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SPECIAL LATTICES OF COMPACTIFICATIONS

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Abstract. Given any compactification αX of a Tychonoff space X, let $f_\alpha : \beta X \longrightarrow \alpha X$ denote the canonical quotient map from the Stone-Cech compactification of X onto αX. It is known that the complete upper semi-lattice $K(X)$ of all compactifications of X becomes a lattice whenever the set $F^*(\alpha X) = \{f_\alpha^{-1}(p) : |f_\alpha^{-1}(p)| > 1\}$ is finite for all $\alpha X \in K(X)$. In this paper we give some necessary and sufficient conditions, in terms of X and $\beta X - X$, for $F^*(\alpha X)$ to be finite for all $\alpha X \in K(X)$.

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Introduction. Let X be a Tychonoff space. Denote by $K(X)$ the family of T_2-compactifications of X. Two compactifications αX and γX are considered equivalent if there is a homeomorphism between αX and γX, which leaves X pointwise fixed; we do not distinguish between equivalent compactifications in $K(X)$. $K(X)$ is partially ordered by the relation: $\alpha_1 X \le \alpha_2 X$ if there is a continuous map from $\alpha_2 X$ onto $\alpha_1 X$, which leaves X pointwise fixed.
It is known that $K(X)$ is always a complete upper semi-lattice and that it is a complete lower semi-lattice (hence a complete lattice) iff X is locally compact (cf.[M]).
In general $K(X)$ is not a lattice, for example when X is first countable but not locally compact (cf.[FV]).

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In this paper we study questions related to the problem of when $K(X)$ is a lattice.

In the following, we will use the term compactification instead of T_2-compactification.

If $\alpha X \in K(X)$, βX will denote the Stone-Cech compactification of X and $f_\alpha : \beta X \longrightarrow \alpha X$ the canonical quotient map. Define the β-family of αX to be $F(\alpha X) = \{f_\alpha^{-1}(p) : p \in \alpha X - X\}$ and set $F^*(\alpha X) = \{F \in F(\alpha X) : |F| > 1\}$. Recall that any family of bounded continuous functions, $S \subset C^*(X)$, which separates points from closed sets, generates a compactification $\alpha_S X = \overline{e_S(X)}$, where $e_S : X \longrightarrow \prod_{f \in S} K_f$, $K_f = \overline{f(X)}$, is the topological embedding defined by $e_S(x) = \{f(x)\}_{f \in S}$.

Moreover, observe that, if $F_1, \ldots, F_n \subset \beta X - X$ are disjoint compact sets with $|F_i| > 1$, then the quotient space αX of βX, obtained by shrinking each compact F_i to a point, is a compactification of X and one has $F^*(\alpha X) = \{F_1, \ldots, F_n\}$. Obviously αX coincides with the compactification generated by $S = \{f \in C^*(X) : f^\beta_{|F_i}$ is constant $\forall\, i=1,\ldots,n\}$, where f^β is the extension of f to βX.

Some topological spaces have the property that all their compactifications are obtained as previously described, that is $F^*(\alpha X)$ is finite for all $\alpha X \in K(X)$. In this case, it is easy to prove that $K(X)$ is a lattice. In fact, if αX and γX are compactifications of X, then $\alpha X \wedge \gamma X$ is generated by the family of continuous functions

$$\{f \in C^*(X) : f^\beta_{|F} \text{ is constant } \forall\, F \in F^*(\alpha X) \cup F^*(\gamma X)\}.$$

In ([C], th.5.6; see also [FV], proof of th. 1) it is pointed out that if $\beta X - X$ is discrete and C^*-embedded in βX, then $F^*(\alpha X)$ is finite for all $\alpha X \in K(X)$. More generally, one obtains the same result when $\beta X - X$ is a P-space and $Cl_{\beta X}(\beta X - X)$ is an F-space (cf. [U]). Recall that a P-space is a space in which every cozero-set is C-embedded and an F-space is a space in which every cozero-set is C^*-embedded (for equivalent definitions cf. 4J, 14.25, 14.29, 14N in [GJ]).

Among the results of the present paper is the following proposition generalizing the above mentioned results: if $\beta X - X$ is a cf-space (that is a space whose compact sets are finite) and every countable discrete subset of $\beta X - X$

is C^*-embedded in βX , then $F^*(\alpha X)$ is finite for all $\alpha X \in K(X)$.
The same conclusion is achieved if the following three conditions are satisfied: a) $\beta X - X$ is C^*-embedded in βX, b) $\beta X - X$ is countably normal (we say that a space is countably normal if any two disjoint countable closed sets are completely separated), and c) every infinite subset of $\beta X - X$ contains an infinite discrete and closed subset of $\beta X - X$.
An application of the last proposition is obtained when $\beta X - X$ is an MI-space (that is, dense in itself and whose dense subsets are open), countably normal and C^*-embedded in βX .
Moreover, we prove that $\beta X - X$ is a cf-space if $F^*(\alpha X)$ is finite for all $\alpha X \in K(X)$ and, under additional hypotheses on X or $\beta X - X$, we give some equivalent conditions for $F^*(\alpha X)$ to be finite for all $\alpha X \in K(X)$.

We will denote with $\mathbb{N}$ and $\mathbb{R}$ the sets of natural numbers and real numbers, respectively.

1. All spaces we deal with are Tychonoff. Let αX be a compactification of a space X and let $f_\alpha : \beta X \longrightarrow \alpha X$ be the canonical quotient map. A subset A of βX is said to be saturated (relative to f_α) when $A = f_\alpha^{-1}(f_\alpha(A))$. Given $F \subset A \subset \beta X$, where $F = f_\alpha^{-1}(p)$ with $p \in \alpha X$ and A is an open subset of βX , then, since f_α is a closed map, there exists an open saturated subset U of βX such that $F \subset U \subset A$.

LEMMA 1. *Let* αX *be a compactification of* X , $\mathcal{G} = \{F_\lambda\}_{\lambda\in\Lambda} \subset F^* = F^*(\alpha X)$ *and let* $A = \{x_\lambda\}_{\lambda\in\Lambda}$ *with* $x_\lambda \in F_\lambda$ *for every* $\lambda \in \Lambda$. *Then*

$$\left(Cl_{\beta X}\left(\bigcup_{\lambda\in\Lambda} F_\lambda\right)\right) - S = \left(Cl_{\beta X}A\right) - S$$

where $S = \bigcup_{F\in F^*} F$.

Proof.

Obviously $\left(Cl_{\beta X} A\right) - S \subset \left(Cl_{\beta X}\left(\bigcup_{\lambda\in\Lambda} F_\lambda\right)\right) - S$. Conversely if $x \notin \left(Cl_{\beta X} A\right) - S$ then we can suppose, without loss of generality, that $x \notin S$. Let $V \subset \beta X$ be an open set such that $x \in V$, $V \cap A = \emptyset$. Then there exists an open saturated subset U of βX with $x \in U \subset V$. It is clear that $U \cap F_\lambda = \emptyset$ for all $\lambda \in \Lambda$, since U is saturated and $x_\lambda \notin U$ for all $\lambda \in \Lambda$. Thus $x \notin \left(Cl_{\beta X}\left(\bigcup_{\lambda\in\Lambda} F_\lambda\right)\right) - S$.

COROLLARY 2. *Let* $G = \{F_\lambda\}_{\lambda\in\Lambda} \subset F^*(\alpha X)$ *and let* $A = \{x_\lambda\}_{\lambda\in\Lambda}$, $B = \{y_\lambda\}_{\lambda\in\Lambda}$, x_λ , $y_\lambda \in F_\lambda$ *for every* $\lambda \in \Lambda$. *Then*

$$(Cl_{\beta X}A) - S = (Cl_{\beta X}B) - S \quad .$$

PROPOSITION 3. *If* $\beta X - X$ *is a cf-space and every countable discrete subset of* $\beta X - X$ *is* C^**-embedded in* βX , *then* $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$.

Proof.

Suppose that, for some $\alpha X \in K(X)$, $F^*(\alpha X)$ is infinite. It follows that $D = f_\alpha(F^*(\alpha X))$ is infinite. If $T = \{p_n\}$ is countably infinite discrete subset of D , set $F_n = f_\alpha^{-1}(p_n)$ for every $n \in N$, and let $A = \{x_n\}$, $B = \{y_n\}$ where x_n , $y_n \in F_n$, $x_n \neq y_n$. Since T is discrete in $\alpha X - X$, it follows that $S = A \cup B$ is a discrete subset of $\beta X - X$. In fact, for every $n \in N$, there is an open set V_n of $\alpha X - X$ such that $p_m \in V_n$ iff $m = n$.
Then setting $U_n = f_\alpha^{-1}(V_n) \supset F_n$, one has $U_n \cap F_k = \emptyset$ for every $k \neq n$, otherwise $p_k \in V_n$.
By assumption S is C^*-embedded in βX , hence A and B , which are completely separated in S , are completely separated in βX . Thus we have $Cl_{\beta X}A \cap Cl_{\beta X}B = \emptyset$; moreover it follows from Corollary 2 that $Cl_{\beta X}A \cap X = Cl_{\beta X}B \cap X$. We conclude that both A and B have no cluster points in X, hence $Cl_{\beta X}A \subset \beta X - X$. This is a contradiction, because $\beta X - X$ was supposed to be a cf-space.

As a consequence of the above proposition we obtain the known results :

COROLLARY 4. ([FV]) *If* $\beta X - X$ *is discrete and* C^**-embedded in* βX , *then* $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$.

COROLLARY 5. ([U]) *If* $\beta X - X$ *is a P-space and* $Cl_{\beta X}(\beta X - X)$ *is an F-space, then* $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$.

Proof. Every countable subset of a P-space is closed and discrete, so every P-space is a cf-space (cf. 4K in [GJ]). Also every countable subset of an F-space is C^*-embedded (cf. 14N in [GJ]). Then apply the Tietze-Urysohn theorem.

We give now another sufficient condition for $F^*(\alpha X)$ to be finite for all $\alpha X \in K(X)$.

PROPOSITION 6. *Let* X *be a space such that :*

a) $\beta X - X$ *is* C^**-embedded in* βX

b) $\beta X - X$ *is countably normal*

c) *every infinite subset of* $\beta X - X$ *contains an infinite discrete and closed subset of* $\beta X - X$.

Then $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$

Proof.

First observe that, if $T \subset \beta X - X$ is infinite, then there exists a countably infinite subset of T , which is closed and discrete.

Now suppose that there is an $\alpha X \in K(X)$ such that $F^*(\alpha X)$ is infinite. Then there exists a countably infinite set $A' \subset \bigcup\{F : F \in F^*(\alpha X)\}$, which is closed and discrete in $\beta X - X$. Since every $F \in F^*(\alpha X)$ is compact, then $A' \cap F$ is finite for all $F \in F^*(\alpha X)$. Thus, one can suppose that A' meets every $F \in F^*(\alpha X)$ in at most one point. If $A' = \{x_n\}$, let $F_n = f_\alpha^{-1}(f_\alpha(x_n))$ for every $n \in \mathbb{N}$. Then consider a countably infinite set $B \subset \bigcup_{n \in \mathbb{N}} F_n - \{x_n\}$ closed and discrete in $\beta X - X$. As above, we can suppose that $|B \cap F_n| \leq 1$ for all $n \in \mathbb{N}$. If $B = \{y_{n_j}\}$ with $y_{n_j} \in F_{n_j}$, let $A = \{x_{n_j}\}$. By arguments similar to those in Proposition 3 , we obtain that A has no cluster points in X and so it is closed in βX . This is a contradiction since A is an infinite discrete set.

COROLLARY 7. *Let* $\beta X - X$ *be an MI-space, countably normal and* C^**-embedded in* βX *, then* $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$.

Proof.

It is easy to prove that, every infinite subset of a Hausdorff MI-space Y contains a countably infinite closed and discrete subset. In fact, if $T \subset Y$ is infinite, consider a copy N of $\mathbb{N}$ in T . N has no interior points, otherwise, since N is discrete, such points would be isolated in Y . Thus $Y - N$ is dense in Y , hence open. We conclude that N is closed and discrete in Y .

Next, we will give an example in which $\beta X - X$ is C^*-embedded in βX , and it is neither a P-space nor an MI-space, but satisfies the hypotheses of Proposition 3 or 6.

Recall that a space is said to be extremally disconnected if every open set has an open closure. It is said to be basically disconnected if every cozero-set has an open closure. One can also give an equivalent definition of an F-space as being a space in which disjoint cozero-sets are completely separated.

Clearly, the following implications hold :

extremally disconnected $\Longrightarrow$ basically disconnected $\Longrightarrow$ F-space .

Let $\Sigma = \mathbb{N} \cup \{\sigma\}$ and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. In Σ define the following topology : a subset A of Σ containing σ is open iff $A = U \cup \{\sigma\}$, $U \in \mathcal{U}$, also all subsets of Σ that do not contain σ are to be open.

It is easy to prove that Σ is a normal, extremally disconnected space (and so an F-space), but it is not a P-space, nor an MI-space (cf. 4M in [GJ]). Since Σ is an F-space, then every subset of Σ is C^*-embedded (cf. 14N in [GJ]). It is known that, if Y is a Tychonoff space, then there is a space X such that $\beta X - X$ is homeomorphic to Y and is C^*-embedded in βX (cf.[C] Cor.4.18). We apply this result to the case $Y = \Sigma$.

Now we show that every infinite subset of Σ contains an infinite closed and discrete subset of Σ , so Σ is a cf-space.

Let T' be an infinite subset of Σ and let $T = T' - \{\sigma\}$. If $T = \{x_n\}$, set $A = \{x_{2n}\}$ and $B = \{x_{2n+1}\}$. Now if $A \in \mathcal{U}$, then $A \cup (\mathbb{N} - T) \in \mathcal{U}$. Otherwise $\mathbb{N} - A \in \mathcal{U}$. In the former case, we obtain that B is closed and discrete. In the latter A is closed and discrete.

2. As we have seen in Proposition 3 and 6, the condition that $\beta X - X$ is a cf-space ensures, together with other conditions, that $F^*(\alpha X)$ is finite for all $\alpha X \in K(X)$. Now we want to prove that this latter condition implies that $\beta X - X$ is a cf-space.

<u>PROPOSITION 8</u>. <u>If</u> $F^*(\alpha X)$ <u>is finite for all</u> $\alpha X \in K(X)$ <u>then</u> $\beta X - X$ <u>is a cf-space.</u>

<u>Proof.</u>

Let K be a compact subset of $\beta X - X$ and suppose that K is infinite.

Then there is a countably infinite discrete subset of K, which we denote by $B = \{x_n\}$. If $\{r_n\}$ is the sequence of real numbers with $r_{2n-1} = r_{2n} = 1/n$, $n \in \mathbb{N}$, then the map $g : Cl_{\beta X} B \longrightarrow \mathbb{R}$ defined by $g(x_n) = r_n$ and $g(x) = 0$, if $x \in (Cl_{\beta X} B) - B$, is continuous and thus it has a continuous extension h to βX.
Now consider the family A of subsets of $\beta X - X$ defined as follows :

$$A = \{h^{-1}(p) \cap K : p \in \mathbb{R}\}$$

and let

$$S = \{f \in C^*(X) : f^\beta{}_{|A} \text{ is constant} \forall A \in A\} .$$

The family S separates points from closed sets of X. In fact if $C \subset X$ is a closed set and $x \in X$, $x \notin C$, let F be a closed set in βX such that $F \cap X = C$. Then there exists $f \in C^*(\beta X)$ such that $f(x) = 0$ and $f(K \cup F) = 1$. The map $f_{|X}$ belongs to S and separates x from C.
The family, S, thus generates a compactification $\alpha X = \alpha_S X = \overline{e_S(X)}$; moreover, if f_α is the canonical quotient map, one has $f_\alpha = e_S^\beta = \prod_{f\in S} f^\beta$.
Now we show that $F^*(\alpha X) = \{A \in A : |A| > 1\}$ and so $F^*(\alpha X)$ is infinite.
If $x, y \in A$, for some $A \in A$, then obviously $f_\alpha(x) = f_\alpha(y)$.
Conversely, suppose that $x, y \in \beta X - X$ do not belong to the same $A \in A$.
If at least one of the two points, say x, does not belong to K, then there is a continuous map $s : \beta X \longrightarrow \mathbb{R}$ such that $s(x) = 0$ and $s(K \cup \{y\}) = 1$. We have $s_{|X} \in S$, and $(s_{|X})^\beta(x) = s(x) \neq s(y) = (s_{|X})^\beta(y)$, therefore $f_\alpha(x) \neq f_\alpha(y)$. Suppose then $x, y \in K$ and $h(x) \neq h(y)$, that is x and y do not belong to the same $A \in A$. The map $\overline{h} = h_{|X} \in S$. Obviously one has $\overline{h}^\beta(x) = h(x) \neq h(y) = \overline{h}^\beta(y)$, and so again we have $f_\alpha(x) \neq f_\alpha(y)$.

We note that the condition that $\beta X - X$ be a cf-space is not enough to imply that $F^*(\alpha X)$ is finite for all $\alpha X \in K(X)$. In fact, it is possible to construct a space X such that $\beta X - X = N \simeq \mathbb{N}$ and $Cl_{\beta X} N \simeq \omega\mathbb{N}$, where $\omega\mathbb{N}$ is the one-point compactification of $\mathbb{N}$. The conclusion follows from the following fact : if there is a sequence in $\beta X - X$ converging to a point of X, then $K(X)$ is not a lattice (cf. example 4.7 in [T]).

The following corollaries are easy consequences of Proposition 3 and 8.

COROLLARY 9. *Let* $\beta X - X$ *be locally compact and* C^**-embedded in* βX .
Then $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$ *if and only if* $\beta X - X$ *is discrete.*

COROLLARY 10. *Let* X *be locally compact.*
Then $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$ *if and only if* $\beta X - X$ *is finite.*

COROLLARY 11. *Let* $Cl_{\beta X}(\beta X - X)$ *be an F-space.*
Then $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$ *if and only if* $\beta X - X$ *is a cf-space.*

Remark: The hypotheses of Corollary 11 are satisfied, for instance, if X is an F-space or $\beta X - X$ is an F-space, C^*-embedded in βX (cf. in [GJ] 14.25 (9) and (10) and 14.26 that every C^*-embedded subspace of an F-space is itself an F-space).

We give now another necessary condition for $F^*(\alpha X)$ to be finite for all $\alpha X \in K(X)$.

PROPOSITION 12. *If* $F^*(\alpha X)$ *is finite for all* $\alpha X \in K(X)$, *then* X *is pseudocompact.*

Proof. If X were not pseudocompact, then it would contain a C-embedded copy N of $\mathbb{N}$, in particular a closed C^*-embedded copy of $\mathbb{N}$. Then we would have $\beta N - N \subset \beta X - X$ and so $\beta X - X$ would not be a cf-space.

Note that a pseudocompact space can contain a closed C^*-embedded copy of $\mathbb{N}$ and so the converse of the above proposition is false. For example, the space $\Lambda = \beta\mathbb{R} - (\beta\mathbb{N} - \mathbb{N})$ is pseudocompact and $\mathbb{N}$ is closed C^*-embedded in Λ (cf. 6P in [GJ]).

COROLLARY 13. *If* X *is realcompact and not compact, then there exists* $\alpha X \in K(X)$ *with* $F^*(\alpha X)$ *infinite.*

We conclude with an open question: is there a space X such that $\beta X - X$ is a cf-space, C^*-embedded in βX and X has a compactification αX with $F^*(\alpha X)$ infinite ?

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