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EXTENSION OF DIFFERENTIABLE FUNCTIONS

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Abstract. Let $H \subset \mathbb{R}^n$ be closed and let $F: H \longrightarrow \mathbb{R}^n$ be differentiable with respect to H. It is shown that

(i) F' is Baire 2 on H;

(iii) F' is not necessarily Baire 1;

(iii) F can be extended to $\mathbb{R}^n$ as an everywhere differentiable function if and only if F' is Baire 1 on H.

Key words: Differentiable functions of several variables, extensions.

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1. Introduction. Let H be a perfect subset of $\mathbb{R}$ and let $f: H \longrightarrow \mathbb{R}$ be differentiable with respect to H. It is easy to see that f' is Baire 1 on H and it is also well-known that f can be extended to $\mathbb{R}$ as an everywhere differentiable function (see e.g. [3],[4]). In this paper we are going to investigate the analogous problems in the n dimensional Euclidean space $\mathbb{R}^n$.

Let $\mathcal{L}(\mathbb{R}^n)$ denote the linear space of all linear forms on $\mathbb{R}^n$ endowed with the usual norm. Let H be a subset of $\mathbb{R}^n$ and $F: H \longrightarrow \mathbb{R}$. F is said to be differentiable at $a \in H$ if there is $L(a) \in \mathcal{L}(\mathbb{R}^n)$ such that

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1) Part of this work was done while the second and third author visited the University of Naples and was completed while they participated in the Special Year in Real Analysis at the University of California, Santa Barbara.

$$\lim_{\substack{x \to a \\ x \in H}} \frac{F(x) - F(a) - (L(a), x - a)}{|x - a|} = 0.$$

The linear form $L(a)$ is called the derivative of F at a; if it is determined uniquely, it is also denoted by $F'(a)$. The function F is said to be differentiable on H, if it is differentiable at every point of H.

Let $H \subset \mathbb{R}^n$ be closed and let $F: H \to \mathbb{R}$ be differentiable on H. Obviously, if its derivative is not determined uniquely, it need not be in the first class. For example, it suffices to consider the segment $H = [0,1]$ as a subspace of $\mathbb{R}^2$, the function $F = 0$ and its derivative $(L(x),(u_1,u_2)) = 0$ if $x \in A$ and $(L(x),(u_1,u_2)) = u_2$ if $x \notin A$, where A is, say, a nonmeasurable subset of H.

A natural conjecture seems to be that the derivative of F is in the first class provided that it is determined uniquely. We prove that this is not the case (Theorem 5). However, if F' is determined uniquely, it is of Baire class 2 (Theorem 4(i)). Also, if the tangent space of H is sufficiently rich, then F' is in the first class (Theorem 4(ii)).

Since the derivative of an everywhere differentiable function is Baire 1, a function $F: H \to \mathbb{R}$, differentiable on the closed set H can be extended to an everywhere differentiable function only if its derivative is Baire 1 on H. We show that this condition is sufficient as well (Theorem 7). For the proof we will need a generalization of the following theorem of L.E. Snyder [5]. If f is Baire 1 on the compact metric space X then there is a function $g: (X \times \mathbb{R}^+) \to \mathbb{R}$ such that f is the limit of g along the Stolz cones $\{(x,y);\ y > \mathrm{dist}(x,a)\}$ $(a \in X)$. We prove that the assertion remains valid if we replace $X \times \mathbb{R}^+$ and

$X \times \{0\}$ by an arbitrary metric space and a nowhere dense closed subset, respectively (Theorem 6).

2. Baire class of derivatives. Let H be a subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. A vector $u \in \mathbb{R}^n$ is called a tangent vector to H at x if

$$\liminf_{r \to 0+} \operatorname{dist}(x + ru, H)/r = 0.$$

The set of all tangent vectors to H at x is denoted by $\operatorname{Tan}(H,x)$.

Lemma 1. Let $L(a)$ be a derivative of the function $f: H \to \mathbb{R}$ at $a \in H$. Then $A \in \mathcal{L}(\mathbb{R}^n)$ is a derivative of F at a if and only if

$$\operatorname{Tan}(H,a) \subset \operatorname{Ker}(A - L(a)).$$

Proof. Let $B = A - L(a)$ and suppose first that A is a derivative of F at a. Then B is a derivative of 0 at a and hence

$$\lim_{\substack{x \to a \\ x \in H}} \frac{(B, x - a)}{|x - a|} = 0. \tag{1}$$

Whenever $u \in \operatorname{Tan}(H,a)$, we can find a sequence r_n of positive numbers converging to zero and a sequence $x_n \in H$ such that

$$\lim_{n \to \infty} |x_n - (a + r_n u)|/r_n = 0.$$

Then

$$|(B,u)| = \lim_{n \to \infty} \frac{|(B, x_n - a)|}{r_n} = \lim_{n \to \infty} \frac{|(B, x_n - a)|}{|x_n - a|} \cdot \frac{|x_n - a|}{r_n} = 0$$

since $|x_n - a| \leq |x_n - (a + r_n u)| + |r_n u| \leq r_n(1 + |u|)$ for n large enough. Therefore $(B,u) = 0$ for every $u \in \operatorname{Tan}(H,x)$ and hence $\operatorname{Tan}(H,x) \subset \operatorname{Ker} B$.

Now let $\operatorname{Tan}(H,x) \subset \operatorname{Ker} B$; first we prove (1). Suppose indirectly that there are $\varepsilon > 0$ and a sequence $x_n \in H \setminus \{a\}$, $x_n \to a$

such that

$$\frac{|(B,x_n - a)|}{|x_n - a|} \geq \varepsilon .$$

There is a subsequence $\{x_{n_k}\}$ such that $\dfrac{x_{n_k} - a}{|x_{n_k} - a|}$ converges to a unit vector u. It is easy to check that $u \in \mathrm{Tan}(H,x)$ and $\langle B,u\rangle \neq 0$, a contradiction. Therefore (1) holds true and hence B is a derivative of 0 and $A = L(a) + B$ is a derivative of F at a.

Corollary 2. Whenever $a \in H \subset \mathbb{R}^n$, the following statements are equivalent.

(i) For every function $F: H \longrightarrow \mathbb{R}$ differentiable at a, the derivative is determined uniquely.

(ii) $\mathrm{Tan}(H,a)$ spans $\mathbb{R}^n$.

Proposition 3. Let H be a subset of $\mathbb{R}^n$ and let, for each $x \in H$, $a_H(x) = \sup \{\det(u^1,\dots,u^n);\ u^1,\dots,u^n$ are unit vectors from $\mathrm{Tan}(H,x)\}$. Then

(i) for every $a > 0$ the set $E_a = \{x \in H;\ a_H(x) \geq a\}$ is a G_δ subset of H, and

(ii) whenever $F: H \longrightarrow \mathbb{R}$ is differentiable on H and $a > 0$, then F' as a map from E_a to $\mathcal{L}(\mathbb{R}^n)$ is of Baire class 1 relative to E_a.

Proof. Let F be a function differentiable on H and let M be a closed subset of $\mathcal{L}(\mathbb{R}^n)$. We intend to prove that the set $B = \{x \in E_a;\ F'(x) \in M\}$ is a G_δ subset of H. This clearly implies the second statement of the proposition and, since one may choose $F = 0$ and $M = \{0\}$, also its first statement.

For each $x \in B$ and each $k = 1,2,\dots$ we find numbers $c_k(x) \in (0,2^{-k})$, $t_k^1(x),\dots,t_k^n(x) \in (0,c_k(x))$ and unit vectors

$u_k^1(x),\dots,u_k^n(x) \in \mathbb{R}^n$ such that $|F(y)-F(x)-\langle F'(x),y-x\rangle| <$ $< 2^{-k}\|y-x\|$ whenever $y\in H$ and $0<\|y-x\|<c_k(x)$, $x+t_k^i(x)u_k^i(x)\in H$ for each $i = 1,\dots,n$, and $\det(u_k^1(x),\dots,u_k^n(x)) > a - 2^{-k}$.

Next we use the continuity of F on H to find $d_k(x) \in (0,2^{-k}\min(t_k^1(x),\dots,t_k^n(x)))$ such that $|F(x + t_k^i(x)u_k^i(x)) - F(y) - \langle F'(x), t_k^i(x)u_k^i(x)\rangle| \le 2^{-k}t_k^i(x)$ for each $i = 1,\dots,n$ and each $y\in H$ with $\|y-x\| < d_k(x)$.

Whenever $y \in H \cap \bigcap_{k=1} \bigcup_{x\in B}\{y\in\mathbb{R}^n;\ \|y-x\| < d_k(x)\}$, we find a sequence $x_k\in B$ such that $\|x_k-y\| < d_k(x_k)$. There is a subsequence $k_1<k_2<\dots$ such that
$u^i = \lim_{j\to\infty} u^i_{k_j}(x_{k_j})$ exists for each $i = 1,\dots,n$. To simplify the notation, we put $z_j = x_{k_j}$, $t_j^i = t^i_{k_j}(z_j)$, $u_j^i = u^i_{k_j}(z_j)$, $v_j^i =$ $= t_j^iu_j^i$, and $d_j = d_{k_j}(z_j)$.

From $z_j + v_j^i \in H$ and from

$$\lim_{j\to\infty}(t_j^i)^{-1}\|(z_j + v_j^i - y) - t_j^iu^i\| \le \lim_{j\to\infty}[(t_j^i)^{-1}\|z_j-y\| + \|u_j^i-u^i\|] \le \lim 2^{-j} = 0$$

we infer that $u^i\in \mathrm{Tan}(H,y)$ for each $i = 1,\dots,n$. Since clearly $\det(u^1,\dots,u^m)\ge a$, $y\in E_a$.

Whenever $\varepsilon > 0$ and j is sufficiently large, we have $|F(z_j + v_j^i) - F(y) - \langle F'(y),(z_j + v_j^i - y)\rangle| \le \varepsilon\,\|z_j + v_j^i - y\| \le$ $\le \varepsilon(t_j^i + d_j) \le 2\varepsilon\, t_j^i$.

Hence

$$|\langle F'(z_j),v_j^i\rangle - \langle F'(y),v_j^i\rangle| \le |F(z_j + v_j^i) - F(y) - \langle F'(z_j),v_j^i\rangle| + 2\,\varepsilon t_j^i + |\langle F'(y),z_j-y\rangle| \le (2^{-j} + 2\varepsilon)t_j^i + \|F'(y)\|d_j$$
$$\le (2^{-j} + 2\varepsilon + 2^{-j}\|F'(y)\|)t_j^i \text{ for each } i = 1,\dots,n.$$

Consequently, $\lim_{j\to\infty}|\langle F'(z_j),u_j^i\rangle - \langle F'(y),u_j^i\rangle| = 0$ for each $i = 1,\dots,n$, which, together with $\lim_{j\to\infty} \det(u_j^1,\dots,u_j^n) > 0$ imply that $F'(y) = \lim F'(z_j)$. Since $F'(z_j) \in M$ for each $j = 1,2,\dots$, $F'(y) \in M$ and therefore $y \in B$. Thus

$$B \supset H \cap \bigcap_{k=1}^{\infty} \bigcup_{x\in B} \{y \in \mathbb{R}^n;\ \|y-x\| < d_k(x)\},$$

and, since the converse inclusion is obvious, B is a G_δ subset of H.

__Theorem 4.__ Let H be a subset of $\mathbb{R}^n$ such that $\mathrm{Tan}(H,x)$ spans $\mathbb{R}^n$ for every $x \in H$.

(i) If $F: H \to \mathbb{R}$ is differentiable on H, then F' as a map from H to $\mathcal{L}(\mathbb{R}^n)$ is of Baire class 2 relative to H.

(ii) If H can be covered by countably many relatively closed subsets H_k such that $\inf \{a_H(x);\ x \in H_k\} > 0$ for each k, then the derivative of every function differentiable on H is of the first class on H.

__Proof.__ Both statements follow immediately from Proposition 3.

__Theorem 5.__ There exist a compact set $H \subset \mathbb{R}^2$ and a function $F: H \to \mathbb{R}$ such that F is differentiable on H, $F'(x)$ is uniquely determined at every point of H and F' is not Baire 1.

__Proof.__ Let C denote the Cantor ternary set in $[0,1]$ and let $]a_n,b_n[$ be the components of $]0,1[\setminus C$. We denote $T_n = \{a_1,\dots,a_n, b_1,\dots,b_n\}$, $T = \bigcup_{m=1}^{\infty} T_n$ and $C' = C \setminus T$.

For every fixed n we construct a set $S_n \subset C'$ with $\overline{S}_n \setminus S_n \subset T_n \cup \{0,1\}$ and such that for every $t \in C \setminus T_n$ there is an $s \in S_n$ with $|t - s| < \mathrm{dist}^4(t,T_n)$.

Let $]\alpha_j, \beta_j[$ $(j = 1,\dots,n+1)$ denote the components of $]0,1[\setminus \bigcup_{i=1}^{n} [a_i,b_i]$. For every $j = 1,\dots,n+1$ we choose an increasing sequence $\{x_k^{(j)}\}_{k=-\infty}^{\infty}$ such that $\lim_{k\to-\infty} x_k^{(j)} = \alpha_j$, $\lim_{k\to\infty} x_k^{(j)} = \beta_j$ and $0 < x_{k+1}^{(j)} - x_k^{(j)} < \min((x_k^{(j)} - \alpha_j)^4, (\beta_j - x_{k+1}^{(j)})^4)$ for every k. We select a point $s \in C' \cap [x_k^{(j)}, x_{k+1}^{(j)}]$ whenever this intersection is nonempty and we denote by S_n the set of these points. It is easy to check that S_n satisfies our requirements.

We denote

$$S_n^* = \bigcup_{s \in S_n} \{(x,y) \in \mathbb{R}^2;\ |x - s| \le y \le \tfrac{1}{2}\operatorname{dist}^2(s,T_n)\}$$

and

$$H_n = \{(x,y);\ a_n \le x \le b_n,\ 0 \le y \le (b_n - a_n)\min(x - a_n, b_n - x)\}$$

$$(n = 1,2,\dots).$$

Finally, we define

$$H = \bigcup_{n=1}^{\infty} (H_n \cup S_n^*) \cup (C \times \{0\})$$

and

$$F(x,y) = \begin{cases} 0 \text{ if } (x,y) \in H \setminus \bigcup_{n=1}^{\infty} H_n \\ y \text{ if } (x,y) \in \bigcup_{n=1}^{\infty} H_n. \end{cases}$$

It is easy to see that H is a compact subset of $\mathbb{R}^2$. We show that $\operatorname{Tan}(H,x)$ spans $\mathbb{R}^2$ for every $x \in H$. This is obvious for $x \in H \setminus (C \times \{0\})$. If $x = (t,0) \in (C \times \{0\})$ and if $t \in T_n$ then x is a vertex of H_n and the assertion is also clear. If $t \in C'$ then, obviously, $(1,0) \in \operatorname{Tan}(H,x)$. We prove that $(0,1) \in \operatorname{Tan}(H,x)$. Let $r_n = \operatorname{dist}^2(t,T_n)$ and choose an $s_n \in S_n$ with $|s_n - t| < r_n^2$. Since $(s_n, \frac{1}{8}r_n) \in H$ if n is sufficiently large,

$$\lim_{n\to\infty} \operatorname{dist}(x + \tfrac{1}{8}r_n(0,1), H)/r_n \le \lim_{n\to\infty} |(t,\tfrac{1}{8}r_n) - (s_n,\tfrac{1}{8}r_n)|/r_n$$

$$= \lim_{n\to\infty} |t - s_n|/r_n \le \lim_{n\to\infty} r_n = 0.$$

Hence (0,1) belongs to $\operatorname{Tan}(H,x)$.

We claim that $F'(x,y) = 0$ for $(x,y) \in H \setminus \bigcup_{m=1}^{\infty} H_n$ and $(F'(x,y), (u,v)) = v$ for $(x,y) \in \bigcup_{m=1}^{\infty} H_n$. This is obvious for all points $(x,y) \in H \setminus (C \times \{0\})$. To prove the remaining case, we first note that, whenever $t \in C$ and $(u,v) \in H_n$, then $0 \leq v \leq (b_n - a_n) \cdot \operatorname{dist}(u, \{a_n, b_n\}) \leq (b_n - a_n)|u - t|$. Since $b_n - a_n \to 0$, this shows $F'(t,0) = 0$ for every $t \in C'$. Now let $t \in \{a_m, b_m\}$, we have to show

$$\lim_{\substack{(u,v) \to (t,0) \\ (u,v) \in H}} \frac{F(u,v) - v}{|(u,v) - (t,0)|} = 0.$$

Since $F(u,v) = v$ if $(u,v) \in \bigcup_{m=1}^{\infty} H_n$ and $F(u,v) = 0$ otherwise, it is enough to prove that

$$\lim_{\substack{(u,v) \to (t,0) \\ (u,v) \in \cup S_n^*}} \frac{v}{u - t} = 0.$$

Since a_m and b_m do not belong to the closure of the set $\bigcup_{m=1}^{m-1} S_n^*$, thus $\delta = \operatorname{dist}((t,0), \bigcup_{m=1}^{m-1} S_n^*) > 0$. If $(u,v) \in \bigcup_{m=1}^{\infty} S_n^*, \operatorname{dist}((u,v), (t,0)) < \delta$, then $(u,v) \in \bigcup_{m=m}^{\infty} S_n^*$ and hence there is $n \geq m$ and $s \in S_n$ such that $|u - s| \leq v \leq \frac{1}{2} \operatorname{dist}^2(s, T_n) \leq \frac{1}{2}(s - t)^2$. Therefore $|u - t| \geq |s - t| - |u - s| \geq |s - t| - \frac{1}{2}(s - t)^2 \geq \geq \frac{1}{2}|s - t|$ and $v \leq \frac{1}{2}(s - t)^2$, which proves our assertion.

Finally, we note that F' is not Baire 1 on H, since $C \times \{0\}$ contains no point of continuity of the restriction of F' to $C \times \{0\}$.

3. <u>Extension of differentiable functions.</u> Our next result will be used in the proof of the extension theorem, but may have some interest in itself. Let H be a nowhere dense, closed subset of the metric space (X,d). By a Stolz cone with vertex $a \in H$ we

mean a set

$$\{x\in X;\ \operatorname{dist}(x,H)\geq c\cdot d(x,a)\},$$

where c is a positive constant. Our theorem implies that if $f:H\longrightarrow \mathbb{R}$ is of Baire class 1 then there is a function $g:X\longrightarrow \mathbb{R}$ such that for every $a\in H$, $\lim\limits_{x\to a} g(x) = f(a)$ relative to any Stolz cone with vertex a. Using a locally finite, continuous partition of unity of $X\setminus H$ subordinated to the system of balls with center $x\in X\setminus H$ and radius $\frac{1}{2}$ dist (x,H), one can easily show that g can be chosen to be continuous. This result is a generalization of a theorem by L.E. Snyder [5].

Theorem 6. Let (X,d) be a metric space, let H be a closed subset of X and let $f:H\longrightarrow \mathbb{R}$ be of Baire class 1 on H. Then there exists a function $g(X\setminus H)\longrightarrow \mathbb{R}$ such that

$$\lim_{\substack{x\to a\\ x\in X\setminus H}} |g(x)-f(a)|\,\frac{\operatorname{dist}(x,H)}{d(x,a)} = 0 \tag{3}$$

for every $a\in\partial H$.

Proof. First we remark that if $f:Y\longrightarrow \mathbb{R}$ is a Baire 1 function defined on the metric space Y then f is the pointwise limit of a sequence of bounded Lipschitz functions. This has been proved by Hausdorff in a more general setting (see [2], § 41, pp. 264-276); or it follows more directly from [1], Proposition 3.9.

Applying this result to $Y = H$ as a subspace of X and to $f:H\longrightarrow \mathbb{R}$, we get a sequence $f_n:H\longrightarrow \mathbb{R}$ of bounded Lipschitz functions converging to f on H. Let $1\leq M_1\leq M_2\leq\ldots$ and $0<K_1\leq K_2\leq\ldots$ be such that $|f_n|\leq M_n$ and $|f_n(x)-f_n(y)|\leq K_n\, d(x,y)$ $(n=1,2,\ldots;\ x,y\in H)$.

Let $x\in X\setminus H$ be fixed. If dist $(x,H)\geq (K_1(M_1+2))^{-1}$, we define $g(x) = 0$. If, for a natural number n,

(4) $\quad [(n+1)K_{n+1}((n+1)M_{n+1}+2)]^{-1} \leq \operatorname{dist}(x,H) < [nK_n(nM_n+2)]^{-1}$

then we select a point $u(x) \in H$ with $d(x,u(x)) < 2 \operatorname{dist}(x,H)$ and define $g(x) = f_n(u(x))$. We prove that if $a \in H$, $x \in X \setminus H$ and (4) holds then

(5) $\quad |g(x) - f(a)| \dfrac{\operatorname{dist}(x,H)}{d(x,a)} \leq \dfrac{1}{n} + \dfrac{|f(a)|}{n} + |f_n(a) - f(a)|.$

Since $f_n(a) \to f(a)$, this will prove (3). We distinguish between two cases.

If $\dfrac{\operatorname{dist}(x,H)}{d(x,a)} \leq \dfrac{1}{nM_n}$ then we have $|g(x) - f(a)| \dfrac{\operatorname{dist}(x,H)}{d(x,a)} \leq$

$$\leq |f_n(u(x)) - f(a)|$$

$\cdot \dfrac{1}{nM_n} \leq \dfrac{M_n}{nM_n} + \dfrac{|f(a)|}{nM_n}$ and thus (5) holds true.

If $\dfrac{\operatorname{dist}(x,H)}{d(x,a)} > \dfrac{1}{nM_n}$ then $d(u(x),a) \leq d(u(x),x) + d(x,a) <$

$< 2 \operatorname{dist}(x,H) + nM_n \operatorname{dist}(x,H) = (nM_n + 2) \operatorname{dist}(x,H) < 1/nK_n$,

and hence

$$|g(x) - f(a)| = |f_n(u(x)) - f(a)| \leq |f_n(u(x)) - f_n(a)| +$$
$$+ |f_n(a) - f(a)| \leq K_n d(u(x),a) + |f_n(a) - f(a)| < \frac{1}{n} +$$
$$+ |f_n(a) - f(a)|.$$

Since $\operatorname{dist}(x,H) \leq d(x,a)$, this implies (5), which completes the proof.

<u>Theorem 7</u>. Suppose that H is a closed subset of $\mathbb{R}^n$, F is a real valued function defined on H and $L: H \to \mathcal{L}(\mathbb{R}^n)$ is a derivative of F on H. Then F can be extended to a function $\hat{F}$ everywhere differentiable on $\mathbb{R}^n$ with

$$\hat{F}'(x) = L(x) \qquad (x \in H)$$

if and only if L is a map of the first class.

__Proof.__ The necessity of the condition is obvious, since, by Theorem 4(ii), the derivative of every function differentiable on $\mathbb{R}^n$ is Baire 1. In order to prove the sufficiency, suppose that $f: H \longrightarrow \mathbb{R}$ and L satisfy the conditions of the theorem.

Then there is a map $A: (\mathbb{R}^n \setminus H) \longrightarrow \mathcal{L}(\mathbb{R}^n)$ such that

$$\lim_{\substack{u \to a \\ u \notin H}} \| A(u) - L(a) \| \frac{\operatorname{dist}(u,H)}{|u - a|} = 0 \tag{6}$$

for every $a \in \partial H$. Indeed, let

$$(L(x),u) = \sum_{i=1}^{n} L_i(x) u_i \qquad (x \in H,\ u = (u_1,\dots,u_n) \in \mathbb{R}^n),$$

then the functions $L_i: H \longrightarrow \mathbb{R}$ are of Baire class 1. By Theorem 6, there are functions $g_i: (\mathbb{R}^n \setminus H) \longrightarrow \mathbb{R}$ such that (3) holds with $X = \mathbb{R}^n$ and $f = L_i$. We define $(A(u),v) = \sum_{i=1}^{n} g_i(u) v_i$ $(u \in \mathbb{R}^n \setminus H,\ v = (v_1,\dots,v_n) \in \mathbb{R}^n)$, then the map $A: (\mathbb{R}^n \setminus H) \to$ $\longrightarrow \mathcal{L}(\mathbb{R}^n)$ satisfies (6).

Let $T: \mathbb{R}^n \longrightarrow H$ be a map with $|u - T(u)| = \operatorname{dist}(u,H)$ $(u \in \mathbb{R}^n)$. Let Φ_j be a locally finite C^1 partition of unity on $\mathbb{R}^n \setminus H$ subordinated to the system of open balls with center $u \in \mathbb{R}^n \setminus H$ and radius $\frac{1}{6} \operatorname{dist}(u,H)$. For every j, let $u_j \in \mathbb{R}^n \setminus H$ be chosen such that $\Phi_j(u_j) \neq 0$. We define $\hat{F}$ by

$$\hat{F}(u) = \begin{cases} F(u) \text{ if } u \in H, \\ \sum_j \Phi_j(u)[F(T(u_j)) + (A(u_j), u - T(u_j))] \text{ if } u \in H. \end{cases}$$

Let $a \in \partial H$. Then for every $u \notin H$

$$\begin{aligned} &|\hat{F}(u) - \hat{F}(a) - (L(a), u - a)| \\ &= |\sum_j \Phi_j(u)[F(T(u_j)) + (A(u_j), u - T(u_j)) - \hat{F}(a) - (L(a), u - a)]| \\ &= |\sum_j \Phi_j(u)[F(T(u_j)) - F(a) - (L(a), T(u_j) - a) - (L(a) - A(u_j), \\ &\qquad u - T(u_j))]| \\ &\leq \sum_{\substack{j \\ \Phi_j(u) \neq 0}} \Phi_j(u) |F(T(u_j)) - F(a) - (L(a), T(u_j) - a)| + \end{aligned}$$

$$+ \sum_{\Phi_j(u) \neq 0} \Phi_j(u) |\langle L(a)-A(u_j), u-T(u_j)\rangle| \overset{\text{def}}{=} \Sigma_1(u) + \Sigma_2(u).$$

Now $\Phi_j(u) \neq 0$ implies $|u - u_j| < \frac{1}{2}$ dist(u,H), from which

$$(7) \quad |u-T(u_j)| \leq |u-u_j| + |u_j-T(u_j)| \leq |u-u_j| + |u_j-T(u)|$$
$$\leq 2|u-u_j| + |u-T(u)| \leq 2 \text{ dist}(u,H).$$

Thus we have

$$|T(u_j)-a| \leq |u-T(u_j)| + |u-a| \leq 2 \text{ dist}(u,H) + |u-a| \leq 3|u-a|.$$

Since $F'(a) = L(a)$, there is a function $w: H \longrightarrow \mathbb{R}$ such that

$$\lim_{\substack{z \to a \\ z \in H}} w(z) = 0 \text{ and}$$

$$|F(z)-F(a)-\langle L(a), z-a\rangle| = w(z)|z-a| \quad (z \in H).$$

Then

$$|\Sigma_1(u)| = \sum_{\Phi_j(u) \neq 0} \Phi_j(u) w(T(u_j)) |T(u_j)-a| \leq 3|u-a|$$

$$\sum_{\Phi_j(u) \neq 0} \Phi_j(u) \sup\{w(z); z \in H, |z-a| \leq 3|u-a|\} = 3|u-a| \sup\{w(z);$$
$$z \in H, |z-a| \leq 3|u-a|\}.$$

Hence

$$\lim_{\substack{u \to a \\ u \notin H}} \frac{|\Sigma_1(u)|}{u-a} = 0.$$

On the other hand, by (7), we get

$$\frac{|\langle L(a)-A(u_j), u-T(u_j)\rangle|}{u-a} \leq \frac{\|L(a)-A(u_j)\| \cdot 2 \text{ dist}(u,H)}{|u-a|}$$

$$\leq \|L(a)-A(u_j)\| \frac{4 \cdot \text{dist}(u_j,H)}{\frac{1}{2}|u_j-a|}$$

which, taking (6) into consideration, implies

$$\lim_{\substack{u \to a \\ u \notin H}} \frac{\Sigma_2(u)}{|u-a|} = 0.$$

Since, from the differentiability of F on H,

$$\lim_{\substack{u \to a \\ u \in H}} \frac{\hat{F}(u)-\hat{F}(a)-(L(a),u-a)}{u-a} = 0,$$

we obtain $\hat{F}'(a) = L(a)$. This finishes the proof of the theorem, since $\hat{F}$ is continuously differentiable on $\mathbb{R}^n \setminus H$.

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