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ON A PRIORI ESTIMATES FOR POSITIVE SOLUTIONS OF A SEMILINEAR BIHARMONIC EQUATION IN A BALL

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Abstract: We deal with a priori estimates in L^∞ for positive, radial symmetric solutions $u \in C^4(\bar{B})$ of the problem

$$\Delta^2 u = g(u) \text{ in } B, \ u = \frac{\partial u}{\partial n} = 0 \text{ at } \partial B,$$

where $B \subset R^N$, $N \geq 1$, is the unit ball, and the nonlinearity $g: R^+ \to R^+$ has superlinear growth at infinity. As a straightforward application some existence results are proved.

Key words: Biharmonic equation, semilinear elliptic equation, positive solution, a priori estimates.

Classification: 35B45, 35P30, 35J65

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1. Introduction. In the present note we are mainly interested in studying L^∞- a priori estimates for positive, radial symmetric solutions of the homogeneous Dirichlet problem for a semilinear biharmonic equation

$$(1) \qquad \begin{aligned} \Delta^2 u &= g(u) \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial n} &= 0 \quad \text{at } \partial\Omega \end{aligned} \qquad (u \in C^4(\bar{\Omega}))$$

in the special case where $\Omega = B$ is the unit ball in R^N.

The motivation for considering this question arises from the extensive literature on analogous problems for second order nonlinear elliptic equations where nearly optimal results have recently been obtained in the case of the Laplace equation. We refer to the paper [1] by D.G. de Figueiredo, P.-L. Lions, and

R.D. Nussbaum (cf. also [2-4] and the further references in [1]). As it was shown in [1], L^{∞} bounds combined with well-known fixed point properties of compact, cone-preserving operators in Banach spaces and variational techniques turn out to be very useful for investigating structural properties of the positive solution set of semilinear problems.

In order to prove a priori L^{∞} bounds for the solutions $u \in C^2(\bar{\Omega})$ of the related semilinear Laplace equation

$$(1)' \qquad \begin{aligned} \Delta u &= g(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{at } \partial\Omega \end{aligned}$$

for more general bounded, smooth domains $\Omega \subset R^N$ and under nearly final conditions on the growth and the regularity of the nonlinearity g, the authors of [1] explored the Pohozaev identity [5] and some monotonicity properties of the solutions of $(1)'$ near the boundary $\partial\Omega$ which follow from results in [6]. The other details were more or less familiar. While identities of Pohozaev type remain valid also for polyharmonic semilinear problems, the results of [6] cannot immediately be carried over to the case under consideration. Thus, we have to look for other techniques which allow to attack higher order problems.

In our special situation (problem (1) with $\Omega = B$ and $u = u(|x|)$) we use an explicit description of the Green's function of the corresponding ordinary differential equation. This yields some analytical properties of positive, radial symmetric solutions of (1) which allow to establish in combination with the ideas used in [1] satisfactory a priori estimates and existence results. A somewhat simplified but typical result for problem (1) is the following:

Let $\Omega = B \subset R^N$, $N \geq 1$, and $g: R^+ \longrightarrow R^+$ be a continuous function satisfying the conditions

(i) $\varliminf_{u\to+\infty} g(u)\cdot u^{-1} > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of Δ^2 with respect to Ω (superlinearity)

(ii) if $N \geq 4$ then $g(u)\cdot u^{-\beta}$ is decreasing for large u and some $\beta < \sigma = (N+4)/(N-4)$ (regularity and growth condition).

Then any positive, radial symmetric solution $u \in C^4(\bar{B})$ of (1) satisfies (with a constant independent of u)

(2) $\|u\|_\infty \leq C < \infty$.

For illustration, consider the pure power case ($\beta > 1$)

$$
(3) \qquad \begin{aligned} \Delta^2 u &= \lambda \cdot u^\beta \quad \text{in } B \\ u = \frac{\partial u}{\partial n} &= 0 \qquad \text{at } \partial B. \end{aligned}
$$

Then, by our results, a priori L^∞ bounds for positive, radial symmetric solutions of (3) hold for arbitrary $\lambda > 0$ and $\beta < \infty$ if $N \leq 4$ resp. $\beta < \sigma$ if $N > 4$. Thus, by the fixed point theorems quoted in [1] (propositions 2.1 - 2.3) the existence of at least one positive solution $u_{\lambda,\beta} = u_{\lambda,\beta}(|x|)$ of (3) follows for all these parameters. Further information on the behaviour of the solutions (e.g., concerning their dependence on λ) can be obtained.

On the other hand, in the remaining cases, i.e., $N > 4$, $\beta \geq \sigma$, and $\lambda > 0$, no positive solutions of (3) exist at all. This is an easy consequence of the Pohozaev type identity given below (cf. Corollary 1). Thus, the growth condition in (ii) seems to be sharp in some sense. It should be mentioned that (in analogy to [1]) it is an open question whether a priori estimates in L^∞ hold under the less restrictive and more natural condition (ii)' $\lim_{u \to +\infty} g(u) \cdot u^{-\sigma} = 0$

instead of (ii).

2. Preliminaries. Let Ω be a bounded, smooth domain in R^N, $B = \{x \in R^N: |x| < 1\}$, $N \geq 1$, $g \in C(R^1)$, and $u = u(x) \in C^4(\bar{\Omega})$ any solution of (1).

Lemma 1. (Pohozaev type identity.) With these assumptions we have

(4) $$\frac{N-4}{2} \cdot \int_\Omega |\Delta u|^2 dx - N \cdot \int_\Omega G(u) dx = -\frac{1}{2} \cdot \int_{\partial\Omega} |\Delta u|^2 \cdot (n \cdot x) dx$$

where $G(u) = \int_0^u g(t) dt$.

Proof. Multiplying equation (1) by $\nabla u \cdot x$ and integrating (over Ω) by part we obtain (we use the notations $n = n(x)$ for the outer unit normal vector at $x \in \partial\Omega$; x_i, n_i for the components of x, n; $u_i = \frac{\partial u}{\partial x_i}$ etc.; $w = \Delta u$, and the summation convention)

$$\int_\Omega g(u) u_i x_i dx = \int_{\partial\Omega} G(u) n_i x_i dx - N \cdot \int_\Omega G(u) dx$$

and

$$\int_\Omega w_{jj} u_i x_i dx = \int_{\partial\Omega} n_j w_j x_i u_i dx - \int_\Omega w_j (u_j + x_i u_{ij}) dx$$

$$= \int_{\partial\Omega} \{ n_j w_j u_i x_i - w(n_j u_j + n_j x_i u_{ij}) \} dx + \int_\Omega w (2u_{jj} + x_i w_i) dx$$

$$= \int_{\partial\Omega} \{ n_j w_j u_i x_i - w(n_j u_j + n_j x_i u_{ij} - \tfrac{1}{2} n_i x_i) \} dx + (2 - \tfrac{N}{2}) \cdot \int_\Omega w^2 dx$$

Thus,

(5) $$\frac{N-4}{2} \cdot \int_\Omega |\Delta u|^2 dx - N \cdot \int_\Omega G(u) dx$$

$$= \int_{\partial\Omega} \{ \tfrac{\partial}{\partial n} \Delta u \cdot (x \cdot \nabla u) - \Delta u (\tfrac{\partial u}{\partial n} + \tfrac{\partial}{\partial n}(x \cdot \nabla u) - \tfrac{1}{2}(x \cdot n)) - G(u)(x \cdot n) \} dx.$$

Finally, taking into account $u = 0$, $\nabla u = 0$ at $\partial\Omega$ we get (4).

Corollary 1. Assume that $\Omega \subset R^N$ is bounded, smooth, and that there exists a point x_0 such that $n\cdot(x-x_0) > 0$ for all $x \in \partial\Omega$ (e.g. let Ω be convex). Let $N > 4$ and suppose $t\cdot g(t) \gneqq 2N/(N-4)\cdot G(t) \geqq 0$ for $t > 0$. Then no positive solutions $u \in C^4(\bar{\Omega})$ of (1) exist at all.

Proof. Without loss of generality, let $x_0 = 0$. Multiplying in (1) by u and integrating by part we get

$$(6) \qquad \int_\Omega |\Delta u|^2 dx = \int_\Omega \Delta^2 u\cdot u \, dx = \int_\Omega g(u)\cdot u \, dx.$$

From our assumptions, (4), and (6), it immediately follows that $w = \Delta u = 0$ at $\partial\Omega$. But $\Delta w = \Delta^2 u = g(u) \gneqq 0$, by the maximum principle this yields $w \lneqq 0$ in Ω. Thus, $\Delta u \lneqq 0$ in Ω, $u = 0$ at $\partial\Omega$, and the Hopf maximum principle (cf. [7]) gives either $u = 0$ in Ω or $\frac{\partial u}{\partial n} < 0$ at $\partial\Omega$ which is the desired contradiction.

Now we specify to the case $\Omega = B$. We need some information concerning the corresponding linear eigenvalue problem.

Lemma 2. There is a $\lambda_1 > 0$ such that the problem

$$(7) \qquad \Delta^2 v = \lambda_1 \cdot v \text{ in } B, \; v = \frac{\partial v}{\partial n} = 0 \text{ at } \partial B$$

possesses a positive, radial symmetric solution $v_1(x)$ which satisfies

$$(8) \quad C_1\cdot(1 - |x|)^2 \leqq v_1(x) \leqq C_2\cdot(1 - |x|)^2, \; x \in \bar{B}, \; C_1 > 0.$$

Lemma 3. Let $u = U(r)$, $r = |x| \in [0,1]$, be a radial symmetric $C^4(\bar{B})$-solution of (1) where $\Omega = B$. Then $U(r) \in C^4(0,1)$ and satisfies

$$U^{(4)} + \frac{2(N-1)}{r} U^{(3)} + \frac{(N-1)(N-3)}{r^2}(U'' - \frac{1}{r} U') = g(U), \quad 0 < r < 1,$$
(9)
$$U'(0) = U^{(3)}(0) = 0, \ U(1) = U'(1) = 0$$

Inversely, any solution $U(r) \in C^4(0,1)$ of (9) gives a radial symmetric solution $u = U(|x|)$ of problem (1).

The proof of Lemma 2 and 3 is obvious. The next lemma contains the desired results concerning the Green's function of the linear problem corresponding to (9). Unfortunately, we have not found these formulae in the literature (except the cases $N = 1, 2$).

<u>Lemma 4</u>. If the kernel function is defined by

(10)
$$k(r,s) = \begin{cases} a_N(s) + r^2 b_N(s), & 0 \le r \le s \le 1 \\ (s/r)^{N-1} (a_N(r) + s^2 b_N(r)), & 0 \le s \le r \le 1 \end{cases}$$

where

(11)
$$a_N(s) = \begin{cases} \frac{s^3}{4(N-2)(N-4)} (2 + (N-4)s^{N-2} - (N-2)s^{N-4}) & \text{if } N \neq 2,4 \\ (s - s^3(1 - \ln s))/8 & \text{if } N = 2 \\ (s^5 - 2s^3 \ln s - s^3)/8 & \text{if } N = 4 \end{cases}$$

and

(12)
$$b_N(s) = \begin{cases} \frac{s}{4N(N-2)} (Ns^{N-2} - (N-2)s^N - 2) & \text{if } N \neq 2,4 \\ (s\cdot(1 + 2 \ln s) - s^3)/8 & \text{if } N = 2 \\ (-s^5 + 2s^3 - s)/16 & \text{if } N = 4 \end{cases}$$

then any solution $U(r) \in C(0,1)$ of the integral equation

(13)
$$U(r) = \int_0^1 k(r,s) \cdot g(U(s)) \, ds, \quad r \in [0,1],$$

actually belongs to $C^4(0,1)$ and solves (9). The following properties hold for arbitrary $r, s \in [0,1]$

(14) $0 \leq k(r,s) \leq C \cdot s^{N-1}(1-s)^2 \cdot \begin{cases} 1 & , N<4 \\ (1+|\ln(\max(r,s))|), & N=4 \\ (\max(r,s))^{4-N} & , N>4 \end{cases}$

(15) $0 \geq \frac{\partial}{\partial r} k(r,s)$

(16) $\frac{\partial^2}{\partial r^2} k(r,s)\Big|_{r=1} = \frac{1}{2} \cdot s^{N-1}(1-s^2).$

The proof of this lemma is a simple but tedious verification of all the properties stated, the details will be omitted.

3. L^∞ <u>a priori estimates</u>. Now we are going to prove the main result.

<u>Theorem</u>. Let $g \in C(R)$ be a given nonlinearity satisfying

(i) $\underline{\lim}_{u \to +\infty} g(u) \cdot u^{-1} > \lambda_1$, where λ_1 is defined in Lemma 2,

(ii)' $\lim_{u \to +\infty} g(u) \cdot u^{-\sigma} = 0$, $\sigma = (N+4)/(N-4)$, if $N > 4$

(resp., $\lim_{u \to +\infty} g(u) \cdot u^{-\beta} = 0$ for some $\beta < \infty$ if $N = 4$)

and

(ii)" if $N > 4$ then there exists $\alpha \in [0, 2N/(N-4))$ such that

$$\overline{\lim}_{u \to +\infty} (u \cdot g(u) - \alpha \cdot G(u)) \cdot (u^{-2} \cdot g(u)^{-4/N}) \leq 0.$$

Then the estimate

(17) $\|u\|_\infty \leq C < \infty$

holds for any positive, radial symmetric solution u of (1) (with $\Omega = B$) where C does not depend on u.

<u>Proof</u>. We mainly proceed in analogy to [1], pp. 44-50. Let $\Omega = B$ and $u = u(x) = U(r)$, $|x| = r \in [0,1]$, be any positive, radial symmetric solution of (1).

<u>Step 1</u>. We prove

(18) $\int_\Omega u \cdot v_1 \, dx \leq C, \quad \int_\Omega |g(u)| \cdot v_1 \, dx \leq C$

under the only condition (i):

$$\int_\Omega |g(u)| \cdot v_1 \, dx \leq C + \int_\Omega g(u) \cdot v_1 \, dx = C + \int_\Omega \Delta^2 u \cdot v_1 \, dx$$

$$= C + \int_\Omega u \cdot \Delta^2 v_1 \, dx = C + \int_\Omega \lambda_1 u \cdot v_1 \, dx \leq C + q \cdot \int_\Omega g(u) \cdot v_1 \, dx$$

with some $q < 1$, and (18) follows.

Let us mention that (18) yields (17) for $N < 4$: According to (8), Lemma 3 and 4 (especially (14)), and (18) we get

$$|u(x)| \leq \max_{r \in [0,1]} \int_0^1 k(r,s)|g(U(s))| \, ds \leq C \int_0^1 s^{N-1}(1-s)^2 |g(U(s))| \, ds$$

$$\leq C \int_0^1 s^{N-1} \cdot v_1(s) |g(U(s))| \, ds \leq C \int_B v_1 \cdot |g(u)| \, dx \leq C.$$

Thus, in the following, let $N \geq 4$.

<u>Step 2</u>. We prove the estimates $U(r) \leq C$ for $r \in [2/3, 1]$ and

(19) $\frac{\partial^2}{\partial n^2} u(x) = U''(1) \leq C, \; x \in \partial B, \quad \int_B |g(u)| \, dx \leq C$

if (i) is fulfilled. For this we introduce the function

$$U(r) \leq U^+(r) = \int_0^1 k(r,s)|g(U(s))| \, ds \leq U(r) + C$$

(the latter inequality easily follows from (i)). Because of (15), $U^+(r)$ is decreasing in r and, therefore, for arbitrary $r \in [2/3, 1]$ we have (cf. also (8),(18))

$$U^+(r) \leq U^+(2/3) \leq 3 \cdot \int_{1/3}^{2/3} U^+(s) \, ds \leq C \cdot \int_0^1 s^{N-1}(1-s)^2 \cdot U^+(s) \, ds$$

$$\leq C \cdot (1 + \int_B v_1 \cdot u \, dx) \leq C.$$

This proves the first inequality which now yields (19) by analogous considerations (use (16) resp. (18) and once again (8)). As an immediate consequence of (19) and the Pohozaev type identity (4) we obtain

$$(20)\quad \left| \frac{N-4}{2} \cdot \int_\Omega |\Delta u|^2 dx - N \cdot \int_\Omega G(u)\,dx \right| \leq C.$$

Step 3. Now we additionally suppose (ii)" to establish

$$(21)\quad \int_\Omega |g(u)|\cdot u\,dx \leq C, \qquad \int_\Omega |\Delta u|^2 dx \leq C.$$

This can be done by a straightforward adaption of step 3 in [1], p. 47/48, the details will be left to the reader (the needed facts from the preceding steps are (19),(20), and (6)). It should be mentioned that now the case $N = 4$ can already be finished by using the growth restriction in (ii)′, the W_2^2 bound from (21), and the usual embedding and regularity results for the (linear) biharmonic equation.

Step 4. Finally, we get (17) for $N > 4$. By the considerations in Step 2 it is clear that (cf. (10) - (11))

$$\|u\|_\infty \leq U^+(0) \leq \int_0^1 k(0,s)|g(U(s))|\,ds \leq C \cdot \int_0^1 s^3 |g(U(s))|\,ds.$$

We denote $g^+(u) = \max_{t\in[0,u]} |g(t)|$ and take an arbitrary $r \in (0,1)$. Then, by Hölder′s inequality, (ii)′, and (21) we have

$$\|u\|_\infty \leq C \cdot \left\{ \int_0^r s^3 \cdot g^+(U(s))\,ds + \int_r^1 s^3 |g(U(s))|\,ds \right\}$$

$$\leq C \left\{ r^4 \cdot g^+(\|u\|_\infty) + \left(\int_r^1 s^{\gamma(\sigma+1)}\,ds \right)^{\frac{1}{\sigma+1}} \cdot \left(\int_0^1 s^{N-1} |g(U(s))|^{\frac{\sigma+1}{\sigma}}\,ds \right)^{\frac{\sigma}{\sigma+1}} \right\}$$

$$\leq C \cdot \left\{ r^4 \cdot g^+(\|u\|_\infty) + \left(\int_r^1 s^{-N-1}\,ds \right)^{\frac{N-4}{2N}} \cdot \left(\int_0^1 s^{N-1} |g(U(s))|\,U(s)\,ds \right)^{\frac{N+4}{2N}} \right.$$

$$\leq C \cdot \left\{ r^4 \cdot g^+(\|u\|_\infty) + r^{2-N/2} \right\}, \text{ where } \gamma = 3 - (N-1) \cdot \frac{\sigma}{\sigma+1}.$$

Taking in this inequality the infimum with respect to r we get

$\| u \|_\infty \leq C \cdot (1 + g^+(\| u \|_\infty)^{1/\sigma})$.
But this estimate yields (17) since (ii)' implies $g^+(t) = o(t^{\sigma})$ for $t \to +\infty$.

Thus, the Theorem is completely proved.

Remark 1. For applications it is important to observe that the constant in (17) can be chosen independent of the parameter $t \in [0, t_0]$, $0 < t_0 < \infty$, if we consider positive, radial symmetric solutions of (1) for the family of nonlinearities $g_t = g(u+t)$.

Remark 2. A careful analysis shows that except Step 2 the proof could be carried out for more general domains Ω. This remark is obvious for Step 1 and 3, in Step 4 you might follow the line of argumentation in [1], p. 49/50, if the identity

$$\int_\Omega |\Delta(u^{2\eta})|^2 dx = \frac{\eta^2}{\eta - 1/4} \cdot \int_\Omega g(u) \cdot u^p dx + \frac{(p-1)^2}{\eta^2} \cdot \int_\Omega |\nabla(u^\eta)|^4 dx,$$

which is satisfied for positive solutions of (1) and $p \geq 3$, $\eta = (p+1)/4$, will be explored.

Remark 3. Clearly, condition (ii)" is technical and not necessary for obtaining a priori estimates. For instance, if $g(u) = u^{\sigma} \cdot (\ln_{++} u)^{-\alpha}$ where $\ln_{++} u = \max(1, \ln u)$ for $u > 0$, $\alpha > 0$, and $N > 4$, then (i),(ii)' hold but condition (ii)" does not. Nevertheless, a slight modification of the above considerations gives (17), at least, for $\alpha > 4/(N-4)$. We only sketch the proof of this statement. A direct verification shows that

$$G(u) - \frac{N-4}{2N}\cdot u\cdot g(u) = \begin{cases} \frac{\alpha}{\sigma+1} \cdot \int_e^u t^{\sigma}\cdot(\ln t)^{-\alpha-1}dt, & u>e \\ 0, & u\leq e \end{cases}$$

Hence, by (6),(20) we have

$$\int_0^1 s^{N-1}\cdot g(U(s))\cdot U(s)\cdot(\ln_{++}U(s))^{-1}ds \leq C,$$

and, repeating the estimations as in Step 4 of the above proof, we obtain

$$\|u\|_{\infty} \leq C\{r^4\cdot g^+(\|u\|_{\infty}) + r^{2-N/2}\cdot(\ln_{++}\|u\|_{\infty})^{\frac{\sigma}{\sigma+1}\cdot\max(0,1-\alpha/\sigma)}\}$$

It remains to check the infimum.

Furthermore, if $N = 4$ then the growth restriction in (ii) can easily be weakened to

$$\overline{\lim_{u\to+\infty}} \ \ln g(u)\cdot u^{-1} < 4.$$

Finally, it should be mentioned that condition (ii) stated in the Introduction obviously yields (ii)$'$, and (ii)".

<u>Remark 4</u>. We only considered radial symmetric solutions but it is not yet clear whether there can exist non-radial symmetric solutions of (1) for $\Omega = B$ at all (concerning (1)$'$ cf. [6]).

We close the exposition by stating an existence result (the analog of Theorem 2.1 in [1]) which immediately follows from our Theorem (for other assertions which can be obtained on the basis of the L^{∞} bounds we refer to [1],[4]).

<u>Corollary 2</u>. Let $\Omega = B$ and $g: R^+ \to R^+$ be continuous. If g satisfies (i),(ii)$'$,(ii)", and

(iii) $\quad \overline{\lim_{u\to 0}} \ g(u)\cdot u^{-1} < \lambda_1$, then there exists at least one positive, radial symmetric solution $u = U(r) \in C^4(\bar{\Omega})$ of (1) which has the additional properties $U'(r) < 0$ for $0 < r < 1$ and $U''(1) > 0$.

Proof. Let us consider the compact map $F: K \times [0,\infty) \to K$ where $K = \{U \in C(0,1): U(r) \geqq 0\}$ is the closed cone of nonnegative functions in $C(0,1)$ given by the formula

$$F(U,t)(r) = \int_0^1 k(r,s) \cdot g(U(s)+t)\, ds.$$

The following properties hold:

(a) Any non-zero solution of the fixed point equation

$$U = \Phi(U) = F(U,0), \quad U \in K,$$

is, actually, a positive solution of (9) and, thus, $u(x) = U(|x|)$ is a positive, radial symmetric solution of (1).

(b) $U \neq \lambda \cdot \Phi(U)$ for arbitrary $\lambda \in [0,1]$ and $U \in K$ with $\|U\|_C = R$ for sufficiently small $R_1 > 0$ since according to (iii) $g(u(x)) \lneqq q \cdot \lambda_1 \cdot u(x)$, $q < 1$, and, therefore,

$$\lambda_1 \int_\Omega u v_1\, dx = \int_\Omega u \cdot \Delta^2 v_1\, dx = \int_\Omega \Delta^2 u \cdot v_1\, dx = \int_\Omega g(u) v_1\, dx \lneqq$$

$$\lneqq q \cdot \lambda_1 \cdot \int_\Omega uv\, dx$$

for sufficiently small solutions of (1) which is a contradiction.

(c) There exists t_0 such that $U \neq F(U,t)$ for arbitrary $U \in K$ and $t \geqq t_0$ because for some finite t_0 we have from (i) $g(u+t) \geqq \lambda \cdot u$ uniformly in $u \geqq 0$ and $t \geqq t_0$ where $\lambda > \lambda_1$ (then proceed as in Step 1 of the proof of the Theorem or as in (b) to obtain a contradiction).

(d) Finally, according to the Theorem, (c), and Remark 1 we can choose a sufficiently large $R_2 > R_1$ such that $U \neq F(U,t)$ for arbitrary $t \in [0,\infty)$ and $U \in K$ with $\|U\|_C = R_2$.

Now, the Krasnosel'skii type fixed point theorem from [1] (cf. Proposition 2.1 and Remark 2.1) can be applied. Hence, the existence statement is proved, the additional properties are obvious consequences of Lemma 4.

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