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POLYADIC SPACES OF ARBITRARY COMPACTNESS NUMBERS

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ABSTRACT: We investigate a compact Hausdorff topology on the set of all subsets of cardinality at most n of a given set S. For each n we construct a polyadic and Eberlein space of compactness number n of weight ω_1 which is the union of finitely many discrete subspaces. Our topology on $[S]^{\le n}$, is such that both the compactness numbers of $[S]^{\le 2n-1}$ and $[S]^{\le 2n}$ are $n+1$ for uncountable S's.

Key words and phrases: Compactness number, polyadic and Eberlein.

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1. Introduction.

In this paper we construct new examples of spaces with arbitrary compactness number. These spaces are particularly simple to construct and most common topological properties are easy to determine. However, the property which we are interested in, compactness number, is not so easy.

One of these spaces answers a question of M. Husek, namely, it is an Eberlein compact space which is not supercompact. It is also a second example of a continuous image of a supercompact space which is not supercompact; the first example being due to C. Mills and J. van Mill. Our example has different properties than theirs, for example, it is polyadic. Another of these spaces answers a question of J. van Mill and the author, namely,

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it is a continuous image of a space of finite compactness number which has infinite compactness number.

Most definitions appear in section 2. Section 3 is devoted to the basic combinatorial set theory that is needed. We define our spaces in section 4 as well as finding upper bounds for their compactness numbers. In section 5 our main Theorem 5.1 produces lower bounds. We thus get our spaces X_n such that $\mathrm{cmpn}(X_{2n}) = \mathrm{cmpn}(X_{2n-1}) = n + 1$.

2. Preliminaries.

If $\mathcal{S}$ is a collection of sets and S is a set then $\mathcal{S}^{(F)}$ denotes the set $\{\cap F : F$ is a finite subset of $\mathcal{S}\}$, $[S]^n$ denotes the set of all subsets of S of size n and $[S]^{\leq n}$ denotes the set of all subsets of S of size at most n; this latter includes the empty set ϕ. If $2 \leq n < \omega$ then $\mathcal{S}$ is said to be n-linked if $\cap F \neq \phi$ for all $F \in [\mathcal{S}]^n$. $\mathcal{S}$ is said to be n-ary if every n-linked subset of $\mathcal{S}$ has a non-empty intersection.

Polyadic (or m-adíc) spaces are Hausdorff continuous images of some power of the Alexandroff one point compactification of a discrete space. These were defined by Mrowka [8] as a good generalization of dyadic spaces. Eberlein spaces, Amir and Lindenstrauss [1], are those spaces homeomorphic to a weakly compact subset of a Banach space. They include, in particular, all compact subspaces of the subspace of 2^κ consisting of functions with finite support, for any cardinal κ.

A space X has compactness number at most n, $\mathrm{cmpn}(X) \leq n$, if X possesses an n-ary closed subbase. If m is the least such n then we write $\mathrm{cmpn}(X) = m$. If no such $n < \omega$ exists and X is compact then we say that

X has infinite compactness number. Spaces of cmpn 2 are called supercompact spaces, de Groot [6]. Bell and van Mill [5] have constructed compact Hausdorff spaces of arbitrary cmpn. Different examples appear in Bell [2]. This paper will produce yet a third class of examples.

If $\mathcal{S}$ is n-ary then so also is $\mathcal{S}^{(f)}$. Hence every space X with $\text{cmpn}(X) \leq n$ possesses an n-ary closed subbase $\mathcal{S}$ with $\mathcal{S} = \mathcal{S}^{(f)}$. The advantage of closing $\mathcal{S}$ under finite intersections is the following: A collection $\mathcal{S} = \mathcal{S}^{(f)}$ of closed subsets of a compact space X is a closed subbase iff for every closed K contained in an open set O there exists a finite $\mathcal{F} \subset \mathcal{S}$ such that $K \subset \cup \mathcal{F} \subset O$.

3. A Free Set Lemma for $[\omega_1]^n$.

For $1 \leq k \leq n < \omega$ and $s \in [\omega_1]^n$ we define $s(k)$ to be the kth element of s under the induced order of ω_1. Therefore for every $s \in [\omega_1]^n$ we have $s = \{s(1),\dots,s(n)\}$ where $s(1) < \dots < s(n)$.

Assume that $f : [\omega_1]^n \to [\omega_1]^{<\omega}$ is such that for every $s \in [\omega_1]^n$ we have $s \cap f(s) = \phi$. A subset A of ω_1 is said to be <u>free</u> if for every $s \in [A]^n$ we have $A \cap f(s) = \phi$ and A is said to be <u>almost free</u> if for every $s \in [A]^n$ we have $A \cap f(s) \subsetneq \{\gamma : s(1) < \gamma < s(n)\}$.

<u>Example</u> (due to S. Todorcevic) There exists $f : [\omega_1]^2 \to [\omega_1]^{<\omega}$ such that for every $s \in [\omega_1]^2$ we have $s \cap f(s) = \phi$ but there is no free subset of cardinality 3 and there is no almost free subset of order type $\omega + 1$.

For every $\gamma < \omega_1$ choose an injection $\varphi_\gamma : \gamma \to \omega$. For each $\beta < \gamma$ define $f(\{\beta,\gamma\}) = \{\pi < \gamma : \varphi_\gamma(\pi) \leq \varphi_\gamma(\beta)\} - \{\beta,\gamma\}$. The basic property of f from which both conclusions follow is that whenever $\alpha < \beta < \gamma$ then either $\alpha \in f(\{\beta,\gamma\})$ or $\beta \in f(\{\alpha,\gamma\})$.

<u>Lemma 3.1</u>. Assume that n is a positive integer and that $f : [\omega_1]^n \to [\omega_1]^{<\omega}$ is such that for every $s \in [\omega_1]^n$ we have $s \cap f(s) = \phi$. Then for every $N \geq n$ there exists an almost free subset A of cardinality N.

Proof: Use induction on $k = 1$ to $k = N$ to choose denumerable subsets A_k of ω_1 such that:

(a) if $j < k$, $\alpha \in A_j$ and $\beta \in A_k$ then $\alpha < \beta$

(b) $A_k \cap f(s) = \phi$ for every $s \in [\bigcup_{j<k} A_j]^n$.

Use induction on $k = N$ to $k = 1$ to choose $\{\alpha_k : 1 \leq k \leq N\}$ such that:

(c) $\alpha_k \in A_k$

(d) $\alpha_k \notin f(s)$ for every $s \in [\{\alpha_j : k < j \leq N\}]^n$. Then $A = \{\alpha_k : 1 \leq k \leq N\}$ is an almost free subset. □

4. <u>The Tychonoff topology on $[S]^{\leq n}$</u>.

If S is an infinite set and $1 \leq n < \omega$ then we define a compact Hausdorff topology on $[S]^{\leq n}$ as follows: If $s \in S$ then put $s^+ = \{F \in [S]^{\leq n} : s \in F\}$ and put $s^- = \{F \in [S]^{\leq n} : s \notin F\}$. Use the collection $\mathcal{S} = \bigcup_{s \in S} \{s^+, s^-\}$ as a closed (also open) subbase for a topology on $[S]^{\leq n}$. This topology is called the Tychonoff topology on $[S]^{\leq n}$. $[S]^{\leq n}$ with this topology is ZF-compact, i.e., it is both defined and provably compact without the aid of any choice principles. If $0 \leq k \leq n$ then $[S]^k$ is a discrete subspace of $[S]^{\leq n}$ Hence $[S]^{\leq n}$ is a space which is the union of $n + 1$ discrete subspaces.

Let $A(S) = S \cup \{\infty\}$ be the Alexandroff one point compactification of the discrete space S and let $A(S)^n$ be the Tychonoff product of n copies of $A(S)$. The mapping $\varphi : A(S)^n \to [S]^{\leq n}$ defined by $\varphi((x_1,\dots,x_n)) = \{x_1,\dots,x_n\} \cap S$ is seen to be continuous and onto. Hence the spaces $[S]^{\leq n}$ are polyadic and Eberlein as well.

The subbase $\mathcal{S}$ as above is $n+1$-ary; because if F is an $n+1$-linked subset of $\mathcal{S}$ then $\{s \in S : s^+ \in F\} \in \cap F$. Hence $\text{cmpn}([S]^{\leq n}) \leq n+1$. This can be improved substantially for totally orderable S's as follows:

<u>Theorem 4.1.</u> To each total order $<$ on S there is a naturally associated $n+1$-ary closed subbase $\mathcal{R}$ of $[S]^{\leq 2n}$.

Proof: If $s \in S$ then put $L_s = \{F \in s^+ : |\{t \in F : t < s\}| \leq n-1\}$ and put $R_s = \{F \in s^+ : |\{t \in F : s < t\}| \leq n-1\}$. Since both L_s and R_s are closed and $s^+ = L_s \cup R_s$ we get that $\mathcal{R} = \bigcup_{s \in S} \{L_s, R_s, s^-\}$ is a closed subbase of $[S]^{\leq 2n}$. Observe also that $\{s\} \in L_s \cap R_s$.

Let $F = \{L_s : s \in A\} \cup \{R_s : s \in B\} \cup \{s^- : s \in C\}$ be an $n+1$-linked subset of $\mathcal{R}$ where $A \cup B \cup C \neq \phi$. We claim that $A \cup B \in \cap F$.

To see this put $T_t = L_t$ if $t \in A$ and put $T_t = R_t$ if $t \in B - A$. If $s \in A$ then $|\{t \in A \cup B : t < s\}| \leq n-1$. This is so because if $D \in [\{t \in A \cup B : t < s\}]^n$ then $\bigcap_{t \in D} T_t \cap L_s = \phi$ but this contradicts $n+1$-linkage of F. Analagously, if $s \in B$ then $|\{t \in A \cup B : s < t\}| \leq n-1$. Both of these implications together imply that $|A \cup B| \leq 2n$ and that $A \cup B \in \bigcap_{s \in A} L_s \cap \bigcap_{s \in B} R_s$. If $t \in A \cup B$ and $s \in C$ then $T_t \cap s^- \neq \phi$. Thus $t \neq s$ and therefore $(A \cup B) \cap C = \phi$. Hence $A \cup B \in \cap F$ and therefore $\mathcal{R}$ is $n+1$-ary. □

The case $n = 1$ in the above theorem was proven in Bell and Ginsburg [4] where it was also proven that for uncountable S, $[S]^{\le 2}$ does not have a 2 - ary closed subbase consisting of clopen sets.

It is shown in [5] that if K is a clopen subspace of a compact space X then $\mathrm{cmpn}(K) \le \mathrm{cmpn}(X)$. If $1 \le k \le 2n$ then $[S]^{\le k}$ is embedded as a clopen subspace of $[S]^{\le 2n}$ as all supersets of a fixed set of size $2n - k$. So we see that if $1 \le k \le 2n$ then $\mathrm{cmpn}([S]^{\le k}) \le n + 1$. In the next section we will show that, in general, this is the best upper bound possible.

5. Lower bound on cmpn $([S]^{\le n})$.

Theorem 5.1. For every n with $2 \le n < \omega$, $[\omega_1]^{\le 2n-1}$ cannot be embedded as a neighbourhood retract in any space K with $\mathrm{cmpn}(K) \le n$.

Proof: Let us put $\kappa = \omega_1$. Assume that $[\kappa]^{\le 2n-1} \subset K$, K is compact, U is an open subspace of K, $r : U \to [\kappa]^{\le 2n-1}$ is a retraction and that $\mathcal{S} = \mathcal{S}^{(f)}$ is a closed subbase for K. For every $\alpha < \kappa$ there exists a finite $\mathcal{S}_\alpha \subset \mathcal{S}$ with $\alpha^+ \subset \cup \mathcal{S}_\alpha \subset r^{-1}[\alpha^+]$. Since κ has uncountable cofinality there exist $m < \omega$ and a subset E of κ of cardinality κ such that if $\alpha \in E$ then $\mathcal{S}_\alpha = \{S_\alpha^1, \ldots, S_\alpha^m\}$. Since each $S \in \mathcal{S}$ is closed for each $s \in [E]^n$ let us choose a finite $F_s \subset \kappa$ such that $F_s \cap s = \phi$ and

$$\bigcap_{\alpha \in s} \alpha^+ \cap \bigcap_{\beta \in F_s} \beta^- \cap \bigcup\{S_\alpha^i : i \le m,\ \alpha \in s \text{ and } s \notin S_\alpha^i\} = \phi .$$

Put $p = m + n^m$. Invoking Ramsey's theorem of the partition calculus, choose $N < \omega$ such that $N \to (2n)^n_p$. Define $f : [E]^n \to [E]^{<\omega}$ by $f(s) = E \cap F_s$. Apply Lemma 3.1 to produce an almost free subset A of E of cardinality N. We now define a partition of $[A]^n$ into p parts. For every $i \le m$ define $Q_i = \{s \in [A]^n : s \in \bigcap_{k=1}^{n} S^i_{s(k)}\}$. For every $\varphi : \{1,\ldots,m\} \to \{1,\ldots,n\}$ define $R_\varphi = \{s \in [A]^n : s \notin \bigcup_{i=1}^{m} S^i_{s(\varphi(i))}\}$. By

Ramsey's theorem, there exists $H \in [A]^{2n}$ and either a Q_i or a R_φ such that $[H]^n \subseteq Q_i$ or $[H]^n \subseteq R_\varphi$. H is an almost free subset because $H \subseteq A$.

We rule out the possibility that $[H]^n \subset R_\varphi$ for some φ. Put $L = H - \{H(1)\}$. Then $L \in [\kappa]^{\leq 2n-1}$. There exists $j \leq m$ such that $L \in S^j_{L(n)}$. Since L has cardinality $2n - 1$ we can choose an s consisting of n consecutive elements of L such that $s(\varphi(j)) = L(n)$. Since $s \in R_\varphi$ we have that $s \notin S^j_{s(\varphi(j))}$. Therefore $s \notin S^j_{L(n)}$ By definition of F_s we get that $\bigcap_{\alpha\in s} \alpha^+ \cap \bigcap_{\beta\in F_s} \beta^- \cap S^j_{L(n)} = \phi$. This is a contradiction since L is in this intersection.

We deduce that $[H]^n \subset Q_i$ for some i. The collection $\{S^i_\alpha : \alpha \in H\}$ is n-linked; because if $s \in [H]^n$ then $s \in Q_i$. Therefore $s \in \bigcap_{k=1}^{n} S^i_{s(k)} = \bigcap_{\alpha\in s} S^i_\alpha$. However $\bigcap_{\alpha\in H} S^i_\alpha \subset \bigcap_{\alpha\in H} r^{-1}[\alpha^+]$ and hence is empty, because there are no sets of size $2n$ in $[\kappa]^{\leq 2n-1}$. Thus $\mathcal{S}$ is not n-ary. □

Corollary 5.2. If $1 \leq n < \omega$ then $\mathrm{cmpn}([\omega_1]^{\leq 2n-1})$ and $\mathrm{cmpn}([\omega_1]^{\leq 2n})$ both equal $n + 1$.

Proof: Use Theorem 5.1 together with the result of section 4 that if $1 \leq k \leq 2n$ then $\mathrm{cmpn}([\omega_1]^{\leq k}) \leq n + 1$.

Examples 5.3. The spaces $[\omega_1]^{\leq 2n-1}$ are polyadic, Eberlein spaces of weight ω_1 and cmpn n+1 and are the union of 2n discrete subspaces.

These spaces answer Problem 3 of M. Hušek [9] on whether every Eberlein space is supercompact. They also serve as second examples of a continuous image of a supercompact space which is not supercompact; the f[illegible] such example was due to C. Mill and J. van Mill [7]. The question of whether every dyadic space is supercompact remains unanswered.

Let K be the Alexandroff one point compactification of the disjoint union of the spaces $[\omega_1]^{\leq 2n-1}$ K is a space of infinite compactness number that is a continuous image of one of finite compactness number, indeed K is polyadic, Eberlein and the union of countably many discrete subspaces. This answers question 4.3. in [5].

Questions 5.4. Is there a compact space K of weight ω_1 that is not a continuous image of any space of finite compactness number? In [5], it is proven that βN is not the continuous image of any space of finite compactness number, so we do not want a consistent example of such a space K. The existence of K would enable one to prove that the hyperspace of closed subsets of 2^{ω_2} is not the continuous image of a space of finite compactness number , cf. Bell [3]. Finally, we repeat question 4.1. of [5]: Is there a sequence of first countable separable spaces X_k for which $cmpn(X_k) = k$?

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