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1-PERFECT CODES OVER SELF-COMPLEMENTARY GRAPHS

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Abstract: We discuss the existence of 1-perfect codes in second powers of graphs. We show a simple lower bound on the cardinality of such a code and we prove that exactly self-complementary graphs satisfy the equality in this bound.

Key words: Graph, perfect code.

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1. Introduction. All graphs considered are undirected, without loops and multiple edges. We use a notation G = (V,E) for a graph G with a vertex set V and edge set E.

By a product of graphs we mean the cartesian product, i.e. the graph on the cartesian product of vertex sets, whose distance function ∂ is the sum of distances in coordinates. The product of n copies of the same graph G is denoted G^n and called an n-th power of G.

Given a graph G = (V,E), any subset C of V is called a code in G. Such a code is called t-perfect iff the closed neighbourhoods of radii t with centres in code-vertices form a partition of V. In particular, C is a 1-perfect code iff i) $\partial(u,v) \geq 3$ for any pair of distinct code-vertices u, v and ii) each vertex not in C is adjacent to at least one code-vertex.

A graph G is called self-complementary iff it is isomorphic with its complement $\overline{G}$, i.e. when a permutation π of V exists, such that for any pair of distinct vertices u, v $\{u,v\}$ is an edge iff $\{\pi(u), \pi(v)\}$ is not. It is well known that there exists a self-complementary graph with n vertices iff $n = 4k$ or $4k+1$.

A maximal possible distance of two vertices in G is denoted d(G) and called a diameter of G.

Biggs showed in [3] a strong necessary condition for the existence of a perfect code in a distance-transitive graph. This was used by Smith [6] to prove the nonexistence of certain perfect codes. Nešetřil suggested another generalization of the classical notion of perfect codes - perfect codes over graphs, i.e. in powers of graphs which correspond to codes over structured alphabets. In this sense the Hamming- and Lee-error correcting codes are codes in powers of complete graphs and cycles, respectively. For known results on the existence of perfect Hamming- and Lee-error correcting codes see [1],[2],[5],[7]. Though in the case of general graphs one can hardly use the algebraic background of powers of graphs as Biggs did, the structure of the product of graphs is sometimes strong enough to forbid the existence of certain perfect codes. For example it is shown in [4] that there are no nontrivial 1-perfect codes in powers of complete bipartite graphs with at least three vertices.

In the following we shall discuss the special case of the existence of 1-perfect codes in second powers of graphs.

2. <u>Known 1-perfect codes</u>. The following two results are easy to obtain:

Proposition 1. A 1-perfect code in the second power of a path exists iff the length of this path is three. Such a code has four code-vertices.

Proposition 2. A 1-perfect code in the second power of a cycle with n vertices exists iff n is divisible by 5. Such a code has $n^2/5$ vertices.

Proposition 2 shows an infinite family of graphs whose second powers contain 1-perfect codes. Another infinite family of such graphs is introduced in [4]:

Proposition 3. If G is a self-complementary graph with n vertices, then G^2 contains a 1-perfect code of size n.

Proof. Let π be the self-complementary permutation of the vertex set of $G = (V,E)$. Then $C = \{(u,\pi(u)) | u \in V\}$ is a 1-perfect code of size n in the graph G^2. □

The self-complementary graphs are interesting because the cardinalities of 1-perfect codes in their second powers reach the lower bound given by the following

Proposition 4. Let C be a 1-perfect code in the second power of a graph G with n vertices. Then card $C \geq n$.

Proof. Suppose a vertex $u \in V$ exists, such that for all $z \in V$ $(u,z) \notin C$. But any vertex (u,z) of G^2 must be adjacent to some code-vertex. So for each $z \in V$ there is $v_z \in V$ such that $(v_z,z) \in C$, and card $C \geq$ card $V = n$.

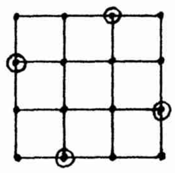
In the opposite case for any $u \in V$ a vertex $v_u \in V$ exists such that $(u,v_u) \in C$, and once again card $C \geq n$. □

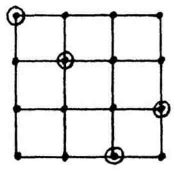
Viewing the previous it is quite natural to ask: Does there exist a non-self-complementary graph G whose second power also contains a 1-perfect code of the same size as the vertex set of G ? The negative answer to this question is proved in the following paragraph.

3. Minimal 1-perfect codes

Definition. A code C in G^2 is called permutational iff for every vertex v of G exactly one vertex u and one vertex w exist such that $(v,u) \in C$ and $(w,v) \in C$, respectively.

Remark. In the usual chess-board-like drawing of cartesian products of graphs a permutational code is a code which picks exactly one vertex from each row and column. In the figure (left), there is shown a 1-perfect permutational code in the second power of the path of length three, while in the right there is a code which is permutational, but not 1-perfect.





Lemma. If a permutational 1-perfect code exists in G^2, then G is a self-complementary graph.

Proof. As the code should be permutational, a permutation π of vertices of G exists, such that $C = \{(u,\pi(u)) \mid u \in V\}$. Take a pair of distinct vertices u, v of G. As C is 1-perfect we have

$$3 \le \partial((u,\pi(u)),(v,\pi(v))) = \partial(u,v) + \partial(\pi(u),\pi(v)),$$

which means $\{u,v\} \in E$ implies $\{\pi(u),\pi(v)\} \notin E$. On the other hand $(u,\pi(v))$ is not in C and so it must be adjacent to some code-vertex $(z,\pi(z))$. But then either $u = z$ and $\{\pi(v),\pi(u)\} \in E$, or $\pi(v) = \pi(z)$, i.e. $v = z$, and $\{u,v\} \in E$. This means that $\{u,v\} \notin E$ implies $\{\pi(u), \pi(v)\} \in E$. Thus π is a self-complementary permutation for G. □

Theorem 1. If a 1-perfect code exists in G^2, where $d(G) = 2$, then G is a self-complementary graph.

Proof. Suppose a 1-perfect code exists in G^2. As $d(G) = 2$, no row or column of the chess-board-like drawing of G^2 may contain more than one code-vertex, otherwise we would have two code-vertices at the distance at most two. But Proposition 4 gives card $C \geq$ card V, and so each row and column contains exactly one code-vertex. So C is a permutational 1-perfect code and G is self-complementary according to Lemma. □

Remark. Notice that in the case of graphs of diameter 2 we did not need the assumption card C = card V.

Now we are prepared to prove the main result.

Theorem 2. If a 1-perfect code of size card V exists in the graph G^2, then G is a self-complementary graph.

Proof. Let C be a 1-perfect code of size card V in G^2.

1) Suppose there is a vertex $v_o \in V$ such that for every $u \in V$ (v_o,u) is not in C. Every vertex (v_o,u) of G^2 must be adjacent to some code-vertex, and so for every $u \in V$ at least one vertex v_u exists, such that $(v_u,u) \in C$. But as card C = card V, it follows that each such v_u is unique. Denote $A = \{v_u | u \in V\}$, obviously A is non-empty, but $v_o \notin A$.

Now for any $v \notin A$ and $u \in V$ we have $(v,u) \notin C$ (otherwise $v = v_u \in A$) and a code-vertex (z,t) exists, such that $\partial((v,u), (z,t)) = 1$. As $(v,t) \notin C$, we have $z \neq v$, and so $u = t$, $z = v_u$ and $\{v,v_u\} \in E$. While u runs through V, v_u runs throughout A, and so for any $v \notin A$ and $w \in A$ we have $\{v,w\} \in E$. As $A \neq \emptyset$ and

$A \neq V$, it follows that $d(G) = 2$, which case was already treated in Theorem 1.

The situation is analogous if there is a vertex v_0 such that for any $u \in V$ $(u,v_0) \notin C$.

2) In the opposite case each row and column of the chess-board-like drawing of G^2 contains at least one code-vertex. But since card C = card V, it follows that each row and column contains exactly one code-vertex, and C is a permutational code. Then G is a self-complementary graph according to Lemma. □

Remark. Notice that the path of length 3 mentioned in Proposition 1 is a self-complementary graph, too.

4. Final remark. We presented a simple lower bound on the cardinality of a 1-perfect code in a second power of a graph with n vertices and we completely characterized the infinite class of 1-perfect codes which meet the equality in this bound.

The similar problem for the upper bound remains open. The simplest upper bound is card $C \leq n^2/3$ (as each neighbourhood of radius 1 in G^2 contains at least 3 vertices), but we only know an infinite class of 1-perfect codes satisfying card $C = n^2/5$. It is still possible that finer methods will enable to prove the sharpest possible bound card $C \leq n^2/5$, $n \geq 5$.

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