3° $\|f(t, x)\| \leq G(t, \|x\|)$ for $(t, x) \in J \times E$, where the function G is nondecreasing in the second variable such that $t \mapsto \int_0^t G(s, u) ds$ is locally bounded for any fixed $u \in J$ and $t \mapsto \int_0^t G(s, y(s)) ds$ is measurable for every continuous bounded function $y: J \rightarrow J$;

- 4° the scalar inequality

$$g(t) \geq \|x_0\| + \int_0^t G(s, g(s)) ds$$

has a bounded solution g existing on J ,

(let us put $r_0 = \sup \{g(t) : t \in J\}$ and $Z_0 = \{x \in E : \|x\| \leq r_0\}$)

- 5° there exist functions m, p of J into itself such that

- (i) m is measurable and integrable on compact subsets of J with

$$M = \sup \left\{ \int_0^t m(s) ds : t \in J \right\} < \infty,$$

- (ii) p is nondecreasing such that $M \cdot p(t) < t$ for $t > 0$,

- (iii) for any $t > 0, \epsilon > 0, X \subset Z_0$ there exists a closed subset $Q \subset [0, t]$ such that $\text{mes}([0, t] \setminus Q) < \epsilon$ and

$$\mu(f[I \times X]) \leq \sup \{m(s) : s \in I\} \cdot p(\mu(X))$$

for each closed subset I of Q .

Then the differential equation

$$y' = f(t, y)$$

with the initial condition $y(0) = x_0$ has at least one solution y defined on J and $\|y(t)\| \leq g(t)$ for $t \in J$.

Proof: Denote by \mathcal{X} the set of all $x \in C(J)$ such that $\|x(t)\| \leq g(t)$ on J and

$$\|x(t_1) - x(t_2)\| \leq \left| \int_{t_1}^{t_2} G(s, r_0) ds \right| \text{ for } t_1, t_2 \in J.$$

The set \mathcal{X} is nonempty closed convex bounded and almost equicontinuous subset of $C(J)$.

Let us put

$$\Phi(X) = \sup \{ \mu(X(t)) : t \in J \} \text{ for a subset } X \subset \mathcal{X}.$$

Obviously $\Phi(X) < \infty$, $\Phi(X_1) \leq \Phi(X_2)$ for $X_1 \subset X_2$ and

$$\Phi(X \cup \{x\}) = \Phi(X) \text{ for } x \in \mathcal{X}.$$

Since

$$(\overline{\text{conv}} X)(t) = (\overline{\text{conv}} X)(t) \subset \overline{(\text{conv} X)(t)} \subset \overline{\text{conv}(X(t))}$$

so $\mu((\overline{\text{conv}} X)(t)) \leq \mu(\overline{\text{conv}(X(t))}) = \mu(X(t))$.

The inverse inequality immediately follows from the inclusion

$X(t) \subset (\overline{\text{conv}} X)(t)$. Hence $\Phi(\overline{\text{conv}} X) = \Phi(X)$. If $\Phi(X) = 0$ then

$\overline{X(t)}$ is compact for every $t \in J$; therefore Ascoli's theorem proves that \overline{X} is compact in $C(J)$.

To apply our fixed-point theorem we define the mapping T as follows:

$$\text{for } y \in \mathcal{X}, (T(y))(t) = x_0 + \int_0^t f(s, y(s)) ds.$$

It is easy to see that T is continuous and $T[\mathcal{X}] \subset \mathcal{X}$.

Let X be a subset of \mathcal{X} such that $\Phi(X) > 0$. To prove the theorem it remains to be shown that $\Phi(T[X]) < \Phi(X)$. To this end, fix t in J . Let $\varepsilon \in (0, 1)$ and $\sigma = \sigma(\varepsilon) > 0$ be a number such that $\int_A G(s, r_0) ds < \varepsilon$ for each measurable $A \subset [0, t]$ with $\text{mes}(A) < \sigma$. By the Luzin theorem there exists a closed subset B_1 of $[0, t]$ with $\text{mes}([0, t] \setminus B_1) < \sigma/2$ such that the function m is continuous on

B_1 . Furthermore, by assumption 5°(iii) there exists a closed subset B_2 of $[0, t]$ such that $\text{mes}([0, t] \setminus B_2) < \sigma/2$ and $\mu(f[I \times X_t]) \leq \sup\{m(s) : s \in I\} \cdot p(\mu(X_t))$ for each closed subset I of B_2 .

Let us put $B = B_1 \cap B_2$, $A = [0, t] \setminus B$. Hence $\text{mes}(A) < \sigma$. Since m is uniformly continuous on B , for any given $\epsilon' > 0$ there exists $\eta > 0$ such that $t', t'' \in B$ and $|t' - t''| < \eta$ implies $|m(t') - m(t'')| < \epsilon'$. Let $t_0 = 0 < t_1 < \dots < t_n = t$ be the partition of the interval $[0, t]$ with $\max\{|t_{j-1} - t_j| : 1 \leq j \leq n\} < \eta$. Moreover, let $I_j = [t_{j-1}, t_j] \cap B$ and s_j be a point in I_j such that $m(s_j) = \sup\{m(s) : s \in I_j\}$.

Putting

$$\int_I f(s, X(s)) ds = \int_I f(s, x(s)) ds : x \in X$$

we get

$$\left\| \int_A f(s, X(s)) ds \right\| \leq \int_A G(s, x_0) ds < \epsilon < 1.$$

By the mean-value theorem, for $x \in X$ we have

$$\begin{aligned} \int_B f(s, x(s)) ds &= \sum_{j=1}^n \int_{I_j} f(s, x(s)) ds \in \\ &\in \sum_{j=1}^n \text{mes}(I_j) \overline{\text{conv}}(\{f(s, x(s)) : s \in I_j\}) \subset \\ &\subset \sum_{j=1}^n \text{mes}(I_j) \overline{\text{conv}}(f[I_j \times X_t]), \\ \text{hence } \int_B f(s, X(s)) ds &\subset \sum_{j=1}^n \text{mes}(I_j) \overline{\text{conv}}(f[I_j \times X_t]). \text{ Thus} \\ \mu(\mathbb{T}[X](t)) &\leq \mu(\{x_0\} + \int_A f(s, X(s)) ds + \int_B f(s, X(s)) ds) \leq \\ &\leq \mu(\{x_0\}) + \left\| \int_A f(s, X(s)) ds \right\| \cdot \mu(K(0, 1)) + \\ &+ \sum_{j=1}^n \text{mes}(I_j) \cdot \mu(f[I_j \times X_t]) \leq \epsilon \cdot \mu(K(0, 1)) + \\ &+ \sum_{j=1}^n \text{mes}(I_j) m(s_j) p(\mu(X_t)) \leq \epsilon \cdot \mu(K(0, 1)) + \\ &+ p(\mu(X_t)) \cdot \left(\sum_{j=1}^n \int_{I_j} |m(s_j) - m(s)| ds + \sum_{j=1}^n \int_{I_j} m(s) ds \right) \leq \\ &\leq \epsilon \cdot \mu(K(0, 1)) + p(\mu(X_t)) \cdot (\epsilon' \cdot t + \int_0^t m(s) ds) \end{aligned}$$

and therefore

$$\mu(T[X](t)) \leq \varepsilon \cdot \mu(K(0,1)) + M \cdot p(\mu(X_t)).$$

Since with respect to Lemma 2

$$\mu(X_t) = \sup \{ \mu(X(s)) : 0 \leq s \leq t \} \leq \Phi(X)$$

we obtain

$$\mu(T[X](t)) \leq \varepsilon \cdot \mu(K(0,1)) + M \cdot p(\Phi(X));$$

as $\varepsilon > 0$ is arbitrary, this implies

$$\mu(T[X](t)) \leq M \cdot p(\Phi(X)).$$

Hence $\Phi(T[X]) \leq M \cdot p(\Phi(X)) < \Phi(X)$, and consequently T has a fixed point in \mathcal{X} . The proof is complete.

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