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CONFIGURATION CONDITIONS OF SMALL POINT RANK IN 3-NETS

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Abstract: There are analyzed all possibilities for closure conditions with at most 7 vertices in 3-nets and the corresponding algebraic identities are found. The method used works also in the general case (with arbitrary number of vertices) but yet for 8 vertices increases rapidly.

Key words: 3-halfnet, 3-net, homomorphism, configuration, closure condition.

Classification: 20N05, 51A20

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§ 1 Some properties of 3-nets

A 3-net (briefly: a net) is defined as a triplet $(P,L,I,(L_1,L_2,L_3))$ where P,L are non-void sets, I is a subset of $P \times L$ and $\{L_1,L_2,L_3\}$ is a decomposition of L (inducing an equivalence relation // on L) such that

(i) for every $a \in L$ there is a $b \in P$ with bIa,

(ii) for every $i \in \{1,2,3\}$ and every $a \in P$ there is just one $b \in L_i$ with aIb, and

(iii) for every $a,b \in L$ not satisfying a//b there is just one $c \in P$ with cIa,b.

If P,L_1,L_2,L_3 are one-element sets then the net is called trivial. Elements of P will be called points, elements of L lines, I incidence and L_1,L_2,L_3 parallelity classes; the cardinality of P will be called point rank, the cardinality of L line rank and the cardinality of $\{p|pI\ell\}$ for any $\ell \in L$ the length of ℓ.

Let $N=(P,L,I,(L_1,L_2,L_3))$, $N'=(P',L',I',(L_1',L_2',L_3'))$ be nets. A couple (π,λ) of bijections $\pi:P\to P'$, $\lambda:L\to L'$ is said to be an isomorphism of N onto N', if $xIy \Rightarrow \pi(x)I'\lambda(y)$ and $\forall i \in \{1,2,3\}$ $(\ell \in L \Rightarrow \lambda(\ell)\in L_i')$. The net isomorphism is an equivalence relation on the class of all nets. The induced equivalence classes are maximal subclasses of mutually isomorphic nets.

From every net $N=(P,L,I,(L_1,L_2,L_3))$ we can obtain nets $N_{ijk}=(P,L,I,(L_i,L_j,L_k))$ (where (i,j,k) are permutations of the set $\{1,2,3\}$) called parastrophs of N.

A three-basic groupoid is defined as a quadruplet $(A,B,C,\cdot)$ where A,B,C are non-empty sets and $\cdot : A\times B\to C$, $(a,b)\mapsto a\cdot b$ is a "three-basic" binary operation. This groupoid is said to be a three-basic quasigroup, if for every $(a,c)\in A\times C$ there exists just one $b\in B$ such that $a\cdot b=c$ and if for every $(b,c)\in B\times C$ there exists just one $a\in A$ such that $a\cdot b=c$. Let $G=(A,B,C,\cdot)$, $G'=(A',B',C',\cdot')$ be three-basic quasigroups. A triplet (α,β,γ) of bijections $\alpha:A\to A'$, $\beta:B\to B'$, $\gamma:C\to C'$ is called an isotopy of G onto G' if for all $x\in A$, $y\in B$ the equation $\alpha(x)\cdot'\beta(y)=\gamma(x\cdot y)$ is valid. The isotopy is an equivalence relation on the class of all three-basic quasigroups. It divides this class onto maximal subclasses of mutually isotopic quasigroups.

THEOREM (cf. [1], pp. 396-398):

a. Every net $N=(P,L,I,(L_1,L_2,L_3))$ canonically determines a three-basic quasigroup $Q_N=(L_1,L_2,L_3,\cdot_N)$ such that for all $\ell_1\in L_1$, $\ell_2\in L_2$, $\ell_3\in L_3$: $\ell_1\cdot_N\ell_2=\ell_3 \Leftrightarrow \{p|pI\ell_1,\ell_2,\ell_3\}\neq\emptyset$.

b. Every three-basic quasigroup $Q=(Q_1,Q_2,Q_3,\cdot)$ with disjoint sets Q_1,Q_2,Q_3 canonically determines a net $N_Q=(Q_1\times Q_2, Q_1\cup Q_2\cup Q_3, I_Q,$

$(Q_1,Q_2,Q_3))$ where for all $x_1\in Q_1, x_2\in Q_2, x\in Q_1\cup Q_2\cup Q_3 : (x_1,x_2)I_Q x \Leftrightarrow$
$\Leftrightarrow (x=x_1 \vee x=x_2 \vee x=x_1\cdot x_2)$.

c. If N is a net then N_{Q_N} is isomorphic to N. If Q is a three-basic quasigroup then Q_{N_Q} is isotopic to Q.

d. Two nets N,N' are isomorphic if and only if $Q_N, Q_{N'}$ are isotopic.

If $Q=(Q_1,Q_2,Q_3,\cdot)$ is a three-basic quasigroup then for all permutations (i,j,k) of the set $\{1,2,3\}$ denote by $\cdot_{ijk}$ the operation $\cdot_{ijk}: Q_i\times Q_j\rightarrow Q_k$ such that $x_i \cdot_{ijk} x_j = x_k \Leftrightarrow x_1\cdot x_2 = x_3$ for all $x_1\in Q_1, x_2\in Q_2$, $x_3\in Q_3$. Evidently all $(Q_i,Q_j,Q_k,\cdot_{ijk})$ are quasigroups (the so called <u>parastrophs</u> of Q). The operations $\cdot_{321}$ or $\cdot_{132}$ will be denoted later also by $/$ $(x_1\cdot x_2=x_3 \Leftrightarrow x_1=x_3/x_2)$ or by $\setminus$ $(x_1\cdot x_2=x_3\Leftrightarrow x_2=x_1\setminus x_3)$.

§ 2 Configurations and closure conditions in 3-nets

A <u>3-halfnet</u> (briefly: a <u>halfnet</u>) is defined as a quadruplet $(P,L,I,(L_1,L_2,L_3))$ where P,L are sets, $I\subseteq P\times L$, $L_1,L_2,L_3\subseteq L$, $L_1\cap L_2=\emptyset$, $L_1\cap L_3=\emptyset$, $L_2\cap L_3=\emptyset$, $L_1\cup L_2\cup L_3=L$ such that

(i) for every $i\in\{1,2,3\}$ and every $p\in P$ there is at most one $\ell\in L_i$ with $pI\ell$, and

(ii) for any two distinct $a,b\in L$ there is at most one $c\in P$ with cIa,b.

The terms points, lines, parallels, parastrophs, ranks etc. for halfnets have a similar meaning as for nets.

We say a halfnet $N=(P,L,I,(L_1,L_2,L_3))$ is a <u>sub-halfnet</u> of a halfnet $N'=(P',L',I',(L_1',L_2',L_3'))$ if $P\subseteq P', I\subseteq I', L_1\subseteq L_1', L_2\subseteq L_2', L_3\subseteq L_3'$ (so that also $L\subseteq L'$). A halfnet $(P,L,I,(L_1,L_2,L_3))$ is said to be a <u>configuration</u> if

(i) P is finite and contains at least four points,

(ii) for every $p\in P$ there are $\ell_1\in L_1, \ell_2\in L_2, \ell_3\in L_3$ such that $pI\ell_1,\ell_2,\ell_3$,

(iii) for every $\ell\in L$ there are distinct $a,b\in P$ such that $a,bI\ell$, and

(iv) for any $a,b \in P$ there is a sequence $(p_0, \ell_0, p_1, \ell_1, \dots, p_m)$ with $p_0, p_1, \dots, p_m \in P$; $\ell_0, \ell_1, \dots, \ell_{m-1} \in L$; $p_0 = a$; $p_m = b$; $p_0, p_1 \, I \, \ell_0$; $p_1, p_2 \, I \, \ell_1$; $\dots$; $p_{m-1}, p_m \, I \, \ell_{m-1}$ (briefly: any two points are connected).

It can be easily seen that every configuration is a sub-halfnet in a convenient net.

A homomorphism of a halfnet $N=(P,L,I,(L_1,L_2,L_3))$ into a halfnet $N'=(P',L',I',(L'_1,L'_2,L'_3))$ is defined as a couple (π, λ) of maps $\pi: P \to P'$, $\lambda: L \to L'$ such that for all $p \in P, \ell \in L$ from $pI\ell$ it follows $\pi(p) \, I' \, \lambda(\ell)$ and for all $i \in \{1,2,3\}$ from $\ell \in L_i$ it follows $\lambda(\ell) \in L'_i$.

Let $\tilde{N} = (\tilde{P}, \tilde{L}, \tilde{I}, (\tilde{L}_1, \tilde{L}_2, \tilde{L}_3))$ be a configuration with a prominent "terminal" line $\tilde{\ell}_0 \in \tilde{L}$ by deleting of which it is obtained a sub-halfnet $\tilde{N}_0$ of $\tilde{N}$. We say that the closure condition associated to $\tilde{N}$ with $\tilde{\ell}_0$ is valid in a net $N=(P,L,I,(L_1,L_2,L_3))$ if every homomorphism of $\tilde{N}_0$ into N can be prolonged onto a homomorphism of $\tilde{N}$ into N. If (π_0, λ_0), (π, λ) is the starting homomorphism and the prolonged one, respectively, then $\pi_0 = \pi$ and $\lambda_0 = \lambda|_{\tilde{L} \setminus \{\tilde{\ell}_0\}}$.

§ 3 Configurations of point rank <8

Using the analysis of more general configurations of point rank <8 in nets of arbitrary finite degree (cf. [3], chap. III) one can deduce all possible configurations of point rank <8 (up to isomorphisms and parastrophs) . The result is as follows:

There is only one configuration of point rank 4. It is described on Fig. 1.

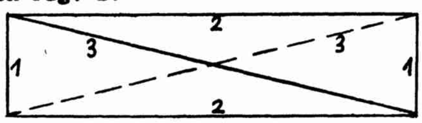

Fig. 1

There is no configuration of point rank 5.

There is exactly one configuration of point rank 6 possessing lines of length 3. It is described on Fig. 2.

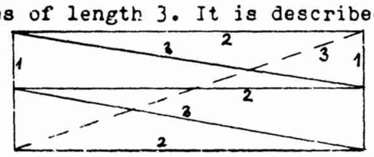

Fig. 2

There are exactly two configurations of point rank 6 with no line of length 3. They are described on Fig. 3 and 4.

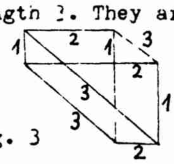

Fig. 3

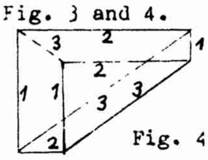

Fig. 4 .

We shall denote configurations of Fig. 1 and 2 as Fano configurations F_2 , F_3 of index 2 and 3, respectively. Configuration on Fig. 3 is Thomsen configuration T and configuration on Fig. 4 is a shattered Desargues configuration D.

There are only three configurations of point rank 7. They are described on Fig. 5-7. We shall denote them as hexagonal configuration H , first hybrid configuration C_1 and second hybrid configuration C_2 .

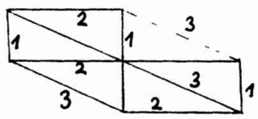

Fig. 5

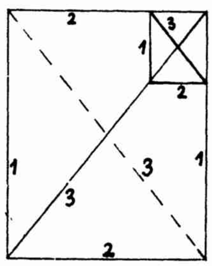

Fig. 6

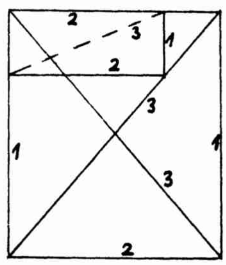

Fig. 7

§ 4 Closure conditions of point rank <8

Now we shall investigate closure conditions associated to configurations $F_2, F_3, T, D, H, C_1, C_2$ with terminal lines denoted in Fig. 1-7 interruptedly. These closure conditions will be denoted by $F_2, F_3, T, D, H, C_1, C_2$ too.

Let $N=(P,L,I,(L_1,L_2,L_3))$ be a net. Then closure condition F_2 is satisfied in N if and only if $a\cdot d=b\cdot c \Rightarrow a\cdot c=b\cdot d$ $(\cdot = \cdot_N)$ for all $a,b\in L_1$ and $c,d\in L_2$. This conditional identity can be rewritten as an identity $a\diagdown(b\cdot c)=b\diagdown(a\cdot c)$ (for all $a,b\in L_1$ and $c\in L_2$). It is well-known ([2], pp. 66-69) that precisely in this case Q_N is isotopic with an abelian group of index 2.

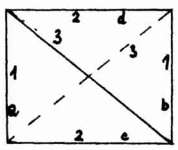

Fig. 8

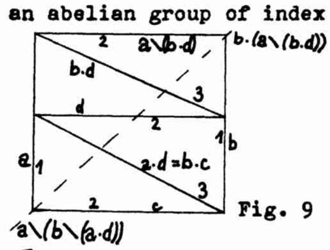

Fig. 9

In other words, closure condition F_2 is satisfied in N if and only if every loop $(Q,\cdot,1)$ isotopic to Q_N is an abelian group satisfying the identity $x\cdot x=1$.

Closure condition F_3 is satisfied in N if and only if $a\cdot d=b\cdot c \Rightarrow a\cdot c=b\cdot(a\diagdown(b\cdot d))$ for all $a,b\in L_1$; $c,d\in L_2$ or, equivalently, if and only if $a\cdot(b\diagdown(a\cdot d))=b\cdot(a\diagdown(b\cdot d))$ for all $a,b\in L_1$ $d\in L_2$. For every loop $(Q,\cdot,1)$ isotopic to Q_N the identity $a\cdot(b\diagdown(a\cdot d))=b\cdot(a\diagdown(b\cdot d))$ is valid, too. Putting $b=1$, $d=1$ we obtain $a\cdot a=a\diagdown 1$, $a\cdot(a\cdot a)=1$. Conversely, if every loop $(Q,\cdot,1)$ isotopic to Q_N satisfies the identity $x\cdot(x\cdot x)=1$ then the points $(1,1),(x,1),(1,x),(x,x),(1,x\cdot x),(x,x\cdot x)$ of N_Q are points of a configuration $\tilde{F}_3$ isomorphic to F_1 (without terminal lines) and the points $(1,1)$, $(1,x\cdot(x\cdot x))$ must coincide because of $x\cdot(x\cdot x)=1$ so that the points $(1,1)$, $(1,x\cdot(x\cdot x))$ must lie on the same line of the third parallelity class of N_Q. If we take all loops isotopic to Q_N then isomorphic images of $\tilde{F}_3$ go over to all positions of configurations isomorphic to F_3 (without terminal lines). Thus the closure condition F_3 is valid in N. It results that N satisfies closure condition F_3 if and only if every loop isotopic to Q_N satisfies the identity $x\cdot(x\cdot x)=1$. Unfortunately we have not reached which is the inner structure of the isotopy class of loops with the identity $x\cdot(x\cdot x)=1$. Remark without proof that in a loop $(Q,\cdot,1)$ the identity $a\cdot(b\backslash(a\cdot d))=b\cdot(a\backslash(b\cdot d))$ is equivalent with the identity $a\cdot(b\cdot(b\cdot(a\cdot(b\cdot(b\cdot(a\cdot c))))))=b\cdot c$ or with two identities $a\cdot(a\cdot(a\cdot c))=c$, $a\cdot(b\cdot(b\cdot(a\cdot c)))=b\cdot(a\cdot(a\cdot(b\cdot c)))$.

It is well-known (cf. [2], pp. 42-43) that N satisfies closure condition T if and only if every loop isotopic to Q_N is an abelian group. This result can be obtained in our description as follows: N satisfies closure condition T if and only if Q_N satisfies the identity $a\cdot(d\backslash(b\cdot c))=b\cdot(d\backslash(a\cdot c))$ for all $a,b,d\in L_1$ and $c\in L_2$. Every loop $(Q,\cdot,1)$ isotopic to Q_N satisfies the identity $a\cdot(d\backslash(b\cdot c))=b\cdot(d\backslash(a\cdot c))$ too. Putting $d=1$ we get $a\cdot(b\cdot c)=$ $=b\cdot(a\cdot c)$.

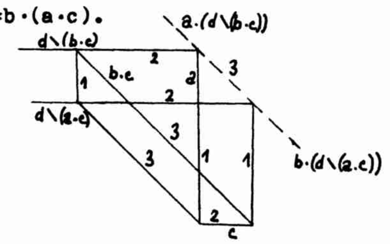

Fig. 10

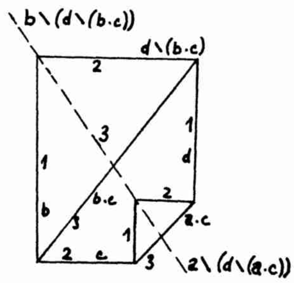

Fig. 11

For $c=1$ we obtain $a\cdot b=b\cdot a$, the commutativity. Using the commutativity, $a\cdot(b\cdot c)=b\cdot(a\cdot c)$ can be rewritten as $(b\cdot c)\cdot a=b\cdot(c\cdot a)$, the associativity. Using the same argumentation as for F_3 we can deduce that N satisfies closure condition T whenever every loop isotopic to Q_N is an abelian group.

N satisfies closure condition D if and only if Q_N satisfies the identity $a\diagdown(d\diagdown(a\cdot c))=b\diagdown(d\diagdown(b\cdot c))$ for all $a,b,d\in L_1$ and $c\in L_2$. In every loop $(L,\cdot,1)$ isotopic to Q_N the preceding identity holds, too. Putting $b=1$, $c=1$ we get $a\diagdown(d\diagdown a)=d\diagdown 1$, $a\cdot(d\diagdown 1)=d\diagdown a$. By the same reasoning as by closure condition F_3 we get the following result: N satisfies closure condition D if and only if every loop $(Q,\cdot,1)$ isotopic to Q_N satisfies the identity $a\cdot(d\diagdown 1)=d\diagdown a$. In loops $(Q,\cdot,1)$ with left inverse property this identity goes over the commutativity.

N satisfies closure condition H if and only if every loop $(Q,\cdot,1)$ isotopic to Q_N satisfies the identity $x\cdot(x\cdot x)=(x\cdot x)\cdot x$ ([2], pp. 46-47) or if and only if in every loop isotopic to Q_N all by one element generated subloops are subgroups ([2],pp.47--50). In our description N satisfies closure condition H if and only if $((c\cdot(a\diagdown(c\cdot b))\diagup b)\ (a\diagdown(c\cdot b))=c\cdot(a\diagdown(c\cdot(a\diagdown(c\cdot b))))$ for all $a,c\in L_1$ and $b\in L_2$. If $(L,\cdot,1)$ is a loop isotopic to Q_N then it satisfies the preceding identity, too. If we put $a=1,b=1$ we get $(c\cdot c)\cdot c=c\cdot(c\cdot c)$. Similarly as for closure condition F_3 we can obtain the result: N satisfies closure condition H if and only if all loops $(Q,\cdot,1)$ isotopic to Q_N satisfy the identity $(x\cdot x)\cdot x=x\cdot(x\cdot x)$.

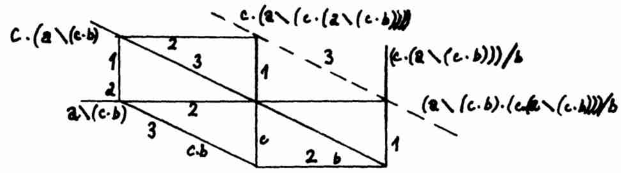

Fig. 12

Both hybrid configurations have only restricted importance: If N satisfies closure condition F_2 then it satisfies consequently closure condition C_1, too. If N does not satisfy closure condition F_2 then closure condition C_1 depends on the existence of a non-void set of all "parallelograms with parallel diagonals" in N and describes some property of this set. We shall not investigate the details here.

As it is easily seen a net N satisfying both closure conditions F_2, C_2 must be necessarily trivial. If N does not satisfy closure condition F_2 then closure condition C_2 describes some property of "triangles inscribed into triangles formed from two sides and one diagonal of parallelograms with parallel diagonals". The detailes are omitted, too.

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