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NOTE ON A PAPER OF McMORRIS AND SHIER

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Abstract: F. R. McMORRIS and D. R. SHIER [3] proved: A graph G is split iff G can be represented as an intersection graph of a set of distinct subtrees of $K_{1,n}$. They give a method for constructing this intersection graph. Here an improved construction with minimum n is described.

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Only finite connected simple graphs are to be considered. For the terminology see [1] and [3].

A graph G is said to be represented on a tree T if and only if G is isomorphic to the intersection graph of a set of distinct subtrees of T.

A graph $G = (V,E)$ is split if and only if there is a partition of the vertex set as $V = I \cup K$, where I is an independent set and K is complete. Furthermore, the partition $V = I \cup K$ can be chosen so that K is a maximum clique [2]. Henceforth we shall assume that K has been chosen in this manner.

Investigating chordal graphs F. R. McMORRIS and D. R. SHIER proved [3]:

Theorem 1. A graph $G = (V,E)$ is split if and only if G can be represented on $K_{1,n}$ for some n.

In their proof F. R. McMORRIS and D. R. SHIER [3] constructed a representation on $K_{1,n}$ for a given split graph. They claim that their method of construction provides a representation of G on $K_{1,n}$ using the smallest possible n . This is not true and I shall give the corrected method.

Before describing it we define: if A , B are sets then $P(A) \underset{\text{def}}{=} \{M/M \subseteq A\}$, where $\emptyset \in P(A)$ and $B \cup P(A) \underset{\text{def}}{=} \{X/X = B \cup M \ , \ M \in P(A)\}$. The subgraph of G induced by H is denoted by $G[H]$. Let $N(x)$ denote the set of all neighbours of the vertex $x \in V$. If $I \subseteq V$ then $N_I(x) \underset{\text{def}}{=} N(x) \cap I$. For real q let $\lceil q \rceil$ denote the smallest integer $\geq q$.

Construction. Suppose $G = (V,E)$ is split, where $V = I \cup K$ and $I = \{x_1, \ldots , x_r\}$. First, label the end vertices (of degree 1) in $T \underset{\text{def}}{=} K_{1,r}$ by the integers $1, \ldots , r$ and the vertex of degree r by 0 . Define the subtree $T(x_i)$, corresponding to vertex x_i , by $T(x_i) = \{i\}$, for all $1 \leq i \leq r$. Next, let L , initially empty, denote a collection of subsets and A , also initially empty, a set of additional vertices of T . For each $y \in K$, we consult L to see if all members of $N_I(y) \cup P(A)$ are in the list L . If not, choose one of the members M not in L , define subtree $T(y) \underset{\text{def}}{=} T[M \cup \{0\}]$ and add M to the list L . If all members of $N_I(y) \cup P(A)$ are in the list L we add a new end vertex α to the current T (joining it to vertex 0) and define $T(y) = T[N_I(y) \cup \{0, \alpha\}]$. We add α to the list A and $N_I(y) \cup \{\alpha\}$ to the list L . This procedure is repeated for all vertices $y \in K$. Upon completion, the process yields a $K_{1,n}$ and a set of distinct subtrees that represent G . □

Applying my construction to K_4 and $\overline{K_6}$ (obtained from K_6

by omitting an edge) I have in both cases a $K_{1,n}$ with an n which is smaller than the one of F. R. McMORRIS and D. R. SHIER.

<u>Theorem 2</u>. For every split graph $G = (V,E)$, $V = I \cup K$, the Construction provides a representation of G on $K_{1,n}$ with minimal n . If m denotes the maximum number of vertices of K having the same neighbourhood $N_I(y)$ in I then the minimal $n = |I| + \lceil \log_2 m \rceil$.

For the simple proof we need the following obvious lemma.

<u>Lemma 3</u>. Let $S_1, \ldots, S_r$ and $T_1, \ldots, T_s$ be the subtrees of $K_{1,n}$ containing precisely 1 vertex or ≥ 2 vertices, respectively. Then

i) in the intersection graph G^* the subtrees $S_1, \ldots, S_r$ form a $\overline{K}_r$ and the subgraph of G^* induced by $T_1, \ldots, T_s$ is a K_s ;

ii) if S_i $(1 \leq i \leq r)$ consists of the "central" vertex (of degree n) of $K_{1,n}$ then S_i is joined to all T_j $(1 \leq j \leq s)$ by edges; i. e. the subgraph of G^* induced by $S_i, T_1, \ldots, T_s$ is a K_{s+1} .

<u>Proof of Theorem 2</u>. Let $G = (V,E)$ be split with partition $V = I \cup K$ such that K is of maximum possible order. Let $I = \{x_1, x_2, \ldots, x_r\}$ and $K = \{y_1, y_2, \ldots, y_s\}$. Let $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ be a representation of G on $K_{1,n}$ such that $x_i \leftrightarrow X_i$ and $y_j \leftrightarrow Y_j$. By Lemma 3 and the maximality of K each subtree X_i consists of an end vertex of $K_{1,n}$. Let the vertices of $K_{1,n}$ be denoted by 0, 1, ... , n so that 0 is the "central" vertex of $K_{1,n}$ and $X_i = \{i\}$ for $1 \leq i \leq r$.

The subtree Y_j contains the vertex i of $K_{1,n}$ $(1 \leq i \leq r)$ iff $(x_i, y_j) \in E$. Thus the subtree $Y_j[\{0,1,\ldots,r\}]$ of Y_j

induced by $\{0, 1, \dots, r\}$ is uniquely determined.

Let m be an integer defined as follows: there are m vertices $y^1, \dots, y^m \in K$ having the same neighbourhood in I and there are no $m+1$ such vertices in K . Let $Y^1, \dots, Y^m$ denote the corresponding subtrees. Then $Y^1[\{0,\dots,r\}] = \dots$
$\dots = Y^m[\{0,\dots,r\}]$.

Since $Y^1, \dots, Y^m$ are pairwise distinct subtrees they contain some of the vertices $r+1, \dots, n$. With these vertices a set $N_I(y_1) \cup P(\{r+1, \dots, n\})$ of 2^{n-r} subtrees of $K_{1,n}$ with fixed $N_I(y_1)$ can be formed. Consequently, the minimal n has to be chosen $n = r + \lceil \log_2 m \rceil$. □

In a further paper I shall investigate intersection graphs of a set S of distinct subtrees of a tree T , where no element of S is contained in an other element of S .

References

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