COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
26,3 (1985)

TWO NON-HOMEOMORPHIC COUNTABLE SPACES HAVING HOMEOMORPHIC SQUARES

M. M. MARJANOVIĆ and A. R. VUČEMILOVIĆ

Abstract: A pair of non-homeomorphic countable metrizable spaces having homeomorphic squares is exhibited. This answers a question of V. Trnková from [4].

Key words: Countable metrizable spaces, homeomorphism, squares of spaces.

Classification: 54B10

---

1. Introduction. A class $\mathcal{K}$ of topological spaces is said to have the unique square root property if for any two objects A and B in $\mathcal{K}$, $A\times A\approx B\times B$ implies $A\approx B$.

Several naturally organized classes of topological spaces do not have this property (see [4]). In [4], V. Trnková asked the following question: Is the unique square root property valid in the class of all countable metrizable spaces?

In this paper, we exhibit a pair of non-homeomorphic countable metrizable spaces having homeomorphic squares.

2. A classification of points of a space. Now we consider a classification of points of a countable metric space, following the case of classification of points of a compact metric 0-dimensional space (see [2]).

When we say "a space", it will mean invariably "a countable metric space".

For a space X, let X_o be the set of all isolated points of X and X_1 the set of those points of X which have a neighborhood without isolated points. Let $X_{(o)} = X \setminus (X_o \cup X_1)$. Since $X_{(o)} \subseteq \subseteq \overline{X}_o$ ($\overline{A}$ denotes the closure of the set A), the set $X_{(o)}$ is split again into two parts $X_2 = X_{(o)} \setminus \overline{X}_1$ and $X_{(o)(1)} = X_{(o)} \cap \overline{X}_1$. In words, the set $X_{(o)}$ is split into the set X_2 of those points which are not accumulation points of X_1 and the set $X_{(o)(1)}$ of those points which are accumulation points of X_1.

Now we have the following inductive definition: Suppose that the sets $X_o, X_1, \dots, X_n$ and $X_{(o)}, X_{(o)(1)}, \dots, X_{(o)(1)\dots(n-1)}$ have been already defined. Put

$$X_{n+1} = X_{(o)(1)\dots(n-1)} \setminus \overline{X}_n,\ X_{(o)(1)\dots(n)} = X_{(o)(1)\dots(n-1)} \cap \overline{X}_n.$$

In this way, we have defined a sequence of sets $X_o, X_1, \dots$ $\dots, X_n, \dots$ which are disjoint and for each n, the set $X_o \cup X_1 \cup$ $\cup \dots \cup X_n$ is open and $X_{(o)(1)\dots(n-1)}$ closed.

Let

$$X_\omega = \bigcap \{X_{(o)(1)\dots(n)}: n \in N\}.$$

The following statement is immediately derived from the given definition.

<u>Statement 1.</u>

(a) $\overline{X}_n = X_n \cup (\bigcup \{X_k: k = n+2, \dots, \omega\})$

(b) <u>If</u> $X_n = \emptyset$, <u>then</u> $X_k = \emptyset$ <u>for</u> $k = n+2, \dots, \omega$.

Call a point $x \in X$ <u>n-point</u> if $x \in X_n$ for some $n = o, 1, \dots$ $\dots, \omega$. The number n is called the <u>accumulation order</u> of x and we write ord (x) = n.

To the space X, for which $X_{n-2} \neq \emptyset$, $X_{n-1} = \emptyset$ and $X_n \neq \emptyset$ (and according to 1 (b), $X_k = \emptyset$ for $k > n$) the sequence

$$s(X) = (o,1,\dots,n-2,\emptyset,n),$$

and to the space X for which $X_{n-1} \neq \emptyset$, $X_n \neq \emptyset$ and $X_k = \emptyset$ for $k > n$, the sequence

$$s(X) = (o,1,\dots,n-1,n)$$

is attached respectively. The sequence s(X) is called the accumulation sequence of the space X (we avoid here the case $X_\omega \neq \emptyset$).

3. Q-full spaces. Denote by Q the space of rational numbers. Every countable metric space without isolated points is homeomorphic to Q (Sierpinski's theorem, [1], p. 290).

Call a space X Q-full if for each $n > o$, $X_n \neq \emptyset$ implies X_n has no isolated point (or $X_n \approx Q$).

Now we construct a sequence of Q-full spaces.

Let $Q_{-1} = \emptyset$ be the empty set, Q_o a one point space and $Q_1 = Q$, where Q is the set of rationals realized geometrically as the set of all end points of removed intervals of the Cantor discontinuum C (when C is constructed in the usual way of removing the middle third intervals).

Suppose the sequence $Q_o, Q_1, \dots, Q_n$ has already been defined (and all the spaces Q_i, $i = o,\dots,n$ are the subspaces of $[o,1]$).

Define Q_{n+1} to be the space Q plus a copy of the disjoint topological sum $Q_{n-2} + Q_{n-1}$ being interpolated in each of the removed intervals. Now by induction, the sequence of spaces

$$Q_o, Q_1, \dots, Q_n, \dots$$

is defined and it is easy to see that all these spaces are Q-full as well as the sums $Q_{n-1} + Q_n$, $(n \in N)$.

As for the accumulation sequences, we have

$$s(Q_o) = (o),\ s(Q_1) = (\emptyset,1),\ s(Q_o + Q_1) = (o,1)$$

and for $n > 1$,

$s(Q_n) = (o,\dots,n-2,\emptyset,n)$, $s(Q_{n-1} + Q_n) = (o,\dots,n-1,n)$.

In particular, $s(Q_2 + Q_3) = (o,1,2,2)$, $s(Q_5) = (o,1,2,3,\emptyset,5)$, what shows that $Q_2 + Q_3 \not\approx Q_5$.

We quote [5] for the following two easily proved statements.

Statement 2.

(a) *A compact space cannot be Q-full.*

(b) *If every infinite sequence in X_0 has an accumulation point then $\overline{X}_0$ is compact.*

Call two Q-full spaces X and Y *equivalent* if their accumulation sequences are finite and equal, and if card (X_0) = card (Y_0).

According to the statement 2.6 in [5], which can be considered as a variation on the already mentioned Sierpinski's theorem, *two equivalent spaces are homeomorphic* (Sierpinski's theorem being the case $s(X) = s(Y) = (\emptyset,1)$).

We give here a sketch of a (new) proof.

In order to simplify the proofs which follow, notice that according to the statement 1, *a space* X *such that* $s(X) = (o,\dots,n-1,n)$ has both parts X_{n-1} and X_n closed in X. Then, it easily follows that X *can be decomposed into two closed and open parts* X' *and* X'' *such* that $s(X') = (o,\dots,\emptyset,n)$, $s(X'') = (o,\dots,\emptyset,n-1)$ (see also 2.3 in [5]).

Notice also that a *closed and open subset of a* Q-*full space* is Q-*full again*.

A *pointed* Q-*full space* is a pair (X,x_0) where X is Q-full space and $x_0 \in X$ a point of highest accumulation order.

Statement 3. Let (X,x_0) *and* (Y,y_0) *be two pointed* Q-*full spaces such that* X *and* Y *are equivalent and* $s(X) = s(Y) = (o,\dots,\emptyset,n)$. *If* $X = X' \cup X''$ *is a decomposition into two closed* *and open subsets such that* $x_0 \in X'$, *then there exists a decomposition of* Y *into two closed and open subsets*, $Y = Y' \cup Y''$ *such that* $y_0 \in Y'$ *and* X' *is equivalent to* Y' *and* X'' *to* Y''.

Proof. The statement is easily seen to be true in the cases $s(X) = o$, $s(X) = (\emptyset,1)$. Suppose $n \geq 2$. We have two cases

a) $s(X') = (o,\dots,\emptyset,n)$, $s(X'') = (o,\dots,\emptyset,m)$

b) $s(X') = (o,\dots,\emptyset,n)$, $s(X'') = (o,\dots,m-1,m)$.

a) If $m = o$ and card $X'' < \aleph_0$, we take $Y'' \subset Y_0$ such that card Y'' = card X'' and $Y' = Y \setminus Y''$.

If $m = o$ and card $X'' = \aleph_0$, then by 2, there exists a closed and open subset $Y'' \subset Y_0$ such that card $Y'' = \aleph_0$ and $Y' = Y \setminus Y''$ has the required properties.

If $1 \leq m < n$, then since $Y_0 \cup Y_1 \cup \dots \cup Y_{m-2} \cup Y_m$ is open, take a small enough closed and open neighborhood Y'' of a point $y \in Y_m$ such that $Y'' \subset Y_1$ if $m = 1$ and $Y'' \subset Y_0 \cup Y_1 \cup \dots \cup Y_{m-2} \cup Y_m$ if $m > 1$. Let $Y' = Y \setminus Y''$.

If $m = n$, let Y' be a small enough closed and open neighborhood of y_0 such that $Y_n \setminus Y' \neq \emptyset$. Take $Y'' = Y \setminus Y'$.

b) If $m = o$, we do the same as under a) (and it is the same case). If $s(X'') = (o,1)$, take a closed and open neighborhood U of a point in Y_1 such that $U \subseteq Y_1$ and let $V \subset Y_0$, closed in Y, be equivalent to $X'' \cap X_0$. Take $Y'' = U \cup V$, $Y' = Y \setminus Y''$.

Now we have left the case $1 < m \leq n - 2$. Take U and V to be closed and open neighborhoods of a point $y_1 \in Y_m$ and $y_2 \in Y_{m-1}$ respectively such that $U \subseteq Y_0 \cup Y_1 \cup \dots \cup Y_m$ and $V \subseteq Y_0 \cup Y_1 \cup \dots \cup Y_{m-1}$. Take $Y'' = U \cup V$ and $Y' = Y \setminus Y''$. This concludes the proof.

Statement 4. *If* X *and* Y *are equivalent spaces, then they are homeomorphic.*

Proof. We can suppose that X and Y are subspaces of the interval $[o,1]$. Since X and Y are countable, we have the enumerations of each of them $X = \{x_1,\dots,x_i,\dots\}$, $Y = \{y_1,\dots,y_i,\dots\}$. Let x_{i_1} and y_{j_1} be the first elements of highest order (i.e. of order n) in the enumerations of X and Y respectively. Consider the pointed spaces (X,x_{i_1}), (Y,y_{j_1}).

Now let the term "to point a closed and open part A" of X or Y mean to form the pair (A,a), where $a \in A$ is the point of highest order in A which stands first in the given enumeration and has not been already used in the process of pointing.

If $s(X) = s(Y) = (o,\dots,n-1,n)$, then both of these spaces can be decomposed into two parts each, so that the accumulation sequences of the parts are $(o,\dots,\emptyset,n)$ and $(o,\dots,\emptyset,n-1)$, and the pointed parts having the sequence $(o,\dots,\emptyset,n)$. Point the non-pointed parts, if any. Then, each of these parts of X, or X itself, if $s(X) = (o,\dots,\emptyset,n)$, can be decomposed into two closed and open parts which are of diameter less than 2/3 of the diameter of X. Applying 3, we also have equivalent parts of the parts of Y. Now the decompositions of X and Y have at most 4 elements and let us point non-pointed parts. The parts of Y, having the sequence $(o,\dots,m-1,m)$ decompose into two parts having each the sequences $(o,\dots,\emptyset,m)$ or $(o,\dots,\emptyset,m-1)$, point them and correspond to each the equivalent parts of the corresponding parts of X. Point also parts of X. Now, we have at most 8 parts in each of the spaces. Finally decompose the parts of Y so that the diameters of the parts are less than 2/3 of diameter of Y. Point non-pointed parts and do the same with the equivalent non-pointed parts of X.

In this way X and Y are decomposed into at most 16 pointed parts. If x_{i_k} points a part of X, denote such a part by $X^1(x_{i_k})$ and the corresponding part of Y with $Y^1(y_{j_k})$. The parts $X^1(x_{i_k})$ and $Y^1(y_{j_k})$ are all of diameter less than 2/3 and they are equivalent pointed Q-full spaces.

Now starting with the pairs $X^1(x_{i_k})$, $Y^1(y_{j_k})$. We decompose them into at most 16 parts $X^2(x_{i_k})$, $Y^2(y_{j_k})$ having the diameters less than $(2/3)^2$.

Proceeding inductively, in the m-th step, we have the parts $X^m(x_{i_k})$, $Y^m(y_{j_k})$ of diameter less than $(2/3)^m$.

Define the mapping $f:X \longrightarrow Y$ by $f(x_{i_k}) = y_{j_k}$. If $x_{i_s} \in X_t$, then by 1, the set $X_o \cup \ldots \cup X_t$ is open, and for a large enough m, there will exist a part X^m of X contained in $X_o \cup \ldots \cup X_t$ and disjoint from the set of those points of order t which precede x_{i_s}. So x_{i_s}, if not already used in pointing, will be used in the m-th step. The same is valid for the points of Y, so that f is a mapping defined from the whole X onto Y. It is easily seen that f is 1-1 and on both sides continuous. Hence, X and Y are homeomorphic.

Thus the term "equivalent Q-full spaces" means topologically equivalent and it was only a working term.

The statement 4 shows that the only Q-full spaces are the spaces

$$Q_n,\ Q_{n-1} + Q_n$$

adding to them at most countable discrete spaces and the topological sums of such a space and the space Q_1.

4. <u>The space</u> $Q_2 + Q_3$ <u>and</u> Q_5 <u>have homeomorphic squares.</u> Consider two spaces X and Y having no point of the accumulation order 4. (Such two spaces are $Q_2 + Q_3$ and Q_5.) If ord $(x) = m, (x \in X)$ and ord $(y) = n, (y \in Y)$, we will denote the order of $(x,y) \in X \times Y$ by $m \times n$. The number $m \times n$ does not depend of the choi-

ce of spaces X and Y as it will become evident from the proofs which follow. The evident homeomorphism of the spaces $X\times Y$ and $Y\times X$ sends the point (x,y) onto (y,x) and so $m\times n = n\times m$.

Now we show that $n\times m$ dependently of n and m is given by the following table

×	o	1	2	3	5
o	o	1	2	3	5
1	1	1	1	1	1
2	2	1	2	5	5
3	3	1	5	3	5
5	5	1	5	5	5

(a) $o\times n = n$: Suppose ord $(x) = o$, ord $(y) = n$. The set $\{x\}\times Y$ is mapped onto Y by a homeomorphism sending (x,y) onto y. Thus, ord (x,y) = ord (y).

(b) $1\times n = 1$: The point x has a neighborhood without isolated points and so the point (x,y) has also such a neighborhood.

(c) $2\times 3 \geq 5$: In an arbitrary neighborhood of the point (x,y), there exist two points (x',y'), (x'',y'') such that ord $(x') = o$, ord $(y') = 3$, ord $(x'') = 2$, ord $(y'') = o$. Thus, ord $(x',y') = 3$, ord $(x'',y'') = 2$ and the point (x,y) is an accumulation point of $(X\times Y)_2$ and $(X\times Y)_3$. By the statement 1 (a), it follows that ord $(x,y)\geq 5$.

(d) $2\times 5\geq 5$: The proof is the same as under (c).

(e) $3\times 3 = 3$: By 1 (a), $\overline{X}_2 = X_2\cup X_4\cup X_5$ and the set $X_o\cup \cup X_1\cup X_3$ is open. Take closed and open neighborhoods U and V of x and y respectively so that $U\subseteq X_o\cup X_1\cup X_3$, $V\subseteq Y_o\cup Y_1\cup Y_3$. Let (x',y') be in $U\times V$. If one of the numbers ord (x'), ord (y') is less than 3, then ord $(x',y') = o,1$ or 3. If ord (x') = or (y') = 3, then ord $(x',y')\geq 3$, since $(x',y')\in\overline{(X\times Y)}_o$ and $(x',y')\in \overline{(X\times Y)}_1$. Thus, no point in $U\times V$ has the order 2. Thus, ord $(x',y') = 3$.

(f) $3\times 5 \geq 5$: The proof is the same as under (c).

(g) $5\times 5 \geq 5$: The proof as under (c).

(h) $2\times 2 = 2$: The proof easier than (e).

Hence, the space $X\times Y$ has no point of order 4. By 1 (b), $X\times Y$ has no point of order greater than 5 and we have $2\times 3 = 5$, $2\times 5 = 5$, $3\times 5 = 5$, $5\times 5 = 5$.

It is immediately seen that the product of two Q-full spaces X and Y is a Q-full space.

Take $X = Q_2 + Q_3$, $Y = Q_5$. Then, $s(X) = (o,1,2,3)$ and $s(Y) = (o,1,2,3,\emptyset,5)$ and X and Y are not homeomorphic. The spaces $X\times X$ and $Y\times Y$ are Q-full and $s(X\times X) = s(Y\times Y) = (o,1,2,3,\emptyset,5)$. By the statement 4, the spaces $X\times X$ and $Y\times Y$ are homeomorphic.

In a full analogy with the case of compact spaces (see [3]), it can be shown that there exists an infinite sequence of pairs of non-homeomorphic separable metric spaces having homeomorphic squares.

References

[1] KURATOWSKI K.: Topology (Russian), vol. 1, Moscow (1966).

[2] MARJANOVIĆ M.M.: Exponentially complete spaces III, Publ. Inst.Math.,Beograd, t. 14(28)(1972), 97-109.

[3] " : Numerical invariants of 0-dimensional spaces and their Cartesian multiplication, Publ. Inst.Math., Beograd, t. 17(31)(1974), 113-120.

[4] TRNKOVÁ V.: Representations of commutative semigroups by products of topological spaces, Proc. Fifth Prague Topol. Symp. 1981, Berlin (1982), 631-641.

[5] VUČEMILOVIĆ A.: On countable spaces, Mathematica Balcanica, 4.127(1974), 669-674.

Math. Institute PMF, Studentski trg 16, 11000 Beograd,
Yugoslavia

(Oblatum 23.1. 1985)