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ON THE COMPLEXITY OF THE SUBGRAPH PROBLEM
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Abstract: The complexity of the problem "Does a given graph contain a complete subgraph with k vertices?" is $O(n^k)$.

Key words: Complexity of the subgraph problem, complete subgraph.

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This note is motivated by the complexity of the following decision problem:

Given a graph G and a positive integer k , does there exist a subgraph of G isomorphic to K_k (= the complete graph with k vertices)?

The following particular question was considered independently by L. Lovász and one of us:

Is the complexity of the above problem $O(n^k)$?

In this note we give a positive answer to this question in a slightly more general form. Let us note that we have been informed by L. Lovász that F.K. Chung and R. Karp obtained independently also a solution to the above problem.

Let us stress that all the solutions are based on the fast matrix multiplication and that it is not clear whether one could devise a purely combinatorial algorithm.

1: Fast recognition of complete subgraphs

1.1: First we show how to detect a triangle:

Let G be a graph with vertices x_1, \dots, x_n and let A be the adjacency matrix of G (i.e. $a_{ij} = 1$ if x_i, x_j form an edge of G , $a_{ij} = 0$ otherwise). Compute the matrix $B := A^2$. Then the graph G does not contain a triangle if and only if $\min(a_{ij}, b_{ij}) = 0$ for all i, j (as b_{ij} is the number of paths of length 2 between x_i and x_j). The complexity of this procedure is $O(n^\alpha)$ providing we use an $O(n^\alpha)$ algorithm for the matrix multiplication. It is well known that one may achieve $\alpha < 3$ (see Concluding remarks). If $0 < a_{ij} \leq b_{ij}$ for some i, j then G contains a triangle of the form $\{x_i, x_j, x_k\}$. The third vertex x_k can be found in $O(n)$ steps by checking all the remaining vertices.

1.2: This procedure may be used for detection of complete subgraphs of size 3ℓ in $O(n^{\ell \cdot \alpha})$ steps as follows:

For a given graph G of size n we construct an auxiliary graph H of size $O(n^\ell)$ with the following property: H contains a triangle iff G contains a complete subgraph of size 3ℓ .

Thus the detection of triangles in H yields an $O(n^{\ell \cdot \alpha})$ algorithm for the detection of a complete subgraph of size 3ℓ .

The graph H may be defined as follows:

$$V(H) = \{Y \subseteq V(G); |Y| = \ell \text{ and } Y \text{ forms a complete subgraph in } G\}$$

$$E(H) = \{\{Y, Y'\}; Y \neq Y' \text{ and } Y \cup Y' \text{ forms a complete subgraph in } G\}.$$

1.3: Let us also remark that the vertex sizes which are not divisible by 3 do not present a difficulty by the following:

For a subset Y of vertices of graph $G = (V, E)$ put $N(Y) = \{v \in V \mid \{y, v\} \in E \text{ for every } y \in Y\}$.

$N(Y)$ is the set of all common neighbors of the set Y .

Consider all graphs G_1, \dots, G_n which are induced by the sets $N(\{x\})$, $x \in V$. Then a graph G_1 contains a complete subgraph of size 3ℓ if and only if G contains a complete subgraph of size $3\ell + 1$.

Similarly if we consider all graphs which are induced by the sets $N(\{x, y\})$, $\{x, y\} \in E$ we can detect a complete subgraph of size $3\ell + 2$.

Thus, using the previous $O(n^{\ell \cdot \alpha})$ for a 3ℓ -complete subgraph, we can detect a $(3\ell + 1)$ -complete subgraph of a graph with n vertices in $O(n^{i + \ell \cdot \alpha})$ steps, $i = 0, 1, 2$.

2: Fast recognition of arbitrary subgraphs

2.1: Here we prove

Proposition. Let F be a fixed graph with k vertices. Let there exist an $O(n^{\alpha(k)})$ algorithm for finding a K_k in a graph with n vertices. Then the following two problems can be solved in $O(n^{\alpha(k)})$ steps for arbitrary graph G with n vertices:

- (1) Does G contain F as an induced subgraph?
- (2) Does G contain F as a (not necessarily induced) subgraph?

We give two proofs.

2.2: Proof I: For a given instance F, G of the problem ((1) or (2)) we construct auxiliary graphs H_1 and H_2 of size $k \times n$ with the property that H_1 contains a complete graph of size k iff the answer to the problem (1) is positive, $i = 1, 2$.

Put $V(H_1) = V(H_2) = V(F) \times V(G)$. Denote by E_1 , E_2 and E_3 the following three sets:

$\{(f_1, g_1), (f_2, g_2)\} \in E_1$ iff $f_1 \in V(F)$, $g_1 \in V(G)$, $f_1 \neq f_2$ and $g_1 \neq g_2$.

$\{(f_1, g_1), (f_2, g_2)\} \in E_2$ iff $f_1 \in V(F)$, $g_1 \in V(G)$, $\{f_1, f_2\} \in E(F)$
and $\{g_1, g_2\} \in E(G)$.

$\{(f_1, g_1), (f_2, g_2)\} \in E_3$ iff $f_1 \in V(F)$, $g_1 \in V(G)$, $\{f_1, f_2\} \notin E(F)$
or $\{g_1, g_2\} \in E(G)$.

Put $E(H_1) = E_1 \cap E_2$, and $E(H_2) = E_1 \cap E_3$.

It is easy to see that these graphs have the desired properties.

2.3: Proof II: We consider only the case $k = 3\ell$.

We construct an auxiliary graph H as follows:

Let $X_1 \cup X_2 \cup X_3$ be a partition of the set $V(F)$ into parts of size ℓ . Denote by F_1 the subgraph of F induced by the set X_1 and denote by $F_{1,j}$ the subgraph of F induced by the set $X_1 \cup X_j$, $i, j = 1, 2, 3$, $i \neq j$.

Denote by V_1 the set of all embeddings of F_1 into G (explicitly: $f \in V_1$ iff $f: X_1 \rightarrow V(G)$ is one-to-one and $\{f(x), f(y)\} \in E(G) \iff \{x, y\} \in E(F_1)$).

Put $V(H) = V_1 \cup V_2 \cup V_3$ and $\{f, f'\} \in E(H)$ iff $f \in V_1$, $f' \in V_j$ ($i \neq j$) and the mapping $f \cup f'$ is an embedding of $F_{1,j}$ into G .

Clearly H contains a triangle if and only if G contains an induced subgraph isomorphic to F .

3: Concluding remarks

3.1: Instead of Strassen algorithm [2] we could use any of its refinements.

The current best performance $n^{2,495364}$ is due to Copper-smith and Winograd, see [1].

3.2: Apart from the problem of finding a combinatorial algorithm (see the introduction) the following question may be of interest:

Does there exist a graph F with k vertices for which the

decision problem

"does G contain F as an induced subgraph"

is easier than the corresponding problem for the complete graph K_k ?

Of course the non-induced subgraph problem is easier (e.g. for forests).

R e f e r e n c e s :

- [1] D. COPPERSMITH, S. WINOGRAD: On the asymptotic complexity of matrix multiplication, in: Proceedings 22nd Symposium on Foundations of Comp. Sci, 1981, p. 82-90.
- [2] V. STRASSEN: Gaussian elimination is not optimal, Num. Math. 13(1969), 354-356.

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