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SOME CLASS OF UNIFORMLY NON-SQUARE ORLICZ-BOCHNER SPACES

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Abstract: It is proved that if X is a uniformly non-square normed space, Φ is a uniformly convex Orlicz function satisfying the respective condition Δ_2 and μ is a non-negative and σ-finite measure, then the Orlicz-Bochner space $L^{\Phi}(\mu,X)$ is uniformly non-square. It is proved also that the assumptions about X and partially about Φ are necessary.

Key words and phrases: Orlicz function, Orlicz-Bochner spaces, uniformly non-square normed spaces, condition Δ_2.

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0. Introduction. (T,Σ,μ) is a measure space with non-negative and σ-finite measure, R denotes the real line, $R_+ = [0,+\infty)$, $(X,\|\cdot\|)$ is a normed space. We assume for simplicity that all atoms are of measure one. A mapping $\Phi: R \longrightarrow R_+$ is called an Orlicz function if it is convex, even, and vanishing only at zero. By $F(\mu,X)$ we denote the space of all equivalence classes of strongly Σ-measurable functions $f: T \longrightarrow X$.

Let Φ be an Orlicz function. We define on $F(\mu,X)$ the convex modular I (for definition see [9]) by

$$I(f) = \int_T \Phi(\|f(t)\|)d\mu .$$

The Orlicz-Bochner space $L^{\Phi}(\mu,X)$ is defined by

$$L^{\Phi}(\mu,X) = \{f \in F(\mu,X): I(kf) < \infty \text{ for some } k>0\}.$$

This space is a normed space under the so-called Luxemburg norm

$\|f\|_\Phi = \inf\{r>0: I(x/r)\leq 1\}$.

We say an Orlicz function Φ is <u>uniformly convex</u> (see [8]) if for every $a\in(0,1)$ there exists $p(a)\in(0,1)$ such that

$$\Phi\left(\frac{u+au}{2}\right)\leq\frac{1-p(a)}{2}\{\Phi(u)+\Phi(au)\}$$

for every $u\in R$. If Φ is a uniformly convex Orlicz function, then the inequality

$$\Phi\left(\frac{u+bu}{2}\right)\leq\frac{1-p(a)}{2}\{\Phi(u)+\Phi(bu)\}$$

holds for all $u\in R$ and $0\leq b\leq a$ (see [1]).

A normed space $(X, \|\cdot\|)$ is called uniformly non-square if there exists $\varepsilon>0$ such that for every $x,y\in X$ satisfying $\max(\|x\|,\|y\|)\leq 1$ we have $\min\left(\left\|\frac{x+y}{2}\right\|,\left\|\frac{x-y}{2}\right\|\right)\leq 1-\varepsilon$ (see [5]).

1. <u>Results</u>

<u>Theorem 1.1</u>. Let Φ be a uniformly convex Orlicz function satisfying the respective condition Δ_2, i.e. there exists a constant $K,a>0$ such that the inequality $\Phi(2u)\leq K\,\Phi(u)$ holds:

(i) for all $u\in R$ if μ is an infinite measure that is not purely atomic,

(ii) for $u\in R$ satisfying $|u|\geq a$ if μ is an atomless and finite measure,

(iii) for $u\in R$ satisfying $|u|\leq a$ if μ is a purely atomic measure.

Let X be a uniformly non-square normed space. Then the Orlicz-Bochner space $L^\Phi(\mu,X)$ is uniformly non-square.

Proof. It follows from the respective condition Δ_2 for Φ that for every $\varepsilon\in(0,1)$ there exists $\delta(\varepsilon)\in(0,1)$ such that for every $f\in L^\Phi(\mu,X)$ the inequality $I(f)\leq 1-\varepsilon$ implies $\|f\|_\Phi\leq 1-$

$-\ \delta(\varepsilon)$ (see [3],[6],[8]).

First, we shall prove the inequality

$$(1)\qquad \Phi\left(\left\|\frac{x+y}{2}\right\|\right)+\Phi\left(\left\|\frac{x-y}{2}\right\|\right)\le\alpha\{\Phi(\|x\|)+\Phi(\|y\|)\}$$

for all $x,y\in X$ (with an absolute constant $\alpha\in(0,1)$). Let $\varepsilon>0$ be the ε in the definition of X being uniformly non-square and let $x,y\in X$. We have

$$\min\left(\left\|\frac{x+y}{2}\right\|,\left\|\frac{x-y}{2}\right\|\right)\le(1-\varepsilon)\max(\|x\|,\|y\|).$$

Without loss of generality we may assume that $\|y\|\le\|x\|$ and $\|x+y\|\le\|x-y\|$. Thus, we have $\|x+y\|\le 2(1-\varepsilon)\|x\|$. We shall consider two cases.

I. $\|x\|\le\|y\|/\sqrt{1-\varepsilon}$. Then, we have

$$\Phi\left(\left\|\frac{x+y}{2}\right\|\right)\le\Phi((1-\varepsilon)\|x\|)\le\Phi\left(\sqrt{1-\varepsilon}\,\frac{\|x\|+\|y\|}{2}\right)\le$$

$$\le\frac{\sqrt{1-\varepsilon}}{2}\{\Phi(\|x\|)+\Phi(\|y\|)\ .$$

II. $\|y\|\le\sqrt{1-\varepsilon}\,\|x\|$. Then, by uniform convexity of Φ, we have

$$\Phi\left(\left\|\frac{x+y}{2}\right\|\right)\le\Phi\left\{\frac{\|x\|+\|y\|}{2}\right\}\le\frac{1-p(\sqrt{1-\varepsilon})}{2}\{\Phi(\|x\|)+\Phi(\|y\|)\ .$$

Denoting $\sigma=\max(\sqrt{1-\varepsilon},\ 1-p(\sqrt{1-\varepsilon}))$ and applying the triangle inequality for the norm $\|\cdot\|$ and convexity of Φ to the term $\Phi\left(\left\|\frac{x-y}{2}\right\|\right)$, we get the inequality (1) with $\alpha=(\sigma+1)/2$.

Now, let $f,g\in L^{\Phi}(\mu,X)$ and $\max(\|f\|_{\Phi},\|g\|_{\Phi})\le 1$. Then $\max(I(f),I(g))\le 1$. Applying the inequality (1), we have for any $t\in T$

$$\Phi\left(\left\|\frac{f(t)+g(t)}{2}\right\|\right)+\Phi\left(\left\|\frac{f(t)-g(t)}{2}\right\|\right)\le\alpha\{\Phi(\|f(t)\|)+$$

$$+\Phi(\|g(t)\|)\ .$$

Integrating this inequality both-side over T, we get

$$I(\tfrac{f+g}{2}) + I(\tfrac{f-g}{2}) \le \alpha(I(f) + I(g)) \le 2\alpha .$$

Thus, we have

$$\min (I(\tfrac{f+g}{2}), I(\tfrac{f-g}{2})) \le \alpha .$$

Hence, we obtain

$$\min (\|\tfrac{f+g}{2}\|_\Phi , \|\tfrac{f-g}{2}\|_\Phi) \le 1 - \delta(1 - \alpha),$$

and the proof is finished.

Theorem 1.2. If the Orlicz-Bochner space $L^\Phi(\mu,X)$ is uniformly non-square, then Φ is an Orlicz function satisfying the respective condition Δ_2 and X is a uniformly non-square normed space.

Proof. If Φ does not satisfy the respective condition Δ_2, then the space $L^\Phi(\mu,X)$ contains an isometric copy of l^∞ (see e.g. [3], [4], [7] and [11]) and so $L^\Phi(\mu,X)$ is not a uniformly non-square, because l^∞ is not, too (see [2]).

If X is not uniformly non-square, then for every $\varepsilon > 0$ there exist $x,y \in X$ such that $\max (\|x\|,\|y\|) \le 1$ and $\min (\|x+y\|,\|x-y\|) > > 2(1 - \varepsilon)$. Let $u_0 > 0$ and $A \in \Sigma$ be such that $\Phi(u_0)\mu(A) = 1$, and let

$$f = u_0 x \chi_A, \quad g = u_0 y \chi_A.$$

We have $\max (\|f\|_\Phi , \|g\|_\Phi) \le 1$ and $\min (\|f+g\|_\Phi , \|f-g\|_\Phi) > 2(1 - \varepsilon)$. Thus, the space $L^\Phi(\mu,X)$ is not uniformly non-square.

Remarks. Theorem 1.1 and inequality (1) are some generalizations of Theorem 15 [10] and of Lemma 14 [10], respectively, in the case n=2. Note that the method of the proof of the inequality (1) is new.

An example of uniformly convex Orlicz function is $\Phi_0(u) = |u|^p$, where $1 < p < \infty$. Then $p(a) = 1-2^{1-p} (1+a^p)$. Moreover, if Φ and Ψ are two Orlicz functions and if at least one of them is uniformly convex, then the Orlicz functions $\Phi \circ \Psi$ and $\Phi \cdot \Psi$ are also uniformly convex (see [3]). The function $\Phi \circ \Psi$ may be uniformly convex even if no function Φ, Ψ is uniformly convex.

Question. Does Theorem 1.1 hold under the weaker assumption $\Phi(u/2) \leq \sigma\, \Phi(u)/2$ for all $u \in R$ with an absolute constant $\sigma \in (0,1)$ instead of the assumption of uniform convexity of Φ ?

This weaker condition is necessary in order that $L^{\Phi}(\mu, X)$ be uniformly non-square.

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