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A NOTE ON THE SOLVABILITY OF NONLINEAR ELLIPTIC PROBLEMS WITH JUMPING NONLINEARITIES

Flavio DONATI *)

Abstract: We study semilinear boundary value problems with nonlinearities crossing a simple eigenvalue. Some criteria for existence and non-existence of solutions are presented; some open questions and connections to a number of papers on the subject are also discussed.

Key words: Nonlinear boundary value problems, cross of a simple eigenvalue, multiplicity of solutions.

Classification: 35J65

Introduction. The aim of this note is to give some contributions to the study of the solvability of semilinear boundary value problems such as

$$(\mathcal{P})\begin{cases} -\Delta u = g(u) + h, & h \in L^2(\Omega) \\ u \in H^2(\Omega) \cap H^1_0(\Omega) \end{cases}$$

where the nonlinearity g interacts, in some sense, with the spectrum of the linear part and $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary.

In the sequel we will not distinguish between the function g and its associated Nemitskyi operator and we shall assume that $g: \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that

$g_\pm = \lim_{r \to \pm\infty} \frac{g(r)}{r}$ exist in $\mathbb{R}$ with $g_- \neq g_+$ that is, following

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[7], g is a "jumping nonlinearity" (with finite jumps). We shall suppose $g_- < g_+$ and the interval (g_-, g_+) containing a simple eigenvalue of the considered linear operator, i.e. the nonlinearity g crosses an eigenvalue.

This type of problems originated from the pioneering work of Ambrosetti and Prodi [3], dealing with the cross of the first eigenvalue, has been extensively investigated in recent years; for an exhaustive bibliography we refer the reader to the survey paper [6]. The cross of a (simple) higher eigenvalue, however, exhibits some particular features as shown, for instance, in [5],[8],[9],[12],[13]. Actually, in this case, the results of Ambrosetti-Prodi type are established only according to the particular nature of the eigenfunction corresponding to the considered eigenvalue; moreover, a complete description of the solvability problems such as $(\mathcal{P})$ seems to be known only for the case N = 1, see [5],[8],[9]. Finally, some "hidden" or nonlinear resonance phenomena can occur, see [9],[13]. For other interesting features on the jumping nonlinearities we refer to recent papers [2],[14].

Here we present, in a simple and unified way, some criteria on g_-, g_+ which allow to decide on the solvability of problem $(\mathcal{P})$(under an additional assumption on g); our results complete and slightly improve analogous results in [5],[12]. The plan is the following: in Section 1 we state the results and briefly discuss some possible refinements and related open questions; in Section 2 we prove some auxiliary lemmas and in Section 3 we give the proofs of the main results.

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1. Notation and statement of the results. We shall study problem ($\mathcal{P}$) in the following, more general, formulation

$$(P) \quad \begin{cases} Au = g(u) + h, & h \in L^2(\Omega) \\ u \in D(A) \end{cases}$$

where

$$(H_1) \quad \left| \begin{array}{l} A: D(A) \subset L^2(\Omega) \longrightarrow L^2(\Omega) \text{ is a densely defined self-} \\ \text{adjoint linear operator with compact resolvent;} \end{array} \right.$$

then A is a closed operator and its domain $D(A)$, equipped with the graph norm $\|u\|' = (\|u\|^2 + \|Au\|^2)^{\frac{1}{2}}$ for $u \in D(A)$, is compactly embedded in $L^2(\Omega)$ (with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$). Moreover, the spectrum of A consists of a countable sequence $(\lambda_k) \subset \mathbb{R}$ of eigenvalues, repeated according to their finite multiplicity, and the corresponding eigenfunctions $\{\varphi_k\}$ are a complete orthonormal basis of $L^2(\Omega)$. In order to simplify the notation we shall set $X = D(A)$, $Y = L^2(\Omega)$ and write λ for the simple eigenvalue crossed by g and φ for the associated normalized eigenfunction; we shall also set $\underline{\lambda} = \sup\{\lambda_k : \lambda_k < \lambda\}$ and $\overline{\lambda} = \inf\{\lambda_k : \lambda < \lambda_k\}$. Then the map $\hat{A} = A - \lambda I : X \subset Y \longrightarrow Y$ is a selfadjoint Fredholm operator (see e.g. [10], p. 239) and the spaces X, Y admit the orthogonal decompositions

$$(1.1) \qquad X = \mathbb{R}\varphi \oplus \hat{X}, \quad Y = \mathbb{R}\varphi \oplus \hat{Y}$$

where $\hat{X} = X \cap (\mathbb{R}\varphi)^{\perp}$ (which is a Hilbert space with the norm $\|\cdot\|'$) and $\hat{Y} = (\mathbb{R}\varphi)^{\perp}$, $(\ )^{\perp}$ being the orthogonal space in Y; it is also known that the restriction of $\hat{A}$ to $\hat{X}$ has an inverse, denoted by $\hat{A}^{-1} : \hat{Y} \longrightarrow \hat{X}$, which is bounded.

For the nonlinear part g, besides the above mentioned general assumptions, we shall require the following Lipschitz condition

$$(H_2)\quad \left| \begin{array}{l} \text{there exists a constant } 0 < L \leq \frac{1}{2} \|\hat{A}^{-1}\|^{-1} \text{ such that} \\ \underline{\lambda} < \lambda - L \leq \dfrac{g(r_1)-g(r_2)}{r_1 - r_2} \leq \lambda + L < \bar{\lambda} \text{ for } r_1 \neq r_2 , \\ \text{and } \lambda - L \leq g_- < \lambda < g_+ \leq \lambda + L; \end{array} \right.$$

finally we shall set $c_+ = g_+ - \lambda$ and $c_- = \lambda - g_-$ while, for a function $u \in Y$, $u^+ = \max\{u,0\}$ and $u^- = -\min\{u,0\}$.

We are now able to state our main results.

Theorem 1. *Let $\int_\Omega |\varphi|\varphi > 0$, i.e. $\|\varphi^+\| > \|\varphi^-\|$; if A and g verify (H_1), (H_2) and*

$$(1.2)\qquad \max\{c_+^2, c_-^2\} < \frac{1}{2\|\hat{A}^{-1}\|} \min\{|c_+\|\varphi^+\|^2 - c_-\|\varphi^-\|^2|,\ |c_-\|\varphi^+\|^2 - c_+\|\varphi^-\|^2|\}$$

then

(i) *when* $\dfrac{\|\varphi^-\|^2}{\|\varphi^+\|^2} < \dfrac{c_+}{c_-} < \dfrac{\|\varphi^+\|^2}{\|\varphi^-\|^2}$, *for all $q \in \hat{Y}$ there exists a real number $T = T(q)$ such that for $h = t\varphi + q$, $t \in \mathbb{R}$, the problem (P) has at least two solutions if $t < T$, at least one solution if $t = T$ and no solutions if $t > T$;*

(ii) *when* $\dfrac{c_+}{c_-} < \dfrac{\|\varphi^-\|^2}{\|\varphi^+\|^2}$ or $\dfrac{\|\varphi^+\|^2}{\|\varphi^-\|^2} < \dfrac{c_+}{c_-}$, *problem (P) is solvable for all $h \in Y$.*

Theorem 2. *Let $\int_\Omega |\varphi|\varphi = 0$; if A and g verify (H_1), (H_2) and $c_+ \neq c_-$ with*

$$(1.3)\qquad \max\{c_+^2, c_-^2\} < \frac{1}{2\|\hat{A}^{-1}\|}\,\frac{|c_+ - c_-|}{2},$$

then problem (P) is solvable for all $h \in Y$.

Of course a result analogous to Theorem 1 is true when $\int_\Omega |\varphi|\varphi < 0$ and both theorems hold, with obvious modifications, for the case $g_- > g_+$ too; on the other hand, one can replace the constant $\frac{1}{2}$ in (H_2) by an arbitrary $K \in (0,1)$ provided $\frac{1}{2}$ in (1.2),(1.3) is replaced by $1 - K$. A result similar to Theorem 1 (i) was proved in [12] by requiring a condition of the type (1.2) for the Lipschitz constant L; our formulation, thanks to (H_2) and (1.2), allows separate controls on L and the behaviour at infinity of g. Moreover, results similar to Theorem 1 (i) and Theorem 2 were proved in [5] by a different method while Theorem 1 (ii) seems to be new.

Despite of the involved form of (1.2), when c_+ and c_- have a common value c (i.e. $\frac{g_+ + g_-}{2} = \lambda$) we simply have

$$c < \frac{1}{2\,\|\hat{A}^{-1}\|} \left| \int_\Omega |\varphi|\varphi \right| .$$

On the other hand, since $\|\hat{A}^{-1}\|^{-1} \leq \min\{\lambda - \underline{\lambda}, \bar{\lambda} - \lambda\}$, it would be interesting to know if the above theorems hold with $\|\hat{A}^{-1}\|^{-1}$ replaced by $\min\{\lambda - \underline{\lambda}, \bar{\lambda} - \lambda\}$ in (1.2),(1.3). Another open question is whether a result of Ambrosetti-Prodi type can occur when $\int_\Omega |\varphi|\varphi = 0$; a negative answer is given in [9], under the stronger assumption that the functions φ^+, φ^- can be obtained one from the other by a translation, and in [5],[8] for the one-dimensional case.

2. <u>Auxiliary lemmas</u>. By the orthogonal decompositions given in (1.1) we can write every $u \in X$ as

$$u = s\varphi + v \quad \text{with} \quad s \in \mathbb{R},\ v \in \hat{X}$$

and every $h \in Y$ as

$$h = t\varphi + q \quad \text{with} \quad t \in \mathbb{R},\ q \in \hat{Y};$$

hence the problem(P) is equivalent to the system

$$\begin{cases} Av = Pg(s\varphi + v) + q & (2.1) \\ s\lambda = (g(s\varphi + v), \varphi) + t & (2.2) \end{cases}$$

where $P: Y \longrightarrow \hat{Y}$ is the orthogonal projection on $\hat{Y}$. As it is known, the equation (2.1) is always solvable, more precisely we have

<u>Lemma 1</u>. <u>If</u> A <u>and</u> g <u>satisfy</u> $(H_1),(H_2)$ <u>then, for every fixed</u> $s \in \mathbb{R}$ <u>and for all</u> $q \in \hat{Y}$, <u>there exists a unique</u> $v = v(s,q) \in \hat{X}$ <u>solution of</u> (2.1).

Though the proof of this lemma is the same of that given in [12], we present it for the reader's convenience.

<u>Proof</u>. Fixed $s \in \mathbb{R}$, we shall prove that the map defined as $\Psi(v) = Av - Pg(s\varphi + v)$, for $v \in \hat{X}$, is a homeomorphism of $\hat{X}$ onto $\hat{Y}$. Since

$$\hat{A}^{-1}\Psi(v) = v - \hat{A}^{-1}P[g(s\varphi + v) - \lambda(s\varphi + v)] \quad (2.3)$$

it suffices to prove that $\hat{A}^{-1}\Psi$ is a homeomorphism on $\hat{X}$; by calling $\Phi(v)$ the second addendum of (2.3), from (H_2) we get

$$\|\Phi(v) - \Phi(\bar{v})\|' \leq \frac{1}{2}\|v - \bar{v}\|' \quad \text{for } v, \bar{v} \in \hat{X},$$

i.e. Φ is a contraction on $\hat{X}$ and then, being $\hat{A}^{-1}\Psi = I + \Phi$, we can conclude by applying the Banach contraction mapping principle.

By this way the solvability of the problem (P) follows from that of equation (2.2) or better, by setting $G(s,q) = s\lambda - (g(s\varphi + v(s,q)),\varphi)$, from the study of the real-valued function $G(s,q)$ for every fixed $q \in \hat{Y}$. The following lemma will enable us to investigate the behaviour at infinity of such a function.

<u>Lemma 2</u>. <u>Let</u> A <u>and</u> g <u>be as in Lemma 1</u>; <u>then for all</u> $q \in \hat{Y}$ <u>there exist</u>

$$\lim_{s \to +\infty} \frac{G(s,q)}{s} = -(c_+(\varphi + \overline{v})^+ + c_-(\varphi + \overline{v})^-, \varphi)$$

$$\lim_{s \to -\infty} \frac{G(s,q)}{s} = (c_-(\varphi + \underline{v})^+ + c_+(\varphi + \underline{v})^-, \varphi),$$

<u>with uniquely determined</u> $\overline{v}$, $\underline{v} \in \hat{X}$ <u>(i.e. which are independent on</u> q) <u>such that</u>

$$\max\{\|\overline{v}\|', \|\underline{v}\|'\} \leq 2\,\|\hat{A}^{-1}\|\,\max\{c_+, c_-\}.$$

<u>Proof.</u> We study only the case $s \to +\infty$ since the proof for the other case is identical. Let $\{s_n\}$ be a positively divergent sequence and, for a fixed $q \in \hat{Y}$, let $v_n = v(s_n, q)$ be the unique solution of the equation (2.1); then v_n, for all $n \in N$, is such that

$$(2.4) \quad v_n = \hat{A}^{-1} P[q(s_n\varphi + v_n) - \lambda(s_n\varphi + v_n)] + \hat{A}^{-1}q.$$

By adding and subtracting the quantity $g(s_n\varphi) - \lambda s_n\varphi$ in the square bracket and using (H_2), after some easy computations, we obtain

$$(2.5) \quad \left\|\frac{v_n}{s_n}\right\|' \leq \frac{\|\hat{A}^{-1}\|}{1 - \|\hat{A}^{-1}\|\,L}\left(\left\|\frac{g(s_n\varphi)}{s_n} - \lambda\varphi\right\| + \left\|\frac{q}{s_n}\right\|\right);$$

next, since $\{\frac{g(s_n\varphi)}{s_n}\}$ converges strongly to $g_+\varphi^+ - g_-\varphi^-$ in Y (see for instance Lemma 2.5 of [9]), we have that

$$\left\|\frac{g(s_n\varphi)}{s_n} - \lambda\varphi\right\| \longrightarrow \left\|c_+\varphi^+ + c_-\varphi^-\right\|$$

and hence the sequence $\left\|\frac{v_n}{s_n}\right\|'$ is bounded.

Then there exist $\overline{v} \in \hat{X}$ and a subsequence of $\{\frac{v_n}{s_n}\}$, still denoted by $\{\frac{v_n}{s_n}\}$, which is weakly convergent to $\overline{v}$ in $\hat{X}$ and from (H_2), (2.5) we get

$$\|\bar{v}\|' \leq 2\,\|\hat{A}^{-1}\| \cdot \|c_+\varphi^+ + c_-\varphi^-\| \leq 2\,\|\hat{A}^{-1}\| \max\{c_+, c_-\}.$$

We have now to show that such a $\bar{v}$ is uniquely determined and independent on the fixed q. For this purpose it suffices to prove that $\bar{v}$ is the unique solution of the equation

$$w \in \hat{X},\ Aw = P\,[\,g_+(\varphi + w)^+ - g_-(\varphi + w)^-]$$

or equivalently

$$(2.6) \qquad w \in \hat{X},\ w = \hat{A}^{-1} P\,[\,c_+(\varphi + w)^+ + c_-(\varphi + w)^-].$$

Since $\left\| \frac{v_n}{s_n} \right\|'$ is bounded and X is compactly embedded in Y, there exists a subsequence of $\{\frac{v_n}{s_n}\}$ which is strongly convergent to $\bar{v}$ in Y; hence, after dividing (2.4) by s_n, we can pass to the limit in (2.4) (again thanks to the quoted lemma in [9]) and conclude that $\bar{v}$ is a solution of (2.6). In order to prove uniqueness let us suppose that there exist two solutions w_1, w_2 of (2.6). By writing (2.6) for w_1 and w_2, subtracting term by term, and using the inequalities

$$-(w_1 - w_2)^- \leq (\varphi + w_1)^+ - (\varphi + w_2)^+ \leq (w_1 - w_2)^+$$

$$-(w_1 - w_2)^+ \leq (\varphi + w_1)^- - (\varphi + w_2)^- \leq (w_1 - w_2)^-\ ,$$

we have, from (H_2),

$$\|w_1 - w_2\|' \leq \|\hat{A}^{-1}\| \max\{c_+, c_-\}\ \|w_1 - w_2\| \leq \frac{1}{2}\ \|w_1 - w_2\|'$$

giving rise to a contradiction.

Finally, the value of $\lim_{n \to +\infty} \frac{G(s_n, q)}{s_n}$ is immediately obtained, since the whole sequence $\{\frac{v_n}{s_n}\}$ converges to $\bar{v}$, by arguing as above for $\{g(s_n(\varphi + \frac{v_n}{s_n}))/s_n\}$.

In the sequel we shall also need the following

Lemma 3. Let A and g be as in Lemma 1; then, for every fixed $q \in \hat{Y}$, $G(s,q)$ is a continuous function of $\mathbb{R}$ into $\mathbb{R}$.

Proof. By the definition of the function $G(s,q)$ and the Lipschitz continuity of g, it suffices to prove the continuity of $v(s,q)$ with respect to s, for every fixed $q \in \hat{Y}$. Then, let $\{s_n\}$ be such that $s_n \longrightarrow s$ and, for every fixed $q \in \hat{Y}$, let $v_n = v(s_n,q)$ be the unique solution of (2.1); by arguing as before in order to obtain (2.5), we get

$$\text{(2.7)} \qquad \| v_n \|' \leq \text{const.}(\| g(s_n \varphi) - \lambda s_n \varphi \| + \| q \|)$$

where the term on the right is bounded.
Hence, after extracting a subsequence, we may assume that $v_n \longrightarrow \tilde{v}$ strongly in Y and by the continuity of the map g in Y we have that $Pg(s_n\varphi + v_n) \longrightarrow Pg(s\varphi + \tilde{v})$ strongly in Y. From (2.1) it follows that $Av_n \longrightarrow Pg(s\varphi + \tilde{v}) + q$ strongly in Y and, since A is a closed operator, we obtain $\tilde{v} \in X$ with $A\tilde{v} = Pg(s\varphi + \tilde{v}) + q$ that is, by Lemma 1, $\tilde{v} \equiv v(s,q)$. Thus the whole sequence $\{v_n\}$ converges to $v(s,q)$ (even w.r.t. the norm $\| \cdot \|'$) and we can conclude.

Remark 1. The result stated in Lemma 2 can be improved when $\lambda = \lambda_1$, the first eigenvalue of A; in fact, in this case it is possible to show that $\bar{v} = \underline{v} = 0$ and, since φ_1 does not change sign on Ω, we have $\lim_{s \to \pm\infty} \frac{G(s,q)}{s} = \lambda_1 - g_{\pm}$. To our knowledge this was firstly observed in [9]; on the other hand, a more direct proof of this result is given in [4].

Remark 2. The proof of Lemma 3 follows essentially by the Lipschitz continuity of g; actually, under this assumption, it is possible to say that $G(s,q)$ has the same regularity of g, see e.g. [4],[11].

3. <u>Proofs of the results.</u> As we already said, the solvability of equation (2.2), and hence that of the problem (P), is an immediate consequence of the behaviour at infinity of $G(s,q)$; more precisely, since by Lemma 3 we know that, for every fixed $q \in \hat{Y}$, $G(s,q)$ is a continuous function, the solvability of equation (2.2) is determined by the sign of the quantities $G_{\pm} = \lim_{s \to \pm\infty} \frac{G(s,q)}{s}$ studied in Lemma 2. Thus, Theorem 1 (i) is readily obtained if we are able to prove that $G_+ < 0$ and $G_- > 0$ since, for a fixed $q \in \hat{Y}$, it suffices to take $T = T(q) \equiv \equiv \max_{\mathbb{R}} G(s,q)$; similarly Theorem 1 (ii) and Theorem 2 wil follow if G_+ and G_- have the same sign.

In order to prove Theorems 1 and 2 we remark that the following estimates hold:

(3.1) $\quad |G_+ + (c_+\varphi^+ + c_-\varphi^-, \varphi)| \leq \max\{c_+, c_-\} \; \|\bar{v}\|'$

(3.2) $\quad |G_- - (c_-\varphi^+ + c_+\varphi^-, \varphi)| \leq \max\{c_+, c_-\} \; \|\underline{v}\|'$

where, besides some simple computations, we used inequalities of the type

$$- w^- \leq (\varphi + w)^+ - \varphi^+ \leq w^+ \quad (\text{with } w = \bar{v} \text{ or } w = \underline{v});$$

from (3.1),(3.2) and the estimate of Lemma 2 on $\|\bar{v}\|'$, $\|\underline{v}\|'$ we get

$$|G_+ + [c_+ \|\varphi^+\|^2 - c_- \|\varphi^-\|^2]| \leq 2 \|\hat{A}^{-1}\| \max\{c_+^2, c_-^2\}$$

$$|G_- - [c_- \|\varphi^+\|^2 - c_+ \|\varphi^-\|^2]| \leq 2 \|\hat{A}^{-1}\| \max\{c_+^2, c_-^2\} .$$

If $\frac{c_+}{c_-}$ satisfies the condition in (i) of Theorem 1, then

$$G_+ \leq - [c_+ \|\varphi^+\|^2 - c_- \|\varphi^-\|^2] + 2 \|\hat{A}^{-1}\| \max\{c_+^2, c_-^2\} < 0$$

$$G_- \geq [c_- \|\varphi^+\|^2 - c_+ \|\varphi^-\|^2] - 2 \|\hat{A}^{-1}\| \max\{c_+^2, c_-^2\} > 0$$

where the strict inequalities follow from (1.2), since the quantities in square brackets are positive, and we can conclude; by the same arguments it is possible to verify that for $\frac{c_+}{c_-} < \frac{\|\varphi^-\|^2}{\|\varphi^+\|^2}$ $\left(\frac{\|\varphi^+\|^2}{\|\varphi^-\|^2} < \frac{c_+}{c_-}\right.$ resp.) we have $G_+ > 0$ and $G_- > 0$ ($G_+ < 0$ and $G_- < 0$ resp.), thus proving (ii) of Theorem 1. Being φ as in Theorem 2 and $c_+ < c_-$ ($c_+ > c_-$ resp.), from (1.3) we have $G_+ > 0$ and $G_- > 0$ ($G_+ < 0$ and $G_- < 0$ resp.) and hence the solvability of (P) for all $h \in Y$.

Remark 3. The statement of part (i) of Theorem 1 can be strengthened, when $A = -\Delta$ and $g \in C^1(\mathbb{R})$, by showing the existence of $T_0 = T_0(q) < T$ such that for $h = t\varphi + q$ with $t < T_0$, the problem (P) has exactly two solutions; this can be proved by arguing as in [1], where such a result was established for the case $c_+ = c_- = L$. On the other hand, by suitably modifying the arguments used in [1], we can also obtain uniqueness of solutions "at infinity" (i.e. for large values of the parameter t) for the situations described in Theorems 1 (ii) and 2.

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Istituto die Matematica, Università dell' Aquila, Via Roma 33, 67100 L'Aquila, Italy

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